# Modular representations of Lie algebras and Humphreys' conjecture

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# Overview

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# Lie algebras of reductive groups

Let $G$ be a reductive algebraic group defined over an algebraically closed field $\Bbbk$ and $\mathfrak{g} = \mathrm{Lie}(G)$.

If $V$ is a rational $G$-module and $v \in V$ we put

$$G^v := \{g \in G \mid g.v = v\} \text{ and } \mathfrak{g}^v := \{x \in \mathfrak{g} \mid x.v = 0\}.$$

It is well-known that $\mathrm{Lie}(G^v) \subseteq \mathfrak{g}_v$ for all $v \in V$ and the equality holds if $\mathrm{char}(\Bbbk) = 0$. If $\mathrm{char}(\Bbbk) = p > 0$ the Lie algebra

$$\mathfrak{g} = \mathrm{Der}_{\text{left inv}}\, \Bbbk[G]$$

carries a natural $[p]$-th power map $\mathfrak{g} \ni x \mapsto x^{[p]} \in \mathfrak{g}$ such that $(\mathrm{ad}\, x)^p = \mathrm{ad}(x^{[p]})$. The *nilpotent cone* $\mathcal{N}(\mathfrak{g})$ is then defined as the set of all $x \in \mathfrak{g}$ such that $x^{[p]^d} = 0$ for $d \gg 0$.

# Standard hypotheses

(H1) the derived subgroup of $G$ is simply connected;
(H2) $p$ is either zero or a good prime for $G$;
(H3) $\mathfrak{g}$ admits a non-degenerate $(\mathrm{Ad}\ G)$-invariant form which we denote $\kappa : \mathfrak{g} \times \mathfrak{g} \to \mathbb{k}$.

If $p > 0$ and $G$ satisfies the standard hypotheses then

$$\mathfrak{g} = \tilde{\mathfrak{g}}_1 \oplus \cdots \oplus \tilde{\mathfrak{g}}_s \oplus \mathfrak{z}, \tag{1}$$

where $\mathfrak{z} \subseteq \mathfrak{z}(\mathfrak{g})$ and $\mathrm{Lie}(G_i)$ is an ideal of codimension $\leqslant 1$ in $\tilde{\mathfrak{g}}_i$ for all $i \leqslant s$. More precisely, $\mathrm{Lie}(G_i) = \tilde{\mathfrak{g}}_i$ unless $G_i$ is of type $\mathrm{A}_{rp-1}$ for some $r \in \mathbb{N}$ in which case we have that $\mathrm{Lie}(G_i) \cong \mathfrak{sl}_{rp}(\mathbb{k})$ and $\tilde{\mathfrak{g}}_i \cong \mathfrak{gl}_{rp}(\mathbb{k})$. Here $G_1, \ldots, G_s$ are the simple components of $G$. All direct summands in (1) are $(\mathrm{Ad}\ G)$-stable and pairwise commute.

Also, $\mathrm{Lie}(G^x) = \mathfrak{g}^x$ for all $x \in \mathfrak{g}$. Condition (H3) implies that for every $\chi \in \mathfrak{g}^*$ there is a unique $x \in \mathfrak{g}$ such that $\chi = \kappa(x, -)$. We say that $\chi$ is *nilpotent* if $x \in \mathcal{N}(\mathfrak{g})$.

# The ring $R$

Let $T$ be a maximal torus of the group $G$ and let $\Phi$ be the root system of $G$ with respect to $T$. Recall that $G_1, \dots, G_s$ are the simple components $G$. The root systems of the $G_i$'s with respect to $T$ are the irreducible components of $\Phi$. Later we are going to view the groups $G_i$ as $\Bbbk$-points of Chevalley group schemes defined and split over $\mathbb{Z}$. Since working over the integers with a fixed nilpotent element in mind is difficult we sacrifice the *bad primes* of $\Phi$ and introduce suitable localisations of $\mathbb{Z}$.

If all components of $\Phi$ have type A, B, C or D we set $R := \mathbb{Z}[\frac{1}{2}]$. If $\Phi$ has components of exceptional types and has no simple components of type $E_8$ we set $R := \mathbb{Z}[\frac{1}{6}]$. If $\Phi$ has a simple component of type $E_8$ we set $R := \mathbb{Z}[\frac{1}{30}]$.

# Reduced enveloping algebras

Suppose $p > 0$. The universal enveloping algebra $U(\mathfrak{g})$ contains a large central subalgebra $Z_p(\mathfrak{g})$ generated by all $x^p - x^{[p]}$ with $x \in \mathfrak{g}$. By the PBW theorem $Z_p(\mathfrak{g})$ is a polynomial algebra in $\dim \mathfrak{g}$ variables and $U(\mathfrak{g})$ is a free $Z_p(\mathfrak{g})$-module of rank $p^{\dim \mathfrak{g}}$. This implies that all irreducible $\mathfrak{g}$-modules have finite dimension. For any irreducible $\mathfrak{g}$-module $V$ there is a linear function $\chi = \chi_V$, called the *p-character* of $V$, such that $x^p - x^{[p]} \in U(\mathfrak{g})$ acts on $V$ as $\chi(x)^p \cdot I_V$ for any $x \in \mathfrak{g}$. Any linear function on $\mathfrak{g}$ serves as a $p$-character of an irreducible $\mathfrak{g}$-module.

