On homogeneous k-nondegenerate CR manifolds

Andrea Santi
UiT The Arctic University of Norway

Seminar “Complex analysis in several variables”
Moscow State University, 14th April 2021

First part of the talk:

- Tanaka prolongation scheme revisited

Second part:

- CR mnfds $(M, \mathcal{D}, \mathcal{J})$ and k-nondegeneracy
- Cores and their prolongations

Last part, applications in $\dim M = 7$:

- Classification of cores
- Building 2- and 3-nondegenerate homogeneous CR mnfds

First Part

History of the problem

- In 1907 H. Poincaré proved that two generic real hypersurfaces M and M' in $\mathbb{C}^2$ are not biholomorphically equivalent
- É. Cartan realized that $M \subset \mathbb{C}^2$ has non-trivial geometric structure: maximal complex distribution $\mathcal{D} \subset TM$ with $\mathcal{J} : \mathcal{D} \to \mathcal{D}$
- He solved equivalence problem for Levi nondegenerate $(M, \mathcal{D}, \mathcal{J})$ in 1932 by associating bundle $\pi : P \to M$ with absolute parallelism Φ s.t. $\mathrm{Aut}\,(M, \mathcal{D}, \mathcal{J}) \cong \mathrm{Aut}\,(P, \Phi)$
- The construction was generalized by S.-S. Chern and J. Moser to *Levi nondegenerate* hypersurfaces $M \subset \mathbb{C}^n$, $n \geqslant 2$, in 1974
- They proposed approach for construction of full system of invariants based on presentation of canonical forms for defining equations

Sternberg's prolongation procedure (60's)

- M = m-dimensional mnfd
- $\mathfrak{m} = \mathbb{R}^m$
- $\pi_0 : \mathrm{Fr}_M \to M$ principal $GL(\mathfrak{m})$-bundle of all linear frames on M

$$\mathrm{Fr}_M = \{\text{linear isomorphism } \varphi : \mathfrak{m} \to T_x M \mid x \in M\}$$

- geometric input: reduction $\pi_0 : \mathrm{F}_0 \subset \mathrm{Fr}_M \to M$ to $G_0 \subset GL(\mathfrak{m})$.
  Vertical bundle $\mathrm{Ker}(d\pi_0) \cong F_0 \times \mathfrak{g}_0$ via

$$\gamma_0 : \mathrm{Ker}(d\pi_0) \to \mathfrak{g}_0 \quad , \quad \text{fundamental v.f. } \zeta_X \mapsto X$$

  Canonical soldering form $\vartheta : TF_0 \to \mathfrak{m}$.
- output: tower

$$\cdots \to F_2 \to F_1 \to F_0 \to M$$

  F_0 is called of finite type if this tower stabilizes.

Sternberg's prolongation procedure (60's)

Def. A horizontal subspace at $\varphi \in F_0$ is $H \subset T_\varphi F_0$ complementary to vertical subspace and the corresponding *frame* $\varphi_H : \mathfrak{m} \oplus \mathfrak{g}_0 \to T_\varphi F_0$ is

$$\varphi_H|_{\mathfrak{m}} = b^{-1} \circ \varphi$$
$$\varphi_H|_{\mathfrak{g}_0} = \gamma_0^{-1}$$

where $b := (d\pi_0)|_H : H \xrightarrow{\cong} T_x M$. The torsion $T_H \in \mathfrak{m} \otimes \Lambda^2 \mathfrak{m}^*$ is

$$T_H(v_1, v_2) = d\vartheta\big(\varphi_H(v_1), \varphi_H(v_2)\big) .$$

Consider exact sequence defining *first prolongation* of $\mathfrak{g}_0 \subset \mathfrak{gl}(\mathfrak{m})$:

$$0 \to \mathfrak{g}_1 \longrightarrow \mathfrak{g}_0 \otimes \mathfrak{m}^* \xrightarrow{\delta} \mathfrak{m} \otimes \Lambda^2 \mathfrak{m}^* \to 0 ,$$

