Holomorphic Rigidity of Totally Nondegenerate CR Manifolds

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Outline of the talk

- Let M be a CR manifold with the CR structure T^cM. Set:

$$\mathfrak{m}^{-1} := T^cM, \qquad \mathfrak{m}^{-k} := [\mathfrak{m}^{-1}, \mathfrak{m}^{-k+1}] \text{ for } k > 1.$$

a) each distribution $\mathfrak{m}^{-k}$ is regular i.e. $\text{rank}(\mathfrak{m}_x^{-k}) = const.$, locally.

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- The vector space $\mathfrak{m}$ equipped with the standard bracket of vector fields constitutes a *depth* μ fundamental graded algebra, called Tanaka (or Levi-Tanaka in the sense of Beloshapka) algebra of M.

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A CR manifold, as above, with the associated Tanaka algebra $\mathfrak{m}$ is totally nondegenerate of depth μ whenever for each $2 \leq k \leq \mu - 1$, the dimension of the component $\mathfrak{m}^{-k}$ is maximum among all CR manifolds which admit a depth μ Tanaka algebra.

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- Equivalently, let M be a CR manifold of CR dimension n with a Tanaka algebra:

$$\mathfrak{m} := \mathfrak{m}^{-\mu} + \mathfrak{m}^{-\mu+1} + \cdots + \mathfrak{m}^{-1}$$

Consider the complexification $\mathbb{C} \otimes \mathfrak{m} \cong \mathbb{C} \otimes TM$ with:

$$\mathbb{C} \otimes \mathfrak{m}^{-1} = \mathfrak{m}^{1,0} + \mathfrak{m}^{0,1}.$$

Then, M is totally nondegenerate of depth μ whenever:

$$\frac{\mathbb{C} \otimes \mathfrak{m}}{\mathbb{C} \otimes \mathfrak{m}^{-\mu}} \cong \frac{\mathfrak{f}_{2n,\mu-1}}{\mathcal{I}}$$

✓ $\mathfrak{f}_{2n,\mu-1}$ is a depth $\mu - 1$ (complex) free Lie algebra of rank $2n$ generated by some $v_1, \cdots, v_n \in \mathcal{V}^{1,0}$ and $\overline{v}_1, \cdots, \overline{v}_n \in \mathcal{V}^{0,1}$,
✓ $\mathcal{I}$ is the ideal generated by the (abstract) elements of:

$$[\mathcal{V}^{1,0}, \mathcal{V}^{1,0}] \qquad \textit{and} \qquad [\mathcal{V}^{0,1}, \mathcal{V}^{0,1}].$$

- Proposition (Beloshapka, Math. Notes (2004))

✓ M: a depth μ totally nondegenerate of CR dimension n with the associated Tanaka algebra $\mathfrak{m} = \mathfrak{m}^{-\mu} + \cdots + \mathfrak{m}^{-2} + \mathfrak{m}^{-1}$.

✓ $r := \dim_{\mathbb{C}} \mathfrak{m}$ and $k_j := \dim_{\mathbb{C}} \mathfrak{m}^{-j}$ for $j = 2, \cdots, \mu$.

✓ let $\mathbf{w}_j$ be a vector coordinate of k_j complex variables.

✓ weights: in coordinates $z_1, \cdots, z_n, \mathbf{w}_2, \cdots, \mathbf{w}_\mu$ of $\mathbb{C}^r$, assign $[z] = [\overline{z}] = 1$ and $[\mathbf{w}_j] = [\mathbf{u}_j] = [\mathbf{v}_j] = j$.

Then:

$$M : \begin{cases} \mathbf{v}_2 = \mathbf{\Phi}_2(z, \overline{z}, \mathbf{u}) + \mathrm{O}(2) \\ \qquad \vdots \\ \mathbf{v}_\mu = \mathbf{\Phi}_\mu(z, \overline{z}, \mathbf{u}) + \mathrm{O}(\mu) \end{cases}$$

where for $j = 1, \cdots, \mu$, each Φ_j is a weighted homogeneous vector polynomial function of weight j and $\mathrm{O}(j)$ means a (possibly infinite) sum over monomials of weights $> j$.