Given $\chi \in \mathfrak{g}^*$ we let $I_\chi$ denote the 2-sided ideal of $U(\mathfrak{g})$ generated by all $x^p - x^{[p]} - \chi(x)^p$ with $x \in \mathfrak{g}$ and define

$$U_\chi(\mathfrak{g}) := U(\mathfrak{g})/I_\chi \cong U(\mathfrak{g}) \otimes_{Z_p(\mathfrak{g})} \Bbbk_\chi.$$

This algebra has dimension $p^{\dim \mathfrak{g}}$ for every $\chi \in \mathfrak{g}^*$.

# Humphreys' conjecture

Suppose $p > 0$ and let $\mathcal{O}(\chi)$ denote the coadjoint $G$-orbit of $\chi \in \mathfrak{g}^*$. It is known that any irreducible $U_\chi(\mathfrak{g})$ has dimension divisible by $p^{d(\chi)}$ where $d(\chi) = \frac{1}{2}\dim \mathcal{O}(\chi)$. This was conjectured by Kac and Weisfeiler in 1971 and proved by the presenter in 1995 with the help of the cohomological theory of support varieties developed for restricted Lie algebras by Friedlander–Parshall and Jantzen. Different proofs of the Kac–Weisfeiler conjecture are now available in the literature, but the following very natural question remained open until now:

**Humphreys' conjecture.** *For every $\chi \in \mathfrak{g}^*$ there exists a $U_\chi(\mathfrak{g})$-module whose dimension equals $p^{d(\chi)}$.*

For $G = \mathrm{GL}_n$ the conjecture is known to be true for quite while. Very recently, all $U_\chi(\mathfrak{gl}_n)$-modules of dimension $p^{d(\chi)}$ are classified by Goodwin and Topley.

# Reduction to the nilpotent case

The direct sum decomposition (1) enables one to quickly reduce proving the conjecture to the case where $G$ is a simple algebraic group of type other than A. So we shall assume that this is the case from now on. We have $\chi = \kappa(x, -)$. Since $\mathfrak{g}$ is restricted, we have that $x = x_s + x_n$ where $[x_s, x_n] = 0$, $x_n \in \mathcal{N}(\mathfrak{g})$ and $x_s$ is $[p]$-semisimple. Then $x_s$ is $(\mathrm{Ad}\ G)$-conjugate to an element of the maximal toral subalgebra $\mathfrak{t} = \mathrm{Lie}(T)$. The centraliser $\mathfrak{g}^{x_s}$ is a Levi subalgebra of $\mathfrak{g}$; we call it $\mathfrak{l}$. We have a triangular decomposition $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{l} \oplus \mathfrak{n}_+$ and $\chi$ vanishes on both $\mathfrak{n}_-$ and $\mathfrak{n}_+$. In this situation an old result of Kac, generalised later by Friedlander–Parshall, says that $U_\chi(\mathfrak{g})$ is isomorphic to $\mathrm{Mat}_{p^{\dim \mathfrak{n}_-}}(U_\chi(\mathfrak{l}))$. Also, $x_n \in \mathcal{N}(\mathfrak{l})$ and the restriction of $\chi$ to $[\mathfrak{l}, \mathfrak{l}]$ coincides with that of $\kappa(x_n, -)$.

# Lusztig–Spaltenstein induction

Let $L$ be a proper Levi subgroup of $G$ and $\mathfrak{l} = \mathrm{Lie}(L)$. Consider a triangular decomposition $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{l} \oplus \mathfrak{n}_+$. Then $\mathfrak{p} := \mathfrak{l} \oplus \mathfrak{n}_+$ is a parabolic subalgebra of $\mathfrak{g}$. Let $e_0 \in \mathcal{N}(\mathfrak{l})$ and write $\mathcal{O}_L(e_0)$ for the adjoint $L$-orbit of $e_0$. The set $\mathcal{O}_L(e_0) + \mathfrak{n}_+$ is contained in $\mathcal{N}(\mathfrak{g})$. As the orbit set $\mathcal{N}(\mathfrak{g})/G$ is finite there is a unique nilpotent orbit in $\mathcal{N}(\mathfrak{g})$ which intersects this set densely; it is denoted $\mathrm{Ind}_{\mathfrak{l}}^{\mathfrak{g}}(\mathcal{O}_L(e_0))$ and referred to as *the orbit induced from* $\mathcal{O}_L(e_0)$. This orbit is, in fact, independent of the choice of a triangular decomposition. The most important property (for us) is as follows:

$$\dim \mathrm{Ind}_{\mathfrak{l}}^{\mathfrak{g}}(\mathcal{O}_L(e_0)) \;=\; \dim \mathcal{O}_L(e_0) + 2\dim \mathfrak{n}_+. \qquad (2)$$

The nilpotent $G$-orbits which cannot be induced from proper Levi subalgebras of $\mathfrak{g}$ are called *rigid*.