with δ Spencer skew-symmetrization and $\mathfrak{g}_1 = \mathrm{Ker}(\delta) = \mathfrak{g}_0 \otimes \mathfrak{m}^* \cap \mathfrak{m} \otimes S^2 \mathfrak{m}^*$.
Fix normalization $\mathfrak{m} \otimes \Lambda^2 \mathfrak{m}^* = N \oplus \mathrm{Im}(\delta)$ and set $F_1 = \big\{\varphi_H \text{ s.t. } T_H \in N\big\}$.
It is reduction of Fr_{F_0} with the affine structure group $\mathfrak{g}_1 \subset GL(\mathfrak{m} \oplus \mathfrak{g}_0)$.
Rinse and repeat.

Sternberg's algebraic prolongation (60's)

Def. The *maximal prolongation* of affine Lie algebra $\mathfrak{m} + \mathfrak{g}_0$ is maximal graded Lie algebra $\mathfrak{g} = \mathfrak{g}_{-1} + \mathfrak{g}_0 + \mathfrak{g}_1 + \cdots$ s.t.

i) $\mathfrak{g}_{-1} + \mathfrak{g}_0 = \mathfrak{m} + \mathfrak{g}_0$,

ii) for any $k \geqslant 0$ and $X \in \mathfrak{g}_k$ one has $[X, \mathfrak{g}_{-1}] = \{0\} \Rightarrow X = 0$.

Concretely $\mathfrak{g}_k = \left(\mathfrak{m} \otimes S^{k+1}\mathfrak{m}^*\right) \cap \left(\mathfrak{g}_0 \otimes S^k \mathfrak{m}^*\right)$, which can be realized as collection of polynomial vector fields on $\mathfrak{m} = \mathbb{R}^m$ of degree k.

Sternberg's algebraic prolongation (60's)

Def. The *maximal prolongation* of affine Lie algebra $\mathfrak{m} + \mathfrak{g}_0$ is maximal graded Lie algebra $\mathfrak{g} = \mathfrak{g}_{-1} + \mathfrak{g}_0 + \mathfrak{g}_1 + \cdots$ s.t.

i) $\mathfrak{g}_{-1} + \mathfrak{g}_0 = \mathfrak{m} + \mathfrak{g}_0$,

ii) for any $k \geqslant 0$ and $X \in \mathfrak{g}_k$ one has $[X, \mathfrak{g}_{-1}] = \{0\} \Rightarrow X = 0$.

Concretely $\mathfrak{g}_k = \left(\mathfrak{m} \otimes S^{k+1}\mathfrak{m}^*\right) \cap \left(\mathfrak{g}_0 \otimes S^k \mathfrak{m}^*\right)$, which can be realized as collection of polynomial vector fields on $\mathfrak{m} = \mathbb{R}^m$ of degree k.

Thm[Kobayashi, Sternberg]

1. If $\dim(\mathfrak{g}) < \infty$ then $\mathrm{Aut}\,(M, F_0)$ is a Lie group of dim. $\leqslant \dim(\mathfrak{g})$,
2. If $\dim(\mathrm{Aut}\,(M, F_0)) = \dim(\mathfrak{g}) < \infty$, then $(M, F_0) \underset{\substack{\text{locally,}\\ \text{up to deform.}}}{\cong} (G/P, F_0^{\text{flat}})$,

– G is Lie group with $Lie(G) = \mathfrak{g}$, P subgroup with $Lie(P) = \sum_{k \geqslant 0} \mathfrak{g}_k$,

– F_0^{flat} is G-invariant reduction of the frame bundle of G/P with fiber G_0.