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- **Model.** Associated with the above totally nondegenerate CR manifold of CR dimension n and codimension k:

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✓ The Lie algebra $\mathfrak{aut}_{CR}(M) := \mathsf{Lie}(\mathsf{Aut}_{CR}(M))$ of infinitesimal CR automorphisms of M is generated by the real parts of all holomorphic vector fields:

$$X = \sum_{j=1}^{n} Z^j(\mathbf{z}, \mathbf{w}) \frac{\partial}{\partial z_j} + \sum_{l=1}^{k} W^l(\mathbf{z}, \mathbf{w}) \frac{\partial}{\partial w_l}$$

which are tangent to M, i.e.:

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1) For each integer t, let $\mathfrak{g}^t$ be the vector space of all infinitesimal CR automorphisms of $\mathbb{M}$ of the precise weight t. Then:

$$\boxed{\mathfrak{aut}_{CR}(\mathbb{M}) = \underbrace{\mathfrak{g}_{-\mu} + \ldots + \mathfrak{g}_{-1}}_{\mathfrak{g}_-} + \mathfrak{g}_0 + \underbrace{\mathfrak{g}_1 + \ldots + \mathfrak{g}_\nu}_{\mathfrak{g}_+}} \qquad (1)$$

is a graded Lie algebra of polynomila type.

2) The subalgebra $\mathfrak{g}_-$ is fundamental. Moreover, the graded algebra $\mathfrak{aut}_{CR}(\mathbb{M})$ is transitive that is: for each $X \in \mathfrak{g}_t$ with $t \geq 0$, the equality $[X, \mathfrak{g}_{-1}] = 0$ implies that $X \equiv 0$.

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3) Let $\mathfrak{g}_- := \mathsf{Lie}(G_-)$. The Lie group G_- acts transitively on $\mathbb{M}$ without fixed point, thus:

$$\mathbb{M} \cong G_-$$

Moreover, $\mathfrak{g}_-$ is isomorphic to the Tanaka algebra $\mathfrak{m}_-$ of $\mathbb{M}$.

4) Let $\mathfrak{g}_0 := \mathsf{Lie}(G_0)$ and $\mathfrak{g}_+ := \mathsf{Lie}(G_+)$. Then $G_0 \cdot G_+$ is the isotropy subgroup of $Aut_{CR}(\mathbb{M})$ at the origion. G_0 consists of linear CR automorphisms $(\mathbb{M}, 0) \to (\mathbb{M}, 0)$ while G_+ consists of nonlinear ones.

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- **Question.** What is the precise dimension of $\mathfrak{g}_+$ in the gradation (1) of $\mathfrak{aut}_{CR}(\mathbb{M})$.

Maximum Conjecture (Beloshapka, Proc. Steklov Inst. Math. (2012))

Totally nondegenerate CR models of depth ≥ 3 never admit origion preserving nonlinear CR automorphism. In other words, if $\mathbb{M}$ is a totally nondegenerate CR model of depth ≥ 3, then the subalgbera $\mathfrak{g}_+$ in the gradation (1) of $\mathfrak{aut}_{CR}(\mathbb{M})$ is trivial.

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Theorem. (Tanaka, J. Math. Kyoto Univ., (1970))

For each graded CR algebra $\mathfrak{m} := \sum_{j=-\mu}^{-1} \mathfrak{m}_j$, let $\mathfrak{g}_0$ to be the space of all degree zero derivations of $\mathfrak{m}$ which are $\mathbb{C}$-linear on $\mathfrak{m}_{-1}$ and respect the complex structure map J. Then, there exists a transitive extension $(\mathfrak{m} + \mathfrak{g}_0)^\infty = \mathfrak{m} + \mathfrak{g}_0 + \sum_{p \geq 1} \mathfrak{g}_p$, called Levi-Tanaka prolongation of the CR Tanaka algebra $\mathfrak{m} + \mathfrak{g}_0$, which is unique up to isomorphism and maximal among all transitive extesions of $\mathfrak{m} + \mathfrak{g}_0$.

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★ $(\mathfrak{m} + \mathfrak{g}_0)^\infty := \mathfrak{m} + \mathfrak{g}_0 + \sum_{j>0} \mathfrak{g}_j$ is the Levi-Tanaka prolongation of $\mathfrak{m} + \mathfrak{g}_0$.