# Reduction to the rigid case

Suppose that $\mathcal{O} \subseteq \mathfrak{g}$ is an induced nilpotent orbit. Let $\mathfrak{l} = \mathrm{Lie}(L)$ be a proper Levi subalgebra and $\mathcal{O}_0 \subseteq \mathfrak{l}$ a rigid nilpotent $L$-orbit such that $\mathcal{O} = \mathrm{Ind}_{\mathfrak{l}}^{\mathfrak{g}}(\mathcal{O}_0)$. Choose a parabolic subalgebra $\mathfrak{p} \subset \mathfrak{g}$ which has $\mathfrak{l}$ as a Levi factor, and write $\mathfrak{n}$ for the nilradical of $\mathfrak{p}$. Let $e_0 \in \mathcal{O}_0$ and pick $e_0 + n \in (\mathcal{O}_0 + \mathfrak{n}) \cap \mathcal{O}$. We write $\bar{\chi} = \kappa(e_0, -)|_{\mathfrak{l}} \in \mathfrak{l}^*$ and $\chi = \kappa(e, -) \in \mathfrak{g}^*$. Observe that $\chi|_{\mathfrak{l}} = \bar{\chi}$ and $\chi(\mathfrak{n}) = 0$. This gives a natural embedding of algebras $U_{\bar{\chi}}(\mathfrak{l}) \hookrightarrow U_{\chi}(\mathfrak{g})$ through which we may inflate $U_{\bar{\chi}}(\mathfrak{l})$-modules to $U_{\chi}(\mathfrak{p})$-modules. If $V$ is a $U_{\bar{\chi}}(\mathfrak{l})$-module of dimension $p^{d(\bar{\chi})}$ where $d(\bar{\chi}) = (\dim \mathcal{O}_0)/2$, then the induced $U_{\chi}(\mathfrak{g})$-module

$$\tilde{V} := U_{\chi}(\mathfrak{g}) \otimes_{U_{\chi}(\mathfrak{p})} V$$

has dimension $p^{\dim \mathfrak{n} + d(\bar{\chi})}$. Applying (2) we deduce that $\tilde{V}$ is a $U_{\chi}(\mathfrak{g})$-module we are looking for.

# The rigid nilpotent orbits

Since any Levi subgroup $L$ of $G$ satisfies the standard hypotheses and the restricted enveloping algebra $U_0(\mathfrak{l})$ contains ideals of codimension 1, our earlier remarks show that in order to prove Humphreys' conjecture we may assume without loss of generality that $\chi = \kappa(e, -)$ for some *nonzero* rigid nilpotent element $e \in \mathfrak{g}$.

In characteristic zero, the rigid nilpotent orbits are classified by Kraft, in type A, by Kempken and Spaltenstein, in types B, C, D, and by Elashvili (and Elashvili–de Graaf) for exceptional Lie algebras. By a recent result of David Stewart and the presenter, the same description holds in arbitrary good characteristic.

Taking into account the direct decomposition (1) and the fact that in type A all nonzero nilpotent orbits are induced we may assume from now on that $G$ is simple and not of type A.

# Generalised Gelfand–Graev modules, I

Suppose $p \geqslant 0$ and let $e$ be a nonzero nilpotent element of $\mathfrak{g}$. Then $e$ is a $G$-unstable vector of $\mathfrak{g}$. If $p$ is good for $G$ then $e$ admits a rational cocharacter $\lambda_e\colon \mathbb{k}^\times \to G$ which is optimal for $e$ in the sense of the Kempf–Rousseau theory and has the property that $e$ has weight 2 with respect to $\lambda_e$. The adjoint action of $\lambda_e(\mathbb{k}^\times)$ gives $\mathfrak{g}$ a $\mathbb{Z}$-grading $\mathfrak{g} = \bigoplus_{i\in\mathbb{Z}} \mathfrak{g}(i)$ such that $e \in \mathfrak{g}(2)$ and $\mathfrak{g}^e \subset \bigoplus_{i\geqslant 0} \mathfrak{g}(i)$. It follows that the subspace $\mathfrak{g}(-1)$ carries a non-degenerate symplectic form $\Psi$ given by $\Psi(x,y) = \kappa(e,[x,y])$ for all $x, y \in \mathfrak{g}(-1)$. There is a basis $\{z_1,\ldots,z_s,z_1',\ldots,z_s'\}$ of $\mathfrak{g}(-1)$ such that $\Psi(z_i,z_j) = \Psi(z_i',z_j') = 0$ and $\Psi(z_i,z_j') = \delta_{i,j}$ for all $i,j \leqslant s$. Let $\mathfrak{g}(-1)_0$ be the $\mathbb{k}$-span of all $z_i'$ and define $\mathfrak{m} := \mathfrak{g}(-1)_0 \oplus (\bigoplus_{i\leqslant -2} \mathfrak{g}(i))$. This is a nilpotent Lie algebra of dimension $d(\chi) = \frac{1}{2}\dim \mathcal{O}(e)$ and $\chi = \kappa(e,-)$ vanishes on $[\mathfrak{m},\mathfrak{m}]$. Let $\mathfrak{g}(0)_1 := \operatorname{span}\{z_i \mid 1 \leqslant i \leqslant s\}$.

# Generalised Gelfand–Graev modules, II

It is known that one can choose all $z_i$ to be positive root vectors with respect to a maximal torus $T$ of $G$ contained in $C_G(\lambda_e)$ and adjust $\mathfrak{g}(-1)_1$ in such a way that it is spanned by negative root vectors with respect to $T$. There exits a connected unipotent subgroup $M$ of $G$ with $\mathrm{Lie}(M) = \mathfrak{m}$.

