GENERALIZED METRIC SPACES AND THE SPACE OF G-PERMUTATION DEGREE

R.M.Zhuraev

March 22, 2021

Introduction

In 1981 on the Prague topological symposium V.V.Fedorchuk put forward the following common problems in the theory of covariant functors, which determined a new direction of research in the field of topology: Let P be some geometrical property and F - some covariant functor. If topological space X has a property P, then $F(X)$ has the same property P? Or on the contrary, i.e. for what functors F, if $F(X)$ possesses a property P, it follows that X possesses the same property P? In our case $F = SP^n_G$ and $X \in T_1$ [1].

1.*V.V. Fedorchuk, Covariant functors in the category of compacts, absolute retracts and Q-manifolds // Uspekhi Matem. Nauk. 36:3 (1981). P.177-195.*

Let X^n be the n-th power of a compact X. The permutation group S_n of all permutations, acts on the n-th power X^n as permutation of coordinates. The set of all orbits of this action with quotient-topology we denote by SP^nX. Consider as a quotient mapping

$$\pi_n^s : X^n \to SP^nX$$

corresponding the point $x = (x_1, x_2, ..., x_n) \in X^n$ with the orbit of this point. Thus, points of the space SP^nX are finite subsets (equivalence classes) of the product X^n. Thus two points $(x_1, x_2, ..., x_n)$, $(y_1, y_2, ..., y_n)$ are considered to be equivalent if there is a permutation $\sigma \in S_n$ such that $y_i = x_{\sigma(i)}$ for all $i = 1, 2, ..., n$.

The space SP^nX is called the nth permutation degree of a space X. Equivalence relations by which we obtained spaces SP^nX and $\exp_n X$ is called the symmetric and hypersymmetric equivalence relations, respectively. Any symmetrically equivalent points X^n are hypersymmetrically equivalent. But inverse is not correct. So, for $x \neq y$ points (x, x, y), $(x, y, y) \in X^3$ are hypersymmetrically equivalent, but not symmetrically equivalent.

Let $f: X \to Y$ be a continuous mapping. For a class equivalence $[(x_1, x_2, ..., x_n)] \in SP^n X$ put

$$SP^n f[(x_1, x_2, ..., x_n)] = [(f(x_1), f(x_2), ..., f(x_n))].$$

Thereby, a mapping is defined

$$SP^n f : SP^n X \to SP^n Y.$$

It is easy to check that the operation SP^n so constructed is a covariant functor in the category of compacts. This functor is called the functor of n-th permutation degree [2].

2.*V.V. Fedorchuk, V.V. Filippov, Topology of hyperspaces and its applications// Moscow: Mathematica, cybernetica. 4 (1989) P.48.*

The concept of a permutation degree has generalizations. Let G be any subgroup of the group S_n. Then it also acts on X^n as permutation group of coordinates. Consequently, it generates a G-symmetric equivalence relation on X^n. The quotient space of the product X^n under the G-symmetric equivalence relation, is called G-permutation degree of the space X and is denoted by $SP^n_G X$. The operation SP^n_G is also the covariant functor in the category of compacts and is said to be a functor of G-permutation degree. If $G = S_n$ then $SP^n_G = SP^n$. If the group G consists only of unique element then $SP^n_G = \Pi^n$. Moreover, if $G_1 \subset G_2$ for subgroups G_1, G_2 of the permutation group S_n then we get a sequence of the factorization of functors [2]

$$\Pi^n \to SP^n_{G_1} \to SP^n_{G_2} \to SP^n.$$

2.*V.V. Fedorchuk, V.V. Filippov, Topology of hyperspaces and its applications// Moscow: Mathematica, cybernetica. 4 (1989) P.48.*

In this case quotient map $\pi^s_{n,G} : X^n \to SP^n_G X$ is determined in the following way:

$$\pi^s_{n,G}(x_1, x_2, ..., x_n) = [x_1, x_2, ..., x_n]_G.$$

Preliminaries

A continuous mapping $f : X \to Y$ is called *closed (open)* if for every closed (open) set $A \subset X$ the image $f(A)$ is closed (open) in Y. Mappings which are simultaneously closed and open are called *closed-and-open* mappings [3].