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$$[X, \mathfrak{m}_{-1}] \neq 0.$$

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• Recall that: If M is a depth μ totally nondegenerate model with the associated Tanaka algebra $\mathfrak{m} := \mathfrak{m}_{-\mu} + \cdots + \mathfrak{m}_{-1}$ then:

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✓ The Lie algebra $\mathfrak{aut}_{CR}(M)$ is a transitive prolongation (cf. (1)):

$$\mathfrak{aut}_{CR}(M) := \mathfrak{m} + \mathfrak{g}_0 + \sum_{j=1}^{\nu} \mathfrak{g}_j$$

of the Tanaka algebra $\mathfrak{m}$

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$$\mathfrak{aut}_{CR}(M) := \mathfrak{m} + \mathfrak{g}_0 + \sum_{j=1}^{\nu} \mathfrak{g}_j$$

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Algebraic version of the maximum conjecture

Let $\mathfrak{m}$ be a totally nondegenerate algebra of depth $\mu \geq 3$, that is the Tanaka algebra associated with a totally nondegenerate CR manifold of this depth. Then

$$(\mathfrak{m} + \mathfrak{g}_0)^\infty = \mathfrak{m} + \mathfrak{g}_0.$$

- In order to prove the above algebraic conjecture, one may show equivalently that the first component $\mathfrak{g}_1$ of $(\mathfrak{m} + \mathfrak{g}_0)^\infty$ is trivial.

$$\mathfrak{g}_1 = \Big\{ L \in \mathrm{Lin}\Big(\mathfrak{m} = \sum_{j=-\mu}^{-1} \mathfrak{m}_j, \sum_{j=-\mu+1}^{-1} \mathfrak{m}_j + \mathfrak{g}_0\Big) : [L(Y), Z] + [Y, L(Z)] = L([Y, Z]), \ \forall Y, Z \in \mathfrak{m} \Big\}.$$

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Lemma.

Denote by $\mathfrak{m}^{1,0}$ and $\mathfrak{m}^{0,1}$ the $\pm i$-eigenspaces of $\widehat{J}$ i.e. $\mathfrak{m}^{\mathbb{C}}_{-1} = \mathfrak{m}^{1,0} + \mathfrak{m}^{0,1}$. Let $\mathbb{D}$ and $\mathbb{L}$ be $\mathbb{C}$-linear extensions of some homomorphisms $D \in \mathfrak{g}_0$ and $L \in \mathfrak{g}_1$. Then:

$$\overline{(\mathbb{L}(X))(Y)} = (\mathbb{L}(\overline{X}))(\overline{Y}) \qquad \textit{and} \qquad \overline{\mathbb{D}(X)} = \mathbb{D}(\overline{X})$$

for each $X, Y \in \mathfrak{m}^{\mathbb{C}}$. In particular $\mathbb{D}(\mathfrak{m}^{1,0}) \subset \mathfrak{m}^{1,0}$.

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• **The goal** is to show that if $\mathfrak{m}$ is the Tanaka algebra associated with a totally nondegenerate model of depth $\mu \geq 3$, if $(\mathfrak{m}+\mathfrak{g}_0)^\infty = \mathfrak{m}+\mathfrak{g}_0+\sum_{j=1}^{\nu}\mathfrak{g}_j$ is its Levi-Tanaka prolongation and $L \in \mathfrak{g}_1$, then its $\mathbb{C}$-linear extension:

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Then, the basis elements of $\mathfrak{m}^{\mathbb{C}}$ are the basis elements of $\frac{\mathfrak{f}_{2n,\mu-1}}{\mathfrak{i}^{1,0}+\mathfrak{i}^{0,1}}$ together with the basis elmements of the Abelian subalgebra $\mathfrak{m}_{-\mu}$.

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- Hall bases (M. Hall, Proc. AMS, (1950)) Let $\mathfrak{f} := \mathfrak{f}_{-1} + \mathfrak{f}_{-2} + \cdots$ be a free Lie algebra of rank n generated by the abstract elements $V := \{v_1, \ldots, v_n\}$. The Hall basis $\mathcal{H} := \bigsqcup_{i \geq 1} \mathcal{H}^i$ of $\mathfrak{f}$ consists of the basis elements defined inductively as:

✓ $\mathcal{H}^{-1} := V$.