As $\chi$ vanishes on $[\mathfrak{m}, \mathfrak{m}]$ we can define an $\mathfrak{m}$-module structure on a 1-dimensional vector space $\Bbbk 1_\chi$ by setting $x.1_\chi = \chi(x)1_\chi$ for all $x \in \mathfrak{m}$. For $p \geqslant 0$, we set

$$Q := U(\mathfrak{g}) \otimes_{U(\mathfrak{m})} \Bbbk_\chi,$$

an induced $\mathfrak{g}$-module, and for $p > 0$, we define

$$Q_\chi := U_\chi(\mathfrak{g}) \otimes_{U_\chi(\mathfrak{m})} \Bbbk_\chi,$$

an induced $\mathfrak{g}$-module with $p$-character $\chi$. Note that $Q$ is infinite-dimensional whilst $\dim Q_\chi = p^{\dim \mathfrak{g}^e + \dim \mathfrak{m}}$.

# Finite $W$-algebras

Let $\mathfrak{m}_\chi$ be the subspace of $U(\mathfrak{g})$ spanned by all $x - \chi(x)$ with $x \in \mathfrak{m}$. Then $Q \cong U(\mathfrak{g})/U(\mathfrak{g})\mathfrak{m}_\chi$ as $\mathfrak{g}$-modules. The unipotent group $\mathrm{Ad}\, M$ preserves the left ideal $U(\mathfrak{g})\mathfrak{m}_\chi$ of $U(\mathfrak{g})$ and hence acts on $Q$ and the fixed point space $Q^{\mathrm{Ad}\, M}$ carries a natural algebra structure.

The *finite W-algebra* associated with the pair $(\mathfrak{g}, e)$ is

$$U(\mathfrak{g}, e) \ := \ Q^{\mathrm{Ad}\, M}.$$

If $p = 0$ then $U(\mathfrak{g}, e) \cong (\mathrm{End}_{\mathfrak{g}}\, Q)^{\mathrm{op}}$ as algebras. If $p > 0$ then $(\mathrm{End}_{\mathfrak{g}}\, Q)^{\mathrm{op}}$ is generated by $U(\mathfrak{g}, e)$ and a large central subalgebra $Z_p(\mathfrak{g}, e)$ induced by the action of the $p$-centre $Z_p(\mathfrak{g})$ on $Q$.
For $p > 0$, the *restricted finite W-algebra* is

$$U_\chi(\mathfrak{g}, e) \ := \ (\mathrm{End}_{\mathfrak{g}}\, Q_\chi)^{\mathrm{op}}.$$

# A Morita theorem

Suppose $p > 0$. Then $Q_\chi$ is a projective generator for $U_\chi(\mathfrak{g})$. More precisely, $\dim U_\chi(\mathfrak{g}, e) = p^{\dim \mathfrak{g}^e}$ and

$$U_\chi(\mathfrak{g}) \;\cong\; \mathrm{Mat}_{p^{d(\chi)}}(U_\chi(\mathfrak{g}, e))$$

as algebras. This result provides another proof of the Kac–Weisfeiler conjecture and implies that Humphreys' conjecture is equivalent to the fllowing problem:

**Conjecture on one-dimensional representations**. All algebras $U(\mathfrak{g}, e)$ and $U_\chi(\mathfrak{g}, e)$ admit one-dimensional representations.
This conjecture was put forward by the presenter in 2007 and is known to hold for $p = 0$ and for $p \gg 0$ with no explicit bound on $p$. The proof was a collective effort of several mathematicians. So our task now is to prove this conjecture for $e$ rigid under our mild assumptions on $p$.

# The final reduction

Suppose $p > 0$. In this case, we have a $p$-centre $Z_p(\mathfrak{g}, e)$ in $U(\mathfrak{g}, e)$ and $U(\mathfrak{g}, e)$ is a free $Z_p(\mathfrak{g}, e)$-module of rank $p^{\dim \mathfrak{g}^e}$. Given a maximal ideal $J_\eta = \ker \eta$ of $Z_p(\mathfrak{g}, e)$ we define the *reduced finite W-algebra*

$$U_\eta(\mathfrak{g}, e) \ := \ U(\mathfrak{g}, e)/J_\eta \cong U(\mathfrak{g}, e) \otimes_{Z_p(\mathfrak{g},e)} \mathbb{k}_\eta.$$

The characters $\eta$ of $Z_p(\mathfrak{g}, e)$ are parametrised by the points of a good transverse slice $\chi + \mathfrak{o} \subset \mathfrak{g}^*$ to the orbit $\mathcal{O}(e)$. There is a $\mathbb{k}^\times$-action on $\chi + \mathfrak{o}$ which contracts all points of the slice to $\chi$.

If $U(\mathfrak{g}, e)$ admits a one-dimensional representation then one of the reduced $W$-algebras $U_\eta(\mathfrak{g}, e)$ has an ideal of codimension 1. Using the contracting $\mathbb{k}^\times$-action one obtains that $U_\chi(\mathfrak{g}, e)$ has an ideal of codimension 1, too.