Sternberg's algebraic prolongation (60's)

Def. The *maximal prolongation* of affine Lie algebra $\mathfrak{m} + \mathfrak{g}_0$ is maximal graded Lie algebra $\mathfrak{g} = \mathfrak{g}_{-1} + \mathfrak{g}_0 + \mathfrak{g}_1 + \cdots$ s.t.

i) $\mathfrak{g}_{-1} + \mathfrak{g}_0 = \mathfrak{m} + \mathfrak{g}_0$,

ii) for any $k \geqslant 0$ and $X \in \mathfrak{g}_k$ one has $[X, \mathfrak{g}_{-1}] = \{0\} \Rightarrow X = 0$.

Concretely $\mathfrak{g}_k = \left(\mathfrak{m} \otimes S^{k+1}\mathfrak{m}^*\right) \cap \left(\mathfrak{g}_0 \otimes S^k\mathfrak{m}^*\right)$, which can be realized as collection of polynomial vector fields on $\mathfrak{m} = \mathbb{R}^m$ of degree k.

Thm[Kobayashi, Sternberg]

1. If $\dim(\mathfrak{g}) < \infty$ then $\mathrm{Aut}\,(M, F_0)$ is a Lie group of dim. $\leqslant \dim(\mathfrak{g})$,
2. If $\dim(\mathrm{Aut}\,(M, F_0)) = \dim(\mathfrak{g}) < \infty$, then $(M, F_0) \underset{\substack{\text{locally,} \\ \text{up to deform.}}}{\cong} (G/P, F_0^{\text{flat}})$,

– G is Lie group with $Lie(G) = \mathfrak{g}$, P subgroup with $Lie(P) = \sum_{k \geqslant 0} \mathfrak{g}_k$,

– F_0^{flat} is G-invariant reduction of the frame bundle of G/P with fiber G_0.

Any real hypersurface $M \subset \mathbb{C}^n$ is of infinite type.

Tanaka-Weisfeiler method (70's)

Let M be mnfd with distribution $\mathcal{D} \subset TM$. Consider *filtration* of Lie algebra $\mathfrak{X}(M)$ of all vector fields defined by $\Gamma(\mathcal{D})^{-1} = \Gamma(\mathcal{D})$ and

$$\Gamma(\mathcal{D})^{-k} = \Gamma(\mathcal{D})^{-k+1} + [\Gamma(\mathcal{D}), \Gamma(\mathcal{D})^{-k+1}]$$

for any $k > 1$. Evaluating at $x \in M$, we get flag

$$\ldots = \mathcal{D}^{-\mu-1}(x) = \mathcal{D}^{-\mu}(x) \supset \mathcal{D}^{-\mu+1}(x) \supset \cdots \supset \mathcal{D}^{-2}(x) \supset \mathcal{D}^{-1}(x) = \mathcal{D}(x)$$

and assuming $T_x M = \mathcal{D}^{-\mu}(x)$, the commutator in $\mathfrak{X}(M)$ induces structure of nilpotent graded Lie algebra on $\mathfrak{m}(x) = \mathrm{gr}(T_x M) = \mathfrak{m}(x)_{-\mu} + \cdots + \mathfrak{m}(x)_{-1}$.

Def.

$i)$ $\mathcal{D}$ is *strongly regular* if we have flag $TM = \mathcal{D}^{-\mu} \supset \cdots \supset \mathcal{D}^{-1} = \mathcal{D}$ of distributions and all $\mathfrak{m}(x)$ are isomorphic to $\mathfrak{m} = \mathfrak{m}_{-\mu} + \cdots + \mathfrak{m}_{-1}$

$ii)$ A *Tanaka structure* is a reduction F_0 of the bundle of graded frames

$$\mathrm{Fr}_M = \{\text{graded Lie algebra isomorphism } \varphi : \mathfrak{m} \to \mathfrak{m}(x) \mid x \in M\}.$$