✓ Suppose that all sets $\mathcal{H}^\ell$, $\ell < k_o$, are defined. Set the order $\prec$ on $\bigsqcup_{1 \leq \ell \leq k_o - 1} \mathcal{H}^\ell$ such that:

$$\text{degree}(X) < \text{degree}(Y) \Rightarrow X \prec Y$$

✓ $\mathcal{H}^{k_o}$:= the collection of all monomials of degree k_o of the form $[U, V]$ with $U, V \in \bigsqcup_{1 \leq \ell \leq k_o - 1} \mathcal{H}^\ell$, with:
(i) $V \prec U$;
(ii) if $U = [U_1, U_2]$ for some $U_i \in \mathcal{H}$, then $U_2 \preceq V$.

• Witt formula. Let $\mathfrak{f} := \mathfrak{f}_{-1} + \mathfrak{f}_{-2} + \cdots$ be a free algebra of rank n. Assume that $\dim \mathfrak{f}_{-\ell} := n_\ell$. Then, the following induction relation holds for each $\ell > 1$:

$$n_\ell - n_{\ell-1} = \frac{1}{\ell} \sum_{d|\ell} \mu(d) n^{\frac{\ell}{d}}$$

where μ is the Möbius function

$$\mu(d) := \begin{cases} 1, & \text{if} d = 1 \\ 0, & \text{if } d \text{ contains square integer factors} \\ (-1)^\nu, & \text{if } d = p_1 \cdots p_\nu \end{cases}$$

• Example. Let $V := \{v_1, v_2, \overline{v}_1, \overline{v}_2\}$ with the order $v_1 \prec v_2$, $\overline{v}_1 \prec \overline{v}_2$ and $\overline{v}_i \prec v_j$ for $i, j = 1, 2$. Then,

✓ $[[v_1, \overline{v}_1], v_2]$ and $[[v_2, \overline{v}_1], v_1]$ are two distinct elements of $\mathcal{H}^3 \subset \mathcal{H}$.

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CR Hall Bases

Definition

Let $\mathfrak{f}$ be a free algebra generated by the n abstract elements $v_1, \ldots, v_n$. For each monomial $X \in \mathfrak{f}$, define the type of X as the n-tuple:

$$\mathrm{type}(X) := (t_1, \ldots, t_n)$$

where t_j is the number of the times the corresponding basis element v_j is used in construction of the monomial X.

- Recall that if:
  - ✓ $\mathfrak{m} := \mathfrak{m}_{-\mu} + \cdots + \mathfrak{m}_{-1}$: a totally nondegenerate algebra of depth μ with the generators $v_1, \cdots, v_n, \overline{v}_1, \cdots, \overline{v}_n$ of $\mathfrak{m}^{\mathbb{C}}_{-1}$
  - ✓ $\mathfrak{f}$: a depth $\mu - 1$ complex free algebra generated by $v_1, \cdots, v_n, \overline{v}_1, \cdots, \overline{v}_n$
  - ✓ $\mathfrak{i}^{1,0} := \langle [v_i, v_j] : i, j = 1, \ldots, n \rangle$ and $\mathfrak{i}^{0,1} := \langle [\overline{v}_i, \overline{v}_j] : i, j = 1, \ldots, n \rangle$

Then:

$$\mathfrak{m}^{\mathbb{C}}_{-\mu+1} + \ldots + \mathfrak{m}^{\mathbb{C}}_{-1} \cong \frac{\mathfrak{m}^{\mathbb{C}}}{\mathfrak{m}^{\mathbb{C}}_{-\mu}} \cong \mathfrak{f}_J = \frac{\mathfrak{f}}{\mathfrak{i}_{10} + \mathfrak{i}_{01}}.$$

Proposition.

In agreement with the above notations, if $X_1, \ldots, X_\ell$ are monomials in $\mathfrak{f}$ (of degree $< \mu$), with distinct type and none of them in $\mathfrak{i} = \mathfrak{i}_{10} + \mathfrak{i}_{01}$, then their classes $X_1 + \mathfrak{i}, \ldots, X_\ell + \mathfrak{i}$ constitute a linearly independent set in $\mathfrak{m}^{\mathbb{C}}$.

