Dynamics of a population with two equal dominated species[1]

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[1]Rozikov U.A., Usmonov J.B., **Qual. Theory Dyn. Syst.**, 19, 2020, 19 pp. (**IF=1.4**)

Introduction

The study of discrete time continuous dynamical systems is well developed[2]. However, for discrete time discontinuous systems, the theory was not exhaustively studied, including significant results such as Sharkovskii's theorem does not hold in the discontinuous case. Some results were given in the study of dynamics of piecewise - continuous systems[3] .

Throughout the talk, we focus on the study of dynamics generated by one-dimensional function with a discontinuity point.

[2] R.L. Devaney, *An Introduction to Chaotic Dynamical System* (Westview Press, 2003).

[3] M. di Bernardo, C.J. Budd, A.R. Champneys, P. Kowalczyk, *Piecewise-smooth Dynamical Systems: Theory and Applications*. Applied Math. Sci., **163** (2008).

Main definitions

Let $f : X \to X$ be a function.
For $x \in X$ denote by $f^n(x)$ the n-fold composition of f with itself :

$$f^n(x) = \underbrace{f(f(f \dots (f}_{n\,\text{times}}(x))) \dots).$$

For arbitrary given $x_0 \in X$ and $f : X \to X$ the discrete-time dynamical system (also called forward orbit or trajectory of x_0) is the sequence of points

$$x_0, x_1 = f(x_0), x_2 = f^2(x_0), x_3 = f^3(x_0), \dots \tag{1}$$

Definition 1

A point $x \in X$ is called a fixed point for $f : X \to X$ if $f(x) = x$. The point x is a periodic point of period p if $f^p(x) = x$. The least positive p for which $f^p(x) = x$ is called the prime period of x.

Denote the set of all fixed points and all periodic points by $\mathrm{Fix}(f)$ and $\mathrm{Per}_p(f)$ respectively.

There are three kinds of periodic points: attracting, repelling and indifferent. Let x^* be a p-periodic point.

- If $|(f^p(x^*))'| < 1$, x^* - attracting;
- $|(f^p(x^*))'| > 1$, x^* - repelling;
- $|(f^p(x^*))'| = 1$, x^* - indifferent.

Evolution operator.

Let S^{m-1} be the simplex:

$$S^{m-1} = \{x = (x_1, \dots, x_m) \in \mathbb{R}^m : x_i \geq 0, \ \sum_{i=1}^{m} x_i = 1\}.$$

Consider a population consisting of m species. Let $x^0 = (x_1^0, \dots, x_m^0) \in S^{m-1}$ be the probability distribution (where $x_i^0 = P(i)$ is the probability of i, $i = 1, 2, \dots, m$) of species in the initial generation, and $P_{ij,k}$ the probability that individuals in the ith and jth species interbreed to produce an individual k, more precisely $P_{ij,k}$ is the conditional probability $P(k|i,j)$ that ith and jth species interbred successfully, then they produce an individual k.

Assume the "parents" ij are independent i.e., $P(i,j) = P(i)P(j) = x_i^0 x_j^0$. Then the probability distribution $x' = (x'_1, \dots, x'_m)$ (the state) of the species in the first generation can be found by the total probability

$$x'_k = \sum_{i,j=1}^{m} P(k|i,j)P(i,j) = \sum_{i,j=1}^{m} P_{ij,k} x_i^0 x_j^0, \quad k = 1, \dots, m. \qquad (2)$$

This means that the association $x^0 \in S^{m-1} \to x' \in S^{m-1}$ (i.e. (2)) defines a map V called the *evolution operator*.

The states of the population described by the following discrete-time dynamical system

$$x^{(0)}, \quad x^{(1)} = V(x^{(0)}), \quad x^{(2)} = V^2(x^{(0)}), \quad x^{(3)} = V^3(x^{(0)}), \dots \qquad (3)$$

where $V^n(x) = \underbrace{V(V(\dots V}_{n}(x))\dots)$ denotes the n times iteration of V to x.

The main problem for a given dynamical system is to describe the limit points of $\{x^{(n)}\}_{n=0}^{\infty}$ for arbitrary given $x^{(0)}$.