The following theorem is known.

Theorem 1[4]

Let X be any topological space. Then the map $\pi^s_{n,G} : X^n \to SP^n_G X$ is closed-and-open

3. *R.Engelking, General topology. Revised and completed edition. Berlin: Helderman, 1986, 752 p.*

4. *Wagner C.H., Symmetric, cyclic and permutation products of manifolds. Warszava : PWN, 1980, 48 p.*

Theorem 2

Let X be a Hausdorff space. Then the space X is homeomorphic to a closed subspace of $SP^n_G X$.

Definition 1[5]

Recall that a space X is said to be Lašnev if it is the closed image of a metric space M.

5.*Fucai Lin, Chuan Liu, The k-spaces property of the free Abelian topological groups over non-metrizable Lasnev spaces // Topology and its Applications 220 (2017), 31-42.*

Main result

Proposition

An arbitrary subspace of a Lašnev space is also Lašnev space.

Corollary 1

Let X be a T_1-space, n a positive integer and let G be an arbitrary subgroup of the permutation group S_n. If $SP^n_G X$ is a Lašnev space, then so is X.

A family $\mathcal{N} = \{M_s\}_{s\in S}$ of subsets of a topological space X is a *network* for X if for every point $x \in X$ and any neighbourhood U of x there exists an element $s \in S$ such that $x \in M_s \subset U$. A family $\{A_s\}_{s\in S}$ of subsets of a topological space X is called *locally finite* if for every point $x \in X$ there exists a neighbourhood U such that the set $\{s \in S : U \bigcap A_s \neq \emptyset\}$ is finite. If every point $x \in X$ has a neighbourhood that intersects at most one set of a given family, than we say that the family is *discrete*. A family of subsets is called *σ-locally finite* (*σ-discrete*) if it can be represented as a countable union of a locally finite (discrete) families [3].

3.*R.Engelking, General topology. Revised and completed edition. Berlin: Helderman, 1986, 752 p.*

Definition 2[6]

A topological space X is called σ-space if it has a σ-locally finite network.

6.*J.Nagata, Modern general topology, second revised edition: North-Holland mathematical library, 1985, 522 p.*

We obtained the following result.

Theorem 3

Let X be a T_1-space, n is a natural number, let and G is an arbitrary subgroup of permutation group S_n. A space X is σ-space if and only if $SP^n_G X$ is σ-space.

Definition 3[7]

A space X is a (strong) Σ-space if there exists a pair $\{\mathcal{F}, \mathcal{C}\}$ of families satisfying the following:
(a) $\mathcal{F}$ is a σ-discrete family of subsets of X;
(b) $\mathcal{C}$ is a cover of X by (respectively compact) countably compact subsets of X;
(c) If $C \in \mathcal{C}$ and U is an open subset of X such that $C \subset U$, then $C \subset F \subset U$ for some $F \in \mathcal{F}$.

7. *T.Mizokami, On hyperspaces of generalized metric spaces // Topology and its Applications 76 (1997), 169-173.*

A topological space X is called a *paracompact space* if X is a Hausdorff space and every open cover of X has a locally finite open refinement [3].

Remark

There exists a paracompact space X^* such that SP^2X^* is not a paracompact space.

Construction. Consider the space "One arrow" P.S.Alexandroffs $X^* = [0,1)$ the base of which is formed by subsets of the form $[\alpha,\beta)$, where $0 \leq \alpha < \beta \leq 1$. It is clear that the space X^* is a paracompact. In this case a space SP^2X^* is not a paracompact space.

3.*R.Engelking, General topology. Revised and completed edition. Berlin: Helderman, 1986, 752 p.*

Theorem 4

Let X be a Hausdorff space, n be a positive integer and G be an arbitrary subgroup of the permutation group S_n. If X is a paracompact Σ-space, then $SP^n_G X$ is also a paracompact Σ-space.

Corollary 2

Hausdorff space X is a paracompact σ-space iff $SP^n_G X$ is paracompact σ-space.