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Two Algebraic Questions

- Question 1. Is there a certain basis for $\mathfrak{f}_J$ like that of Hall for the free algebra $\mathfrak{f}$?
- Question 2. Is there any certain formula for computing the dimension of the components of $\mathfrak{f}_J$ like that of Witt for the free algebra $\mathfrak{f}$?

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CR version of Warhurst' lemma

- Notations. Let $\mathfrak{m} + \mathfrak{g}_0 + \sum_{p \geq 1} \mathfrak{g}_p$ be the Levi-Tanaka prolongation of a CR algebra $\mathfrak{m}$. Let $X, Y \in \mathfrak{m}$ and $L \in \mathfrak{g}_t$ for $t \geq 0$. Denote
  - ✓ $X^r \cdot Y := [\underbrace{X, [X, [\cdots, [X}_{r-times}, Y] \cdots]]]$,
  - ✓ $L_X := [L, X]$ and $L_{X|Y} := [[L, X], Y]$.

CRW Lemma. (cf. Warhurst, Geom. Ded. (2007))

Let $\mathfrak{m} = \sum_{t=-\mu}^{-1} \mathfrak{m}_t$ be a CR algebra with $\mathfrak{m}_{-1}^{\mathbb{C}} := \mathfrak{m}^{1,0} + \mathfrak{m}^{0,1}$. Given $E \in \mathfrak{m}^{1,0}$ and $L \in \mathfrak{g}^1$ (indeed a $\mathbb{C}$-linear extension), then for any other element $W \in \mathfrak{m}^{\mathbb{C}}$ and $r \geq 1$,

$$L_{E^r \cdot W} = E^{r-1} \cdot (rL_{E|W} - L_{W|E}) + \frac{r(r-1)}{2} E^{r-2} \cdot (L_{E|E} \cdot W).$$

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- Somple simple relations.

Let $\mathfrak{m} := \mathfrak{m}_{-\mu} + \cdots + \mathfrak{m}_{-1}$ be a totally nondegenerate algebra with the Levi-Tanaka prolongation

$$(\mathfrak{m} + \mathfrak{g}_0)^\infty = \mathfrak{m} + \mathfrak{g}_0 + \sum_{j \geq 1} \mathfrak{g}_j.$$

Let $L \in \mathfrak{g}_1$ and $E, F \in \mathfrak{m}^{1,0}$ (and hence $\overline{E}, \overline{F} \in \mathfrak{m}^{0,1}$). Then:

$$\begin{array}{rr} [E, F] = 0, & [\overline{E}, \overline{F}] = 0, \\ L_{E|F} \in \mathfrak{m}^{1,0}, & L_{\overline{E}|F} \in \mathfrak{m}^{1,0}, \\ L_{E|\overline{F}} = \overline{L_{\overline{E}|F}} \in \mathfrak{m}^{0,1} & L_{\overline{E}|\overline{F}} = \overline{L_{E|F}} \in \mathfrak{m}^{0,1} \end{array}$$

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♡ To prove: if $L \in \mathfrak{g}_1$ then $L \equiv 0$.

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♠ Sketch of the proof: Let $\mathfrak{m} := \mathfrak{m}_{-\mu} + \cdots + \mathfrak{m}_{-1}$ be a totally nondegenerate algebra with the basis elements $X_1, \cdots, X_n$ of $\mathfrak{m}^{1,0}$. Let $L \in \mathfrak{g}_1$.

1) Set:

$$L_{X_i|X_j} := \sum_{k=1}^{n} a_{ij}^k X_k \quad \text{and} \quad L_{X_i|\overline{X}_j} := \sum_{k=1}^{n} b_{ij}^k \overline{X}_k$$

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2) Computing $L_{L_X} \in \mathfrak{m}^{\mathbb{C}}_{-\mu+1}$ by means of the CRW lemma where $X \equiv 0$ is some monomial of degree $\mu + 1$.

3) Equating to zero the coefficients of the CR Hall basis elements in $\mathfrak{m}^{\mathbb{C}}_{-\mu+1}$ in the expression $L_{L_X} = 0$ and constructing a certain system of equations with the unknowns a_{ij}^k, b_{ij}^k.

♠ Sketch of the proof: Let $\mathfrak{m} := \mathfrak{m}_{-\mu} + \cdots + \mathfrak{m}_{-1}$ be a totally nondegenerate algebra with the basis elements $X_1, \cdots, X_n$ of $\mathfrak{m}^{1,0}$. Let $L \in \mathfrak{g}_1$.