Consider a population consisting of two species, i.e. $m = 2$. Denote the set of species by $E = \{1, 2\}$.
For a parameter $a \in [-1, 1]$ define the operator $V_a : S^1 \to S^1$ as

$$V_a : \begin{cases} x' = x(1 + ay) \\ y' = y(1 - ax) \end{cases}$$

For an initial point $z = (x, y) \in S^1$ we consider the trajectory $z^{(n)} = (x^{(n)}, y^{(n)}) = V^n(z)$ and

$$\lim_{n\to\infty} z^{(n)} = \begin{cases} (0,1), & \text{if } \ a < 0 \\ (1,0), & \text{if } \ a > 0 \end{cases} \tag{4}$$

Thus (4) means that if $a > 0$ (resp. $a < 0$) then the specie 2 (resp. 1) will extinct and the specie 1 (resp. 2) will dominate.

To ensure that both species will have equal domination we define an evolution operator, $V_{a,b} : z = (x, y) \in S^1 \to z' = (x', y') \in S^1$ by

$$V_{a,b} = \begin{cases} V_a(z), & \text{if } \ x \leq \frac{1}{2} \\ V_b(z), & \text{if } \ x > \frac{1}{2} \end{cases} \tag{5}$$

We note that the probabilities $P_{ij,k}$ mentioned in (2) are independent on $x \in S^{m-1}$, but for operator (5) we have

$$P_{11,1} = 1 - P_{11,2} = P_{22,2} = 1 - P_{22,1} = 1$$

and the remaining coefficients depend on the points $z = (x, y)$ of the simplex S^1:

$$P_{12,1}(z) = 1 - P_{12,2}(z) = \begin{cases} \frac{1+a}{2}, & \text{if } \ x \leq \frac{1}{2} \\ \frac{1-b}{2}, & \text{if } \ x > \frac{1}{2}. \end{cases} \tag{6}$$

A function with unique discontinuity point

Consider the dynamical system generated by the evolution operator $V_{a,b}$. Using the equality $x+y=1$, this operator can be reduced to the function $f_{a,b}:[0,1]\to[0,1]$ defined by

$$f_{a,b}(x)=\begin{cases} x(1+a-ax), & 0\leq x\leq \frac{1}{2} \\ x(1-b+bx), & \frac{1}{2}<x\leq 1, \end{cases} \tag{7}$$

where by the symmetry of parameters we can assume that $a,b\in[0,1]$. It is clear that, the function is a piecewise-continuous, that's, it is discontinuous at the point $x=\frac{1}{2}$ when $(a,b)\neq(0,0)$ and, is smooth at each semi interval.

Topological conjugacy

The notion of topological conjugacy is important in the study of iterated functions and more generally dynamical systems. Because mappings which are topologically conjugate are completely equivalent in terms of their dynamics.

Definition 2

Let $f : A \to A$ and $g : B \to B$ be two maps. f and g are said to be topologically conjugate if there exists a homeomorphism $h : A \to B$ such that, $h \circ f = g \circ h$. The homeomorphism h is called a topologically conjugacy.

Proposition 1

Two distinct functions $f_{a,b}$ and $f_{\tilde{a},\tilde{b}}$ are topologically conjugate if and only if $\tilde{a} = b$ and $\tilde{b} = a$.

Remark 1

According to Proposition 1 it is sufficient to study the dynamical system generated by (7) in the domain $a \in [0,1]$, $a \leq b$.

For $a \leq b$ we consider the following possible cases:

1. $a = b = 0$;
2. $a = 0,\ b \neq 0$;
3. $a \neq 0,\ b \neq 0$.

Remark 2

If $a = b = 0$ *then the function has the form* $f_{0,0}(x) = x,\ 0 \leq x \leq 1.$ *This case is not interesting.*

The case $a = 0,\ b \neq 0$

$$f_{0,b}(x) \equiv f(x) = \begin{cases} x, & 0 \leq x \leq \frac{1}{2}; \\ x(1 - b + bx), & \frac{1}{2} < x \leq 1. \end{cases} \tag{8}$$

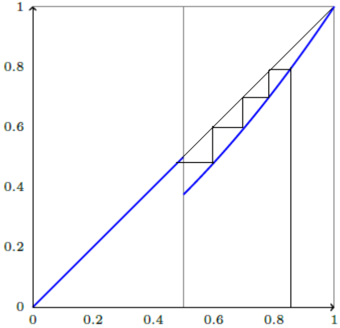

Figure 1 : The graph and phase portraits of (8) with $b = \frac{1}{2}$.