Main example: strongly regular CR structures

- A *CR structure* on mnfd M of dimension $m = 2d + c$ is pair $(\mathcal{D}, \mathcal{J})$ given by distribution $\mathcal{D} \subset TM$ of rank $2d$ and smooth family of complex structures $\mathcal{J} : \mathcal{D} \to \mathcal{D}$ s.t. $\mathcal{J} = +i$ eigenspace distribution $\mathcal{D}^{10} \subset T^{\mathbb{C}}M$ is involutive, i.e., $[\mathcal{D}^{10}, \mathcal{D}^{10}] \subset \mathcal{D}^{10}$. Equivalently

$$[\mathcal{J}X, \mathcal{J}Y] - [X, Y] \in \Gamma(\mathcal{D}) ,$$
$$[\mathcal{J}X, \mathcal{J}Y] - [X, Y] = \mathcal{J}([\mathcal{J}X, Y] + [X, \mathcal{J}Y]) ,$$

  for all $X, Y \in \Gamma(\mathcal{D})$.

- If each $(\mathfrak{m}(x), \mathcal{J}|_x)$ is isomorphic to $(\mathfrak{m}, J)$, with $J : \mathfrak{m}_{-1} \to \mathfrak{m}_{-1}$ s.t. $[Jv, Jw] = [v, w]$ for $v, w \in \mathfrak{m}_{-1}$, then geometry of $(M, \mathcal{D}, \mathcal{J})$ is encoded in Tanaka structure

$$F_0 = \left\{\varphi : \mathfrak{m} \to \mathfrak{m}(x) \text{ s.t. } \varphi^* \mathcal{J}|_x = J\right\} \subset \mathrm{Fr}_M \ .$$

Tanaka-Weisfeiler method (70's)

- $(M, \mathcal{D})$ = m-dimensional mnfd with strongly regular $\mathcal{D}$ of type $\mathfrak{m}$
- $\mathfrak{m} = \mathfrak{m}_{-\mu} + \cdots + \mathfrak{m}_{-1}$
- geometric input: reduction $\pi_0 : \mathrm{F}_0 \subset \mathrm{Fr}_M \to M$ to $G_0 \subset \mathrm{Aut}\,(\mathfrak{m})$.

  Vertical bundle $\mathrm{Ker}(d\pi_0) \cong F_0 \times \mathfrak{g}_0$ via

$$\gamma_0 : \mathrm{Ker}(d\pi_0) \to \mathfrak{g}_0 \quad , \quad \text{fundamental v.f. } \zeta_X \mapsto X$$

Canonical soldering form Via pullback, the filtration on TM induces a filtration on TF_0:

$$\begin{array}{ccccccccc} TF_0 = & T^{-\mu}F_0 & \supset & \ldots & \supset & T^{-1}F_0 & \supset & T^0 F_0 := \mathrm{Ker}(d\pi_0) \\ & \big\downarrow & & & & \big\downarrow & & \\ TM = & \mathcal{D}^{-\mu} & \supset & \ldots & \supset & \mathcal{D}^{-1} & & \end{array}$$

so that $\mathrm{gr}_-(TF_0) \cong \mathrm{gr}(TM)$ and we may define soldering form

$$\vartheta_0 = \{\vartheta_0^i\}_{i<0} : \mathrm{gr}_-(TF_0) \xrightarrow{\cong} \mathfrak{m}$$

Tanaka-Weisfeiler method revisited

Tanaka's original construction of the tower

$$\cdots \to F_2 \to F_1 \to F_0 \to M$$

is rather technical. I will outline an *alternative procedure* developed by I. Zelenko in '09 and recently refined by B. Kruglikov, D. The and myself, which is closer in spirit to Sternberg's construction and has been proven to be more amenable to modifications (e.g., to deal with certain classes of 2-nondegenerate CR mnfds and with supermnfds).