1) Set:

$$\boxed{L_{X_i|X_j} := \sum_{k=1}^{n} a_{ij}^k X_k \quad \text{and} \quad L_{X_i|\overline{X}_j} := \sum_{k=1}^{n} b_{ij}^k \overline{X}_k}$$

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- **Proof in CR dimension 2.**

Let $\mathfrak{m}$ be a totally nondegenerate algebra of depth $\mu > 3$ and CR dimension 2, with $\mathfrak{m}_{-1}^{\mathbb{C}} = \langle X_1, X_2, \overline{X}_1, \overline{X}_2 \rangle$. Let $L \in \mathfrak{g}^1$. Set:

$$\begin{aligned} L_{X_1|X_1} &:= a_1\, X_1 + b_1\, X_2, & L_{X_1|\overline{X}_1} &:= a_5\, \overline{X}_1 + b_5\, \overline{X}_2, \\ L_{X_1|X_2} &:= a_2\, X_1 + b_2\, X_2, & L_{X_1|\overline{X}_2} &:= a_6\, \overline{X}_1 + b_6\, \overline{X}_2, \\ L_{X_2|X_1} &:= a_3\, X_1 + b_3\, X_2, & L_{X_2|\overline{X}_1} &:= a_7\, \overline{X}_1 + b_7\, \overline{X}_2, \\ L_{X_2|X_2} &:= a_4\, X_1 + b_4\, X_2, & L_{X_2|\overline{X}_2} &:= a_8\, \overline{X}_1 + b_8\, \overline{X}_2. \end{aligned} \tag{2}$$

✓ Observation:

$$0 = L_{[X_1, X_2]} = L_{X_1|X_2} - L_{X_2|X_1} = (a_2 - a_3)X_1 + (b_2 - b_3)X_2$$

$$\Downarrow$$

$$\boxed{a_3 = a_2, \qquad b_3 = b_2}$$

✓ first application of CRW.

$$0 = L_{X_1^{\mu} \cdot \overline{X}_1} = \mu\Big(\big(a_5 + \frac{\mu - 1}{2}\, a_1\big)\, X_1^{\mu-1} \cdot \overline{X}_1 + b_5\, X_1^{\mu-1} \cdot \overline{X}_2 + \\ + \frac{\mu - 1}{2}\, b_1\, X_2 \cdot X_1^{\mu-2} \cdot \overline{X}_1\Big).$$

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✓ once again CRW.

$$
\begin{aligned}
0 &= L_{L_{X_1^\mu\cdot\overline{X}_1}} \\
&= \Big(-a_5^2-\frac{1}{2}\,a_1^2+a_5\,a_1\,\mu^2-\frac{5}{2}\,a_5\,a_1\,\mu+b_5\,a_6\,\mu-\frac{3}{2}\,b_1\,a_2\,\mu+\frac{1}{2}b_1\,a_7\,\mu+\frac{1}{2}\,b_1\,a_2\,\mu^2 \\
&+a_5^2\mu+\frac{3}{2}\,a_5\,a_1+\frac{1}{4}\,a_1^2\mu^3-a_1^2\mu^2+\frac{5}{4}\,a_1^2\mu-b_5\,a_6+b_1\,a_2-\frac{1}{2}b_1\,a_7\Big)\,X_1^{\mu-2}\cdot\overline{X}_1+ \\
&+\Big(a_5\,b_5\,\mu+a_1\,b_5\,\mu^2-\frac{5}{2}\,a_1\,b_5\,\mu+b_5\,b_6\,\mu+\frac{1}{2}\,b_1\,b_7\,\mu-a_5\,b_5+\frac{3}{2}\,a_1\,b_5-b_5\,b_6- \\
&-\frac{1}{2}\,b_1\,b_7\Big)X_1^{\mu-2}\cdot\overline{X}_2+\Big(a_5\,b_1\,\mu^2-3\,a_5\,b_1\,\mu+\frac{1}{2}\,b_1\,a_1\,\mu^3-\frac{5}{2}\,a_1\,b_1\,\mu^2+ \\
&+4\,a_1\,b_1\,\mu-\frac{3}{2}\,b_1\,b_2\,\mu+\frac{1}{2}\,b_1\,b_2\,\mu^2+2\,a_5\,b_1-2\,a_1\,b_1+b_1\,b_2\Big)\,X_1^{\mu-3}\cdot X_2\cdot\overline{X}_1+ \\
&\qquad+\Big(b_1\,b_5\,\mu^2-3\,b_1\,b_5\,\mu+2\,b_1\,b_5\Big)\,X_1^{\mu-3}\cdot X_2\cdot\overline{X}_2 \\
&\qquad+\Big(\frac{3}{2}\,b_1^2+\frac{3}{2}\,b_1^2\mu^2-\frac{11}{4}\,b_1^2\mu-\frac{1}{4}\,b_1^2\mu^3\Big)\,X_2\cdot X_1^{\mu-4}\cdot\overline{X}_1\cdot X_2.
\end{aligned}
$$