Proposition 2

For the dynamical system generated by function (8) the following hold:

1) $\mathrm{Fix}(f) = [0, \frac{1}{2}] \bigcup \{1\}$;
2) *if* $x \in (\frac{1}{2}, 1)$, *then there exists* $n \in N$ *such that*

$$f^n(x) = p, \quad f^{n+1}(x) = f(p) = p,$$

   where $p \in (\frac{1}{2} - \frac{b}{4}, \frac{1}{2}]$;
3) *for any initial point* $x_0 \in (\frac{1}{2} - \frac{b}{4}, \frac{1}{2}]$ *the following recurrence formula expresses the orbit of the points which tends to* x_0:

$$x_{n+1} = \frac{1}{b}\left(\sqrt{bx_n + \left(\frac{1-b}{2}\right)^2} + \frac{b-1}{2}\right), \; n \geq 0.$$

The case $a \neq 0,\ b \neq 0$

$$f_{a,b}(x) \equiv f(x) = \begin{cases} x(1+a-ax), & 0 \leq x \leq \frac{1}{2}; \\ x(1-b+bx), & \frac{1}{2} < x \leq 1. \end{cases} \tag{9}$$

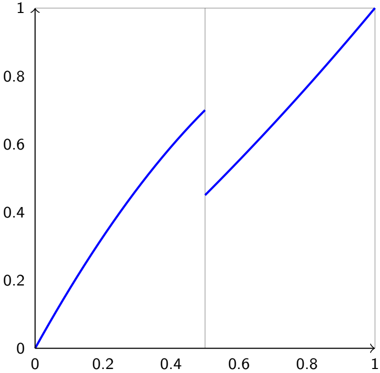

Figure 2 : The graphics of (9) with $a = \frac{1}{5},\ b = \frac{4}{5}$.

Proposition 3

Let $A = (\frac{1}{2} - \frac{b}{4}, \frac{1}{2} + \frac{a}{4}]$ *be subset of* $[0, 1]$. *Then* $f(A) = A$.

Lemma 1

For any $x \in [0, 1] \setminus A$ *there exists* $n_0(x) \in N$, *such that* $f^{n_0(x)}(x) \in A$.

Remark 3

Obviously, the set of fixed points is $\mathrm{Fix(f)} = \{0, 1\}$ *for* $ab \neq 0$. *Besides we have* $|f'(0)| = 1 + a > 1, |f'(1)| = 1 + b > 1$. *Thus both fixed points are repeller. Therefore both species will always survive.*

Theorem 1

The dynamical system generated by the function (9) has 2-periodic points if and only if the parameters a, b satisfy the following conditions

$$a \in (0,1), \ \frac{2(\sqrt{a^2+1}-1)}{a} < b < \frac{4a}{4-a^2}. \tag{10}$$

Theorem 2

If f has a periodic point then the point is a repelling.

Let's consider the following case:

$$a \in (0,1], \quad a \le b \le \frac{4a}{4-a^2} \tag{11}$$

Lemma 2

If the dynamical system associated by the function (9) satisfies (11) then the followings hold for sets A_1, A_2, A_3, A_4:

- $f(A_1) \subset A_2 \cup A_3$;
- $f(A_2) \subset A_4$;
- $f(A_3) \subset A_1$;
- $f(A_4) \subset A_2 \cup A_3$.

Where $A_1 = (\frac{1}{2} - \frac{b}{4}; (\frac{1}{2} + \frac{a}{4})(1 - \frac{b}{2} + \frac{ab}{4})(1 - \frac{b}{2} + b(\frac{a-b}{4} + \frac{a^2 b}{16}))]$,

$A_2 = ((\frac{1}{2} - \frac{b}{4})(1 + \frac{a}{2} + \frac{ab}{4}); \frac{1}{2}], \quad A_3 = (\frac{1}{2}, (\frac{1}{2} + \frac{a}{4})(1 - \frac{b}{2} + \frac{ab}{4})]$,

$A_4 = ((\frac{1}{2} - \frac{b}{4})(1 + \frac{a}{2} + \frac{ab}{4})(1 + \frac{a}{2} - a(\frac{a-b}{4} - \frac{ab^2}{16})); \frac{1}{2} + \frac{a}{4}]$.

Proposition 4

If the dynamical system generated by the function (9) satisfies the condition (11), then the dynamical system has no odd periodic points.

The following example shows that if the condition (11) does not hold then there may exist odd periods. Besides, this example shows that Sharkovskii's theorem does not hold for discontinuous systems.

Example. We have 3-period points when the function is

$$f_{\frac{1}{2},1}(x) = \begin{cases} \frac{1}{2}x(3-x), & 0 \le x \le \frac{1}{2}; \\ x^2, & \frac{1}{2} < x \le 1. \end{cases} \tag{12}$$

The points $0,47556611\ldots$, $0,60026760\ldots$, $0,36032119\ldots$ are 3-periodic points of (12).