Tanaka-Weisfeiler method revisited

- Short-hand notation: $T_0^i = T^i F_0$. Let $i < 0$ and consider the exact sequence

$$0 \longrightarrow T_0^{i+1}/T_0^{i+2} \longrightarrow T_0^i/T_0^{i+2} \longrightarrow T_0^i/T_0^{i+1} \longrightarrow 0$$

  Because all terms are distributions, there exists $H_0^i \subset T_0^i/T_0^{i+2}$ with *splitting* $T_0^i/T_0^{i+2} = H_0^i \oplus T_0^{i+1}/T_0^{i+2}$. (For $i = 0$, simply $H_0^0 := T_0^0$.)
- For $i < 0$, any choice of H_0^i gives an identification

$$b_0^i := (d\pi_0)|_{H_0^i} : H_0^i \xrightarrow{\cong} \mathcal{D}^i/\mathcal{D}^{i+1} .$$

- Take 1-*shifted graded tangent bundle* $\mathrm{gr}^{[1]}(TF_0) = \bigoplus_{i\leqslant 0} T_0^i/T_0^{i+2}$. Given choice of complements $H_0 = \{H_0^i\}_{i\leqslant 0}$, we define the 1-frame $\varphi_{H_0} : \mathfrak{m} \oplus \mathfrak{g}_0 \to \mathrm{gr}^{[1]}(TF_0)$ via

$$\varphi_{H_0}|_{\mathfrak{m}} = \left(b_0^i\right)^{-1} \circ \varphi$$

$$\varphi_{H_0}|_{\mathfrak{g}_0} = \gamma_0^{-1}$$

Tanaka-Weisfeiler method revisited

- Any splitting $H_0 = \{H_0^i\}_{i\leqslant 0}$ of the exact sequences

$$0 \longrightarrow T_0^{i+1}/T_0^{i+2} \longrightarrow T_0^i/T_0^{i+2} \longrightarrow T_0^i/T_0^{i+1} \longrightarrow 0$$

  determines set $h_0 = \{h_0^i\}_{i\leqslant 0}$ of projections $h_0^i : T_0^i/T_0^{i+2} \longrightarrow T_0^{i+1}/T_0^{i+2}$ and allows to define *1st structure function* $c_{H_0} \in \mathfrak{m} \otimes \Lambda^2\mathfrak{m}^*$ by

$$c_{H_0}(v_i, v_j) = \vartheta_0\Big(h_0\big(\big[\varphi_{H_0}(v_i), \varphi_{H_0}(v_j)\big] \bmod T_0^{i+j+2}\big)\Big)$$

  for all $v_k \in \mathfrak{m}_k$, $k < 0$. Since the filtration on TF_0 is respected by the Lie bracket, c_{H_0} is well-defined and it has weighted degree 1.
- **Lemma** Under a change of splitting: $c_{\widetilde{H}_0} = c_{H_0} + \delta\psi$, where δ is the Spencer differential from $C^{1,1}(\mathfrak{m}, \mathfrak{g})$ to $C^{1,2}(\mathfrak{m}, \mathfrak{g})$.

Weisfeiler's maximal prolongation

Def. The *maximal prolongation* of Lie alg. $\mathfrak{m} + \mathfrak{g}_0$ is the maximal graded Lie alg. $\mathfrak{g} = \mathfrak{g}_{-\mu} + \cdots + \mathfrak{g}_{-1} + \mathfrak{g}_0 + \mathfrak{g}_1 + \cdots$ s.t.

i) $\mathfrak{g}_{-\mu} + \cdots + \mathfrak{g}_0 = \mathfrak{m} + \mathfrak{g}_0$,

ii) for any $k \geqslant 0$ and $X \in \mathfrak{g}_k$ one has $[X, \mathfrak{g}_{-1}] = \{0\} \Rightarrow X = 0$.

It can be defined by $\mathfrak{g}_k = \{A \in \mathrm{Der}\,(\mathfrak{m}, \sum_{h<k} \mathfrak{g}_h) \mid A(\mathfrak{m}_j) \subseteq \mathfrak{g}_{j+k}\}$ for $k \geqslant 0$, which can be realized as polynomial vector fields on $\mathfrak{m}$ of weighted degree k.