Corollary

In the pre-assumptions (2) we have:

$$\boxed{a_4 = b_1 = 0}$$

✓ second application of CRW.

$$0 = L_{X_1^{\mu} \cdot \overline{X}_2} = \mu \big(a_6 \, X_1^{\mu-1} \cdot \overline{X}_1 +$$

$$+ \left(b_6 + \frac{\mu-1}{2} \, a_1 \right) X_1^{\mu-1} \cdot \overline{X}_2 + \frac{\mu-1}{2} \, b_1 \, X_2 \cdot X_1^{\mu-2} \cdot$$

$$0 = L_{X_1^{\mu-2} \cdot \overline{X}_2 \cdot X_1 \cdot \overline{X}_2} = \left(2\overline{a}_7 + \overline{b}_4 \right) X_1^{\mu-1} \cdot \overline{X}_2 + 2\overline{b}_7 \, X_1^{\mu-2} \cdot X_2 \cdot \overline{X}_2 +$$

$$+ 2\,(\mu-2)\, a_6 \, X_1^{\mu-3} \cdot \overline{X}_2 \cdot X_1 \cdot \overline{X}_1 +$$

$$+ \left((\mu-2)(a_1 + 2\, b_6) + \frac{(\mu-2)(\mu-3)}{2} \, a_1 \right) X_1^{\mu-3} \cdot \overline{X}_2 \cdot X_1$$

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$$0 = L_{X_1^{\mu-2} \cdot \overline{X}_2 \cdot X_1 \cdot \overline{X}_2} = \big(2\overline{a}_7 + \overline{b}_4\big)\, X_1^{\mu-1} \cdot \overline{X}_2 + 2\overline{b}_7\, X_1^{\mu-2} \cdot X_2 \cdot \overline{X}_2 + \\ + 2\,(\mu-2)\, a_6\, X_1^{\mu-3} \cdot \overline{X}_2 \cdot X_1 \cdot \overline{X}_1 + \\ + \big((\mu-2)(a_1 + 2\, b_6) + \frac{(\mu-2)(\mu-3)}{2}\, a_1\big)\, X_1^{\mu-3} \cdot \overline{X}_2 \cdot X_1$$

✓ once again CRW.

$$0 = L_{L_{X_1^\mu \cdot \overline{X}_2}} =$$

$$\Big(a_6 a_5 \mu - a_6 a_5 + a_6 a_1 \mu^2 - \frac{5}{2}\, a_6 a_1 \mu + \frac{3}{2}\, a_6 a_1$$
$$+ b_6 a_6 \mu - b_6 a_6\Big)\, X_1^{\mu-2} \cdot \overline{X}_1$$
$$+ \Big(b_5 a_6 \mu - b_5 a_6 + b_6^2 \mu - b_6^2 + b_6 a_1 \mu^2 - \frac{5}{2}\, b_6 a_1 \mu + \frac{3}{2}\, b_6 a_1 + \frac{1}{4}\, a_1^2 \mu^3$$
$$- a_1^2 \mu^2 + \frac{5}{4}\, a_1^2 \mu - \frac{1}{2}\, a_1^2\Big)\, X_1^{\mu-2} \cdot \overline{X}_2,$$