For discrete time dynamical system $x_{n+1} = f(x_n)$, for an orbit starting with x_0 the **Lyapunov exponent** can be defined as follows:

$$\lambda(x_0) = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \ln |f'(x_i)|. \tag{13}$$

The Lyapunov exponent "λ", is useful for distinguishing among the various types of orbits. It works for discrete as well as continuous systems.

Proposition 5

Let $x_0 \in [0,1] \setminus \{\frac{1}{2}\}$. Then the Lyapunov exponent $\lambda(x_0)$ is non negative for (9).

By Proposition 5 we have $\lambda \geq 0$, in the case

- $\lambda = 0$ it indicates that the trajectory converges to an indifferent fixed point.
- $\lambda > 0$ it follows that the dynamical system is chaotic, i.e., nearby points, no matter how close, will diverge to any arbitrary separation. All neighborhoods in the phase space will eventually be visited.

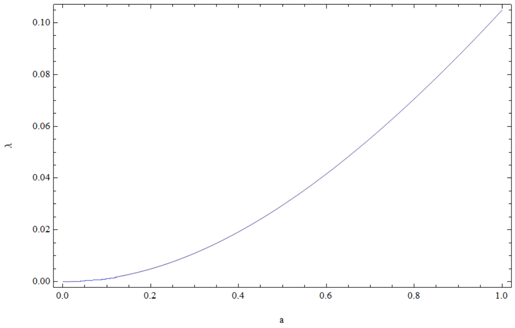

Figure 3 : Typical graphics of the Lyapunov exponent for the system function $f_{a,b}(x)$ for different values of the parameter b related to a: $b = a$.

Bifurcation

In mathematics, particularly in dynamical systems, a bifurcation diagram shows the values visited or approached asymptotically (fixed points, periodic orbits, or chaotic attractors) of a system as a function of a bifurcation parameter in the system. It is usual to represent stable values with a solid line and unstable values with a dotted line, although often the unstable points are omitted.

The bifurcation diagram of Logistic map[4]

$x_{n+1} = rx_n(1 - x_n).$

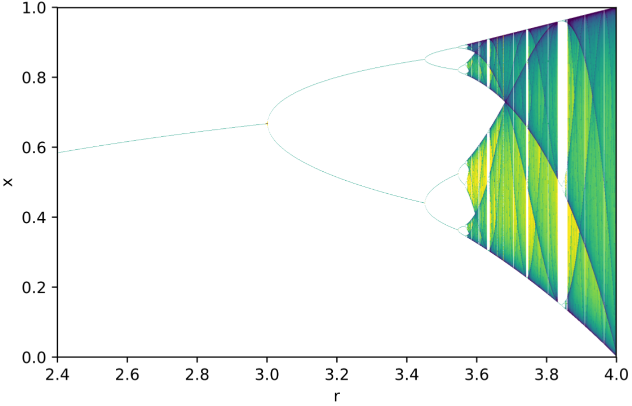

Figure 4 : Bifurcation diagram of the logistic map. The attractor for any value of the parameter r is shown on the vertical line at that r.

In Figure 6 the limit set consists of three intervals as we saw in Proposition 4 for each value of the parameter a in $[0, 1]$ (The sets A_2 and A_3 are symmetric about $\frac{1}{2}$). Because if $b = a$ then the condition (11) holds. That's, the trajectory of any initial point in $[0, 1]$ (except periodic points) is dense in these intervals.

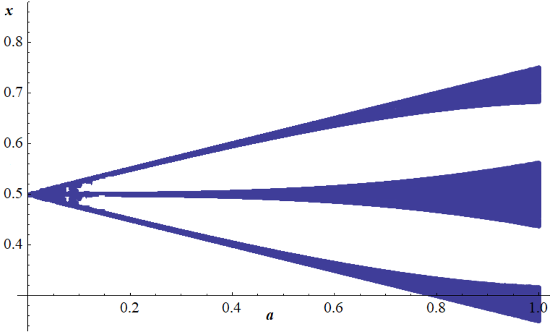

Figure 5 : Bifurcation diagram for the system function $f_{a,b}(x)$ for the case $b = a$.

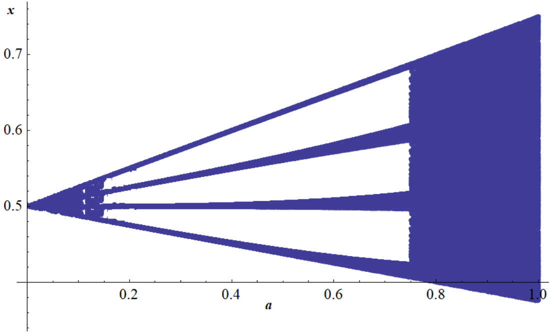

Figure 6 : Bifurcation diagram for the system function $f_{a,b}(x)$ for the case $b = \frac{a}{2}$.