# The Bala–Carter theory over the integers, I

Let $G_{\mathbb{Z}}$ be a split reductive $\mathbb{Z}$-group scheme with root datum $(\Phi, X(T_{\mathbb{Z}}), X_*(T_{\mathbb{Z}}), \Phi^\vee)$ where $T_{\mathbb{Z}}$ is a fixed maximal torus of $G_{\mathbb{Z}}$ split over $\mathbb{Z}$. We assume that $\Phi$ is irreducible of type other than A. Let $\Pi = \{\alpha_1, \ldots, \alpha_l\}$ be a basis of simple roots in $\Phi$ and assume further that the lattice of coweights $X_*(T_{\mathbb{Z}})$ is a free $\mathbb{Z}$-module with basis $\{\alpha^\vee \mid \alpha \in \Pi\}$. The Lie algebra $\mathfrak{g}_{\mathbb{Z}}$ of the group scheme $G_{\mathbb{Z}}$ is a lattice in $\mathfrak{g}_{\mathbb{C}} = \mathrm{Lie}(G_{\mathbb{C}})$ spanned over $\mathbb{Z}$ by a Chevalley basis $\{h_\alpha \mid \alpha \in \Pi\} \cup \{e_\alpha \mid \alpha \in \Phi\}$. For a subset $I \subseteq \{1, \ldots, l\}$ we put $\Pi_I := \{\alpha_i \mid i \in I\}$ and let $L_{\mathbb{Z},I}$ be the standard Levi subgroup scheme of $G_{\mathbb{Z}}$ corresponding to $\Pi_I$. Given a subset $J$ of $I$ we denote by $P_{I,J,\mathbb{Z}}$ the standard parabolic subgroup scheme of $L_{I,\mathbb{Z}}$ associated with $J$. Note that $\mathfrak{p}_{I,J,\mathbb{Z}} = \mathrm{Lie}(P_{I,J,\mathbb{Z}})$ has a natural $\mathbb{Z}$-grading $\mathfrak{p}_{I,J,\mathbb{Z}} = \bigoplus_{k\geqslant 0} \mathfrak{p}_{I,J,\mathbb{Z}}(k)$ such that $\mathfrak{p}_{I,J,\mathbb{Z}}(0) = \mathrm{Lie}(L_{J,\mathbb{Z}})$.

# The Bala–Carter theory over the integers, II

Let $\mathcal{P}(\Pi)$ the set of all pairs $(I, J)$ with $J \subseteq I \subseteq \{1, \ldots, l\}$ such that $P_{I,J,\mathbb{C}}$ is a *distinguished* parabolic subgroup of $L_{I,\mathbb{C}}$ Two pairs $(I, J)$ and $(I', J')$ ar said to be equivalent if there is $w \in W(\Phi)$ such that $w(\Pi_I) = \Pi_{I'}$ and $w(\Pi_J) = \Pi_{J'}$. Let $[\mathcal{P}(\Pi)]$ for the set of all equivalence classes. In good characteristic, this set parametrises the nilpotent orbits of $\mathfrak{g} = \mathfrak{g}_{\mathbb{Z}} \otimes_{\mathbb{Z}} \Bbbk$. For any $(I, J) \in \mathcal{P}(\Pi)$ there is a unique $h_{I,J} = \sum_{i \in I} a_i h_{\alpha_i}$ such that $\alpha_k(h_{I,J}) = 0$ for all $k \in J$ and $\alpha_k(h_{I,J}) = 2$ for all $k \in I \backslash J$. We set $\lambda_{I,J} := \sum_{i \in I} a_i \alpha_i^\vee$, an element of $X_*(T_{\mathbb{Z}})$ (it is important here that $a_i \in \mathbb{Z}$ for all $i \in I$). There is $w \in W(\Phi)$ such that $w(\lambda_{I,J})$ lies in the dual Weyl chamber associated with $\Pi$. Placing the nonnegative integers $\alpha(w(\lambda_{I,J}))$ with $\alpha \in \Pi$ atop of the corresponding vertices of the Dynkin graph of $\Pi$ one obtains the Dynkin label, $\Delta_{I,J}$, of $\mathcal{O}_{\mathbb{C}}(I, J)$ shared by $\mathcal{O}_{\Bbbk}(I, J)$ for any good $p$.

# Selecting $\tilde{e} \in \mathfrak{g}_{\mathbb{Z}} \cap \mathcal{O}_{\mathbb{C}}(I, J)$ for $e \in \mathcal{O}_{\Bbbk}(I, J)$

## Theorem

*Let $\Bbbk$ be an algebraically closed field whose characteristic is good for the root system $\Phi$ and let $\lambda_{I,J} \colon \mathbb{G}_{\mathrm{m}} \to T_{\mathbb{Z}}$ be the cocharacter associated with a pair $(I, J) \in \mathcal{P}(\Pi)$.*

(i) *There exists a principal Zariski open subscheme $\mathfrak{p}_{I,J,\mathbb{Z}}(2)_{\mathrm{reg}}$ of $\mathfrak{p}_{I,J,\mathbb{Z}}(2)$ such that $\mathfrak{p}_{I,J,\Bbbk}(2)_{\mathrm{reg}} = \mathfrak{p}_{I,J,\Bbbk}(2) \cap \mathcal{O}_{\Bbbk}(I, J)$ for any field $\Bbbk$ as above.*

(ii) *The cocharacter $\lambda_{I,J}$ is optimal in the sense of the Kempf–Rousseau theory for any element of $\mathfrak{p}_{I,J,\Bbbk}(2)_{\mathrm{reg}}$.*