Definition 4[8]

A topological space X is called a stratifiable space if X is T_1-space and to each open $U \subset X$, one can assign a sequence $\{U_n\}_{n=1}^{\infty}$ of open subsets of X such that

1) $[U_n] \subset U$

2) $\bigcup\limits_{n=1}^{\infty} U_n = U$

3) $U_n \subset V_n$ *whenever* $U \subset V$ *for each* $n \in N$.

8.*C.R.Borges, On stratifiable spaces // Pacific Journal of Mathematics 17(1966), 1-16.*

Theorem 5

Let X be a T_1-space, n be a positive integer and G be an arbitrary subgroup of the permutation group S_n. A space X is stratifiable if and only if $SP^n_G X$ is a stratifiable space.

Definition 5[5]

A family $\mathcal{P}$ of subsets of a space X is called k-network if for every compact subset K of X and an arbitrary open set U containing K in X there is a finite subfamily $\mathcal{P}' \subset \mathcal{P}$ such that $K \subset \cup\mathcal{P}' \subset U$.

A regular space has a σ-locally finite (resp. countable) k-network is called $\aleph$-space (resp.$\aleph_0$-space).

5.*Fucai Lin, Chuan Liu, The k-spaces property of the free Abelian topological groups over non-metrizable Lasnev spaces // Topology and its Applications 220 (2017), 31–42.*

Theorem 6

Let X be a T_1-space, $n \in N$ and G be an arbitrary subgroup of the permutation group S_n. A space X is an $\aleph$-space ($\aleph_0$-space) if and only if $SP^n_G X$ is an $\aleph$-space ($\aleph_0$-space).

Corollary 3

Hausdorff space X is a paracompact $\aleph$-space iff $SP^n_G X$ is paracompact $\aleph$-space.

Definition 6[9]

A topological space X is semi-stratifiable if, to each open set $U \subset X$, one can assign a sequence $\{F_n\}_{n=1}^{\infty}$ of closed subsets of X such that

1. $\bigcup\limits_{i=1}^{\infty} F_n = U$;

2. $F_n \subset G_n$ *whenever* $U \subset V$ *for each* $n \in N$, *where* $\{G_n\}_{n=1}^{\infty}$ *is the sequence assigned to* V.

9.*G.D.Creede. Concerning semi-stratifiable spaces // Pacific Journal of Mathematics. Vol. 32, No. 1, 1970, p.47-54.*

It is obviously the following theorem.

Theorem 7[9]

The closed image of a semi-stratifiable space is semi-stratifiable. The class of semi-stratifiable spaces is hereditary and countably productive.

From theorem 7 [9] we obtain the following result:

Theorem 8

Let X be a T_1-space, n be a positive integer and G be an arbitrary subgroup of the permutation group S_n. A topological space X is semi-stratifiable if and only if the space $SP^n_G X$ is semi-stratifiable.

Definition 7[10]

A topological space X is called semi-metrizable if there exists a mapping $d : X \times X \to [0, \infty)$ such that

1. for all points $x, y \in X$ implies $d(x, y) = d(y, x)$;

2. $d(x, y) = 0$ if and only if $x = y$;

3. a subset $V \subset X$ is open if and only if for each point $x \in V$ there exists $n \in N$ such that $B(n, x) = \{y \in X : d(x, y) < \frac{1}{n}\} \subset V$;

4. each $B(n, x)$ is a neighborhood of x.

10. *T.G.Rghavan and I.L.Reilly, On semi-metrizable-closed spaces // Indian Journal of Pure and Applied Mathematics, 18(3); 219–225, March 1987.*

Give an example of a semi-metrizable but not metrizable space.
One arrow P.S.Alexandroffs is not metrizable space, but it is semi-metrizable space.

Theorem 9[9]

A topological space $X \in T_1$ is semi-metrizable if and only if it is a first countable semi-stratifiable space.

9. *G.D.Creede. Concerning semi-stratifiable spaces // Pacific Journal of Mathematics. Vol. 32, No. 1, 1970, p.47-54.*

Theorem 10

A T_1-space X is semi-metrizable if and only if $SP_G^n X$ is semi-metrizable.