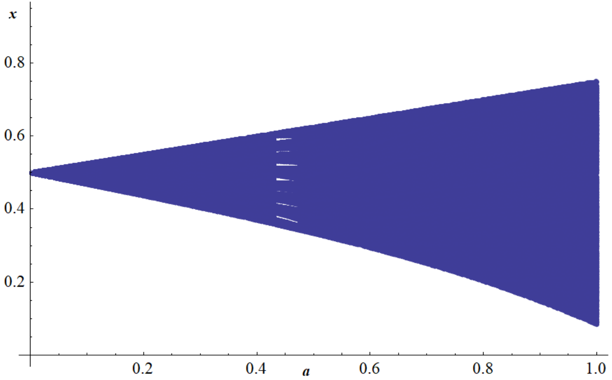

Figure 7 : Bifurcation diagram for the system function $f_{a,b}(x)$ for the case $b = \frac{5a}{4-a^2}$.

THANKS FOR YOUR ATTENTION!

List of Publications

1. U.A. Rozikov, I.A. Sattarov, J.B. Usmonov, **Journal of Applied Nonlinear Dynamics**, 5(2), 2016, pp. 185-191. (**IF=0.487**)
2. U.A. Rozikov, J.B. Usmonov, **Qualitative Theory of Dynamical Systems**, 19, 62, 2020, 19 p. (**IF=1.4**)
3. J.B. Usmonov, **Uzbek Mathematical Journal**, No 2, 2019, pp. 127-134.
4. J.B. Usmonov, M.A. Kodirova, **Bull. Ins. Math.**, No 3, 2020, pp. 98-107.
5. J.B. Usmonov, **Uzbek Mathematical Journal**.(pre-print)

List of Conference Abstracts

1. Usmonov J.B. The dynamical system generated by the bounded floor function [f(x)]. // Conference Actual problems of differential equations and their applications. 15-17 December, 2017. Tashkent.
2. Usmonov J.B. On a two dimensional dynamical system generated by the floor function. // Conference New results of mathematics and their applications, 14-15 May, 2018, Samarkand, p. 14-15.
3. Rozikov U.A., Usmonov J.B. Dynamical system of a piecewise-continuous function. // Actual problems and implementation of the analysis Republican scientific conference, 4-5 October, 2019, Karshi, p. 72-76.

4. Rozikov U.A., Usmonov J.B. On a piecewise-continuous dynamical system. // Modern problems of geometry and topology and its applications International conference, 21-23 November, 2019, Tashkent, p. 75-76.
5. Usmonov J.B. The limit set of trajectories for a discontinuous Volterra operator. // Modern problems of differential equations and related branches of mathematics International scientific conference, 12-13 March, 2020, Fergana, p. 395-396.
6. Usmonov J.B. On the dynamics of discontinuous QSO Volterra. // 41th International Conference on Quantum Probability and Related Topics, 28 March 1 April, 2021, UAE.

Sharkovskii's theorem

Let's define the following ordering (Sharkovskii ordering) of the set of natural numbers:

$$3 \succ 5 \succ 7 \succ \dots \quad \text{(all odd numbers, except 1)} \succ$$

$$3 \cdot 2 \succ 5 \cdot 2 \succ 7 \cdot 2 \dots \quad \text{(all odd numbers multplied to 2, except 1)} \succ$$

$$3 \cdot 2^2 \succ 5 \cdot 2^2 \succ 7 \cdot 2^2 \succ \dots \text{(all odd numbers multplied to } 2^2\text{, except 1)} \succ$$

$$3 \cdot 2^3 \succ 5 \cdot 2^3 \succ 7 \cdot 2^3 \succ \dots \text{(all odd numbers multplied to } 2^3\text{, except 1)} \succ$$

$$\succ \dots \succ 2^n \succ \dots \succ 2^4 \succ 2^3 \succ 2^2 \succ 2 \succ 1.$$

Here every positive integer appears exactly once somewhere on this order.

Theorem 3 (Sharkovskii)

Let f be a continuous function on $\mathbb{R}$. If f has a periodic point of least (prime) period p and p precedes q in the above ordering (i.e. $p \succ q$), then f also has a periodic point of least period q.