(iii) *There exits $e_{I,J} \in \mathfrak{p}_{I,J,\mathbb{Z}}(2)_{\mathrm{reg}} \cap \mathcal{O}_{\mathbb{C}}(I, J)$ whose image $e_{I,J} \otimes_{\mathbb{Z}} 1$ in $\mathfrak{g}_{\Bbbk} = \mathfrak{g}_{\mathbb{Z}} \otimes_{\mathbb{Z}} \Bbbk$ lies in $\mathfrak{p}_{I,J,\Bbbk}(2)_{\mathrm{reg}}$.*

# Working over $R$

Suppose $p$ if good for $G$. By the above, we may assume that $G = G_{\mathbb{Z}}(\Bbbk)$, $\mathfrak{g} = \mathfrak{g}_{\mathbb{Z}} \otimes_{\mathbb{Z}} \Bbbk$, and $e = e_{I,J} \otimes_{\mathbb{Z}} 1$ for some $(I, J) \in \mathcal{P}(\Pi)$. Set $\tilde{e} := e_{I,J}$ and $\mathfrak{g}_R := \mathfrak{g}_{\mathbb{Z}} \otimes_{\mathbb{Z}} R$. We have to make sure that our bilinear form $\kappa$ is $R$-valued on $\mathfrak{g}_R$.

Let $(\cdot\,|\,\cdot)$ be the $W(\Phi)$-invariant scalar product on the Euclidean space $\mathbb{R}\Phi$ such that $(\alpha|\alpha) = 2$ for every *short* root $\alpha \in \Phi$. Put $d := (\tilde{\alpha}|\tilde{\alpha})/(\alpha_0|\alpha_0)$ where $\tilde{\alpha}$ and $\alpha_0$ are the highest root and the maximal short root in $\Phi_+(\Pi)$, respectively. We take for $\kappa$ the *normalised Killing form* on $\mathfrak{g}_{\mathbb{C}}$ which has the property that $\kappa(e_\alpha, e_{-\alpha}) = 2d/(\alpha|\alpha)$ and $\kappa(e_\alpha, e_\beta) = 0$ if $\beta \neq -\alpha$, where $\alpha, \beta \in \Phi$. This form is $\mathbb{Z}$-valued on $\mathfrak{g}_{\mathbb{Z}}$ and $(\mathrm{Ad}\; G_{\mathbb{Z}})$-invariant. If $\Phi$ is not of type A, it enables us to identify $\mathfrak{g}_R$ with the dual $R$-module $\mathfrak{g}_R^*$.

# The $R$-form of $Q$

We can now define $\Psi\colon\ \mathfrak{g}_R(-1) \times \mathfrak{g}_R(-1) \to R$ as before and show that exist a basis $z_i, z_i' \in \mathfrak{g}_R(-1)$ of the free $R$-module $\mathfrak{g}_R(-1)$ such that $\Psi(z_i, z_j) = \Psi(z_i', z_j') = 0$ and $\Psi(z_i, z_j') = \delta_{i,j}$ for all $i, j \leqslant s$. This enables us to define $\mathfrak{m}_R \subset \mathfrak{g}_R$ to be the $R$-module generated by $\bigoplus_{i \leqslant -2} \mathfrak{g}_R(i)$ and all $z_i'$ with $1 \leqslant i \leqslant s$. Note that $\mathfrak{m}_R$ is a direct summand of $\mathfrak{g}_R$. Let $\mathfrak{m}_{R,\chi}$ be the $R$-submodule of $U(\mathfrak{g}_R)$ generated by all $x - \chi(x)$ where $\chi(x) = \kappa(\tilde{e}, x)$ for all $x \in \mathfrak{g}_R$. We set $Q_R \ := \ U(\mathfrak{g}_R)/U(\mathfrak{g})\mathfrak{m}_{R,\chi}$. This is an induced $\mathfrak{g}_R$-module and $Q \cong Q_R \otimes_R \Bbbk$. This holds provided $\mathrm{char}(\Bbbk)$ is good for the root system $\Phi$.

The subquotient $Q_R^{\mathrm{ad}\,\mathfrak{m}_R}$ of $U(\mathfrak{g}_R)$ inherits from $U(\mathfrak{g}_R)$ a natural associative ring structure, but it seems difficult to describe this ring explicitly for a general $\tilde{e} = e_{I,J}$. Luckily, we only need to do this in the case where $\tilde{e}$ is rigid.

# Centralisers over $R$

Choosing $\tilde{e} = e_{I,J}$ as above has various advantages. It turns out that the centraliser $\mathfrak{g}_R^{\tilde{e}}$ and all its graded components $\mathfrak{g}_R^{\tilde{e}}(i)$ are direct summands of $\mathfrak{g}_R$. The reductive Lie algebra $\mathfrak{g}_{\mathbb{Q}}^{\tilde{e}}(0)$ is split over $\mathbb{Q}$, the field of rationals, and $\mathfrak{g}_R^{\tilde{e}}(0)$ is a nice $R$-form of $\mathfrak{g}_{\mathbb{Q}}^{\tilde{e}}(0)$ which behaves as a Chevalley $\mathbb{Z}$-form of $\mathfrak{g}_{\mathbb{C}}$. This means that