$$
\begin{aligned}
0 = &\left((2\overline{a}_7 + \overline{b}_4)(a_6\mu - a_6) + 2\, a_8\overline{b}_7 + (2(\mu-2))a_6(\overline{a}_7 + \overline{a}_2)\right) X_1^{\mu-2}\cdot\overline{X}_1 \\
&+ \Big((2\overline{a}_7 + \overline{b}_4)(\mu b_6 - b_6 + \frac{1}{2}\, a_1\mu^2 - \frac{3}{2}\, a_1\mu + a_1) \\
&+ 2\overline{b}_7(a_2\mu - 2a_2 + b_8) + (2(\mu-2))a_6(\overline{b}_2 + \overline{a}_5) + \\
&+ (-\frac{3}{2}\, a_1\mu + 2\mu b_6 + a_1 - 4\, b_6 + \frac{1}{2}\, a_1\mu^2)(2\overline{a}_7 + \overline{b}_4)\Big) X_1^{\mu-2}\cdot\overline{X}_2 \\
&+ \left(2\overline{b}_7(a_6\mu - 2a_6) + (2(\mu-2))a_6\overline{b}_7\right) X_1^{\mu-3}\cdot X_2\cdot\overline{X}_1 \\
&+ \Big(2\overline{b}_7(\mu a_6 + \mu b_2 - 2a_6 - 2b_2 + \frac{1}{2}\, a_1\mu^2 - \frac{5}{2}\, a_1\mu + 3\, a_1) + (2(\mu-2))a_6 \\
&+ (2(-\frac{3}{2}\, a_1\mu + 2\,\mu b_6 + a_1 - 4\, b_6 + \frac{1}{2}\, a_1\mu^2))\overline{b}_7\Big) X_1^{\mu-3}\cdot X_2\cdot\overline{X}_2 \\
&+ (2(\mu-2))(\mu-3)a_6^2\, X_1^{\mu-4}\cdot\overline{X}_1\cdot X_1\cdot\overline{X}_1 + \cdots
\end{aligned}
$$

Corollary

In the pre-assumptions (2) we have:

$$\boxed{a_1 = b_4 = a_5 = a_6 = b_6 = a_7 = b_7 = b_8 = 0.}$$

- Similar argument on:

✓ $X := X_1^{\mu-2} \cdot \overline{X}_1 \cdot X_1 \cdot \overline{X}_1 \Rightarrow \boxed{b_5 = a_8 = 0}$.

✓ $X := X_1^{\mu-2} \cdot X_1 \cdot X_2 \cdot \overline{X}_2 \Rightarrow \boxed{a_2 = b_2 = 0}$.

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Theorem

The maximum conjecture is correct in CR dimension 2.

Corollary

In the pre-assumptions (2) we have:

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Theorem

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Further investigations

- Conjecture (Beloshapka, Russian J. Math. Phys. (2020)). Let

$$\mathfrak{g}_{-\mu} + \cdots + \mathfrak{g}_{-1} + \mathfrak{g}_0 + \cdots + \mathfrak{g}_\nu$$

be the infinitesimal CR automorphism algebra associated with a certain regular nondegenerate (not necessarily total) manifold. Is it correct that:

$$\mu \geq \nu?$$

Proof in CR dimension $n = 1$

Proposition (Medori-Nacinovich, Compos. Math. (1997))

Let $\mathfrak{m} = \sum_{j=-\mu}^{-1} \mathfrak{m}_j$ be a nondegenerate CR algebra of CR dimension $n = \frac{1}{2}\dim \mathfrak{m}_{-1} = 1$. Then, the Levi-Tanaka prolongation $(\mathfrak{m} + \mathfrak{g}_0)^\infty = \mathfrak{m} + \mathfrak{g}_0 + \sum_{j \geq 1} \mathfrak{g}_j$ is either solvable with $\mathfrak{g}_j = 0$ for every $j \geq 1$ or is simple and isomorphic to $\mathfrak{su}(2,1) = \mathfrak{m}_{-2} + \mathfrak{m}_{-1} + \mathfrak{g}_0 + \mathfrak{g}_1 + \mathfrak{g}_2$.

As long as Algebra and Geometry proceeded along seperate paths, their advance was slow and their applications limited. But when these sciences joined company they drew from each other fresh vitality and thenceforward marched on at rapid pace towards perfection

Joseph L. Lagrange (1736-1813)

Thanks for your attention...