$$\mathfrak{g}_R^{\tilde{e}}(0) = \mathfrak{t}_R^{\tilde{e}} \oplus \sum_{\gamma \in \Phi_e} \mathfrak{g}_{R,\gamma}^{\tilde{e}}(0)$$

where $\mathfrak{t}_R^{\tilde{e}}$ is the $R$-form of a $\mathbb{Q}$-split maximal toral subalgebra of $\mathfrak{g}_{\mathbb{Q}}^{\tilde{e}}(0)$. Each root space $\mathfrak{g}_{R,\gamma}^{\tilde{e}}(0)$ with respect to $\mathfrak{t}_R^{\tilde{e}}$ is a free $R$-module of rank 1. All weight spaces $\mathfrak{g}_{R,\gamma}^{\tilde{e}}(i)$ with respect to $\mathfrak{t}_R^{\tilde{e}}$, where $i \geqslant 0$, are direct summands of $\mathfrak{g}_R$.

# $R$-forms of rigid finite $W$-algebras

For $\tilde{e}$ rigid we set $U(\mathfrak{g}_R, \tilde{e}) \;:=\; Q_R^{\operatorname{ad}\mathfrak{m}_R}$.

## Theorem

(i) *The ring $U(\mathfrak{g}_R, \tilde{e})$ is freely generated over $R$ by a PBW basis of $U(\mathfrak{g}_{\mathbb{C}}, \tilde{e})$ and*

$$U(\mathfrak{g}_R, \tilde{e}) \otimes_R \mathbb{C} \;\cong\; U(\mathfrak{g}_{\mathbb{C}}, \tilde{e}), \;\; U(\mathfrak{g}_R, \tilde{e}) \otimes_R \Bbbk \;\cong\; U(\mathfrak{g}, e)$$

*as algebras over the respective fields.*

(ii) *The ring $U(\mathfrak{g}_R, \tilde{e})$ contains a 2-sided ideal $I_R$ such that $U(\mathfrak{g}_R, \tilde{e}) \;=\; R\,1 \oplus I_R$.*

(iii) *The 2-sided ideal $I_{\Bbbk} \;:=\; I_R \otimes_R \Bbbk$ of $U(\mathfrak{g}, e) \;\cong\; U(\mathfrak{g}_R, \tilde{e}) \otimes_R \Bbbk$ has codimension 1 in $U(\mathfrak{g}, e)$.*

# The main result

Let $p$ be good prime for $G = G_{\mathbb{Z}}(\Bbbk)$ and suppose that $e = e_{I,J} \otimes_{\mathbb{Z}} 1$ is rigid in $\mathfrak{g}$. Then so is $\tilde{e} = e_{I,J} \in \mathfrak{g}_{\mathbb{C}}$. By the theorem from the previous frame, $U(\mathfrak{g}, e)$ has an ideal of codimension 1. It follows that so does $U_\chi(\mathfrak{g}, e)$.

## Theorem (P-Topley)

*Suppose the derived subgroup of G is simply-connected, the Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$ admits a non-degenerate G-invariant symmetric bilinear form, and $p = \mathrm{char}(\Bbbk)$ is a good prime for the root system of G. Then for any $\chi \in \mathfrak{g}^*$ there exists an irreducible $U_\chi(\mathfrak{g})$-module of dimension $p^{d(\chi)}$ where $d(\chi)$ is half the dimension of the coadjoint orbit G-orbit of $\chi$.*

# An application

Suppose $\mathfrak{g} = \mathfrak{g}_{\Bbbk}$ and $\chi = \kappa(e, -)$, where $e = e_{I,J} \otimes_{\mathbb{Z}} 1$. Denote by $\mathfrak{z}$ the coadjoint stabiliser $\mathfrak{g}^{\chi}$ and write $\mathcal{F}(\mathfrak{g}, \mathfrak{z})$ for the (restricted) coadjoint $\mathfrak{g}$-module $\mathrm{Hom}_{U_0(\mathfrak{z})}(U_0(\mathfrak{g}), \Bbbk)$. The comultiplication of $U(\mathfrak{g})$ endows $\mathcal{F}(\mathfrak{g}, \mathfrak{z})$ with a commutative associative $\Bbbk$-algebra structure and the Lie algebra $\mathfrak{g}_{\Bbbk}$ acts on $\mathcal{F}(\mathfrak{g}, \mathfrak{z})$ as derivations. Let $S_{\chi}(\mathfrak{g})$ denote the quotient of the symmetric algebra $S(\mathfrak{g})$ by its ideal generated by all $(x - \chi(x))^p$ with $x \in \mathfrak{g}$. Clearly, $S_{\chi}(\mathfrak{g})$ is a local $\Bbbk$-algebra of dimension $p^{\dim \mathfrak{g}}$ and $\mathfrak{g}$ acts on $S_{\chi}(\mathfrak{g})$ as derivations. The algebra $S_{\chi}(\mathfrak{g})$ has a unique maximal $\mathfrak{g}$-invariant ideal $\mathcal{I}$ and $S_{\chi}(\mathfrak{g})/\mathcal{I} \cong \mathcal{F}(\mathfrak{g}, \mathfrak{z})$ as $\Bbbk$-algebras and $\mathfrak{g}$-modules. Furthermore, the Lie–Poisson structure of $S(\mathfrak{g})$ induces that on $S_{\chi}(\mathfrak{g})/\mathcal{I}$ and the Lie algebra $\mathfrak{g}$ acts on $S_{\chi}(\mathfrak{g})/\mathcal{I}$ by Poisson derivations.

# An application (continued)

Let $t$ be a variable. Following Premet–Skryabin, denote by $U_{\chi,t^2}(\mathfrak{g})$ the quotient of the $\Bbbk[t]$-tensor algebra of $\mathfrak{g}[t]$ by the 2-sided ideal generated by all $x \otimes y - y \otimes x - t^2[x, y]$ and $x^p - t^{2p-2}x^{[p]} = \chi(x)^p \cdot 1$ with $x, y \in \mathfrak{g} \subset \mathfrak{g}[t]$. This is a free $\Bbbk[t]$-module of rank $p^{\dim \mathfrak{g}}$ and the above-mentioned Poisson structure on $S_\chi(\mathfrak{g})$ can be obtained by taking commutators in $U_{\chi,\,t^2}(\mathfrak{g})$, dividing the result by $t^2$ and picking the remainder in $U_{\chi,\,t^2}(\mathfrak{g})/tU_{\chi,\,t^2}(\mathfrak{g}) \cong S_\chi(\mathfrak{g})$. Recall the $\mathbb{Z}$-grading $\mathfrak{g} = \oplus_{i\in\mathbb{Z}}\, \mathfrak{g}(i)$ given by the optimal cocharacter $\lambda_{I,J}$. The $\Bbbk[t, t^{-1}]$-algebra $U_{\chi,\,t^2}(\mathfrak{g})[t, t^{-1}]$ admits a diagonalisable automorphism $\tau$ such that $\tau(x) = t^{i+2}x$ for all $x \in \mathfrak{g}(i) \subset \mathfrak{g}[t] \subset U_{\chi,\,t^2}(\mathfrak{g})$ and all $i$. Let $\mathcal{J}_\chi$ be the annihilator of a small $U_\chi(\mathfrak{g})$-module. We may regard it as a $\Bbbk$-subspace of $U_{\chi,\,t^2}(\mathfrak{g})$.

# Quantising $\mathcal{F}(\mathfrak{g}, \mathfrak{z})$

Applying $\tau$ to the $\Bbbk[t, t^{-1}]$-module generated by $\mathcal{J}_\chi$ one obtains a 2-sided ideal of $U_{\chi, t^2}(\mathfrak{g})[t, t^{-1}]$ denoted $\mathcal{J}_{\chi, t}$. Using a $\Bbbk$-basis of $\mathcal{J}_\chi$ compatible with the (finite) Kazhdan $\mathbb{Z}$-filtration of $U_\chi(\mathfrak{g})$ it is straightforward to see that the $\Bbbk[t]$-module $\mathcal{J}_{\chi, t} \cap U_{\chi, t^2}(\mathfrak{g})$ is a direct summand of $U_{\chi, t^2}(\mathfrak{g})$. As a result, the $\Bbbk[t]$-algebra

$$\mathcal{A} := U_{\chi, t^2}(\mathfrak{g})/\mathcal{J}_{\chi, t} \cap U_{\chi, t^2}(\mathfrak{g})$$

arises as a quantisation of the Poisson algebra $\mathcal{F}(\mathfrak{g}, \mathfrak{z})$. By construction, $\mathcal{A}$ is a free $\Bbbk[t]$-module of rank $p^{\dim \mathfrak{g}}$. When $c \in \Bbbk^\times$ the quotient $\mathcal{A}/(t - c)\mathcal{A}$ is a full matrix algebra over $\Bbbk$ and $\mathfrak{g}$ acts on it as inner derivations. When $c = 0$ we have that $\mathcal{A}/t\mathcal{A} \cong \mathcal{F}(\mathfrak{g}, \mathfrak{z})$ as Poisson algebras. Note that the construction of $\mathcal{A}$ depends of the choice of a small $U_\chi(\mathfrak{g})$-module.

# Methods

Let $\ell$ be the smallest good rime of $\Phi$. Set $e := e_{I,J}$ and $\lambda := \lambda_{I,J}$. Suppose $e$ is rigid.

- We show that each graded component of $\mathfrak{g}_R^e = \bigoplus_{i \geqslant 0} \mathfrak{g}_R^e(i)$ is a direct summand of $\mathfrak{g}_R$.
- We show that the Lie ring $\bigoplus_{i>0} \mathfrak{g}_R^e(i)$ of $\mathfrak{g}_R^e$ is generated $\mathfrak{g}_R(i)$ with $i \leqslant \ell - 2$.
- We study arithmetic properties of the PBW generators $\Theta(x) \in U(\mathfrak{g}_{\mathbb{C}}, e)$ with $x \in \mathfrak{g}_R^e(i)$ and show that for $i \leqslant \ell - 2$ all of them preserve $Q_R$.
- When $\mathfrak{g}_{\mathbb{C}}^e \neq [\mathfrak{g}_{\mathbb{C}}^e, \mathfrak{g}_{\mathbb{C}}^e]$, use the fact that there is a multiplicity-free primitive ideal $I \subset U(\mathfrak{g}_{\mathbb{C}})$ with $\mathrm{VA}(I) = \overline{\mathcal{O}(e)}$ whose central character is defined over $R$.

THE END