Def. The *maximal prolongation* of Lie alg. $\mathfrak{m} + \mathfrak{g}_0$ is the maximal graded Lie alg. $\mathfrak{g} = \mathfrak{g}_{-\mu} + \cdots + \mathfrak{g}_{-1} + \mathfrak{g}_0 + \mathfrak{g}_1 + \cdots$ s.t.

i) $\mathfrak{g}_{-\mu} + \cdots + \mathfrak{g}_0 = \mathfrak{m} + \mathfrak{g}_0$,

ii) for any $k \geqslant 0$ and $X \in \mathfrak{g}_k$ one has $[X, \mathfrak{g}_{-1}] = \{0\} \Rightarrow X = 0$.

It can be defined by $\mathfrak{g}_k = \{A \in \mathrm{Der}\,(\mathfrak{m}, \sum_{h<k} \mathfrak{g}_h) \mid A(\mathfrak{m}_j) \subseteq \mathfrak{g}_{j+k}\}$ for $k \geqslant 0$, which can be realized as polynomial vector fields on $\mathfrak{m}$ of weighted degree k.

Spencer complex of $\mathfrak{g}$ is *Chevalley-Eilenberg complex* $C^\bullet(\mathfrak{m}, \mathfrak{g}) = \mathfrak{g} \otimes \Lambda^\bullet \mathfrak{m}^*$ of the Lie algebra $\mathfrak{m}$ with values in $\mathfrak{g}$:

$$\cdots \longrightarrow \mathfrak{g} \otimes \Lambda^{j-1}\mathfrak{m}^* \xrightarrow{\delta} \mathfrak{g} \otimes \Lambda^{j}\mathfrak{m}^* \xrightarrow{\delta} \mathfrak{g} \otimes \Lambda^{j+1}\mathfrak{m}^* \longrightarrow \cdots$$

It is naturally bi-graded $C^\bullet(\mathfrak{m}, \mathfrak{g}) = \bigoplus_{k \in \mathbb{Z}} C^{k,\bullet}(\mathfrak{m}, \mathfrak{g})$, where k is weighted degree.

Tanaka-Weisfeiler method revisited

- **Lemma** Under a change of splitting: $c_{\widetilde{H}_0} = c_{H_0} + \delta\psi$, where δ is the Spencer differential from $C^{1,1}(\mathfrak{m}, \mathfrak{g})$ to $C^{1,2}(\mathfrak{m}, \mathfrak{g})$.
- Fix *normalization* $C^{1,2}(\mathfrak{m}, \mathfrak{g}) = N_1 \oplus \delta C^{1,1}(\mathfrak{m}, \mathfrak{g})$ and set

$$F_1 = \left\{\varphi_{H_0} \text{ s.t. } c_{H_0} \in N_1\right\}$$

It is principal bundle on F_0 with affine structure group $\mathfrak{g}_1 \subset GL(\mathfrak{m} \oplus \mathfrak{g}_0)$.

~~Rinse and repeat.~~

- The construction is iterative but F_1 is not a Tanaka structure on F_0!

Tanaka-Weisfeiler method revisited

- Suppose we have constructed principal bundle $\pi_\ell : F_\ell \to F_{\ell-1}$ with affine structure group $\mathfrak{g}_\ell$ and consisting of ℓ-frames

$$\varphi_\ell : \mathfrak{g}_{\leqslant \ell-1} \to \mathrm{gr}^{[\ell]}(TF_{\ell-1}),$$

  where $\mathrm{gr}^{[\ell]}(TF_{\ell-1}) := \bigoplus_{i\leqslant \ell-1} T^i_{\ell-1}/T^{i+\ell+1}_{\ell-1}$ is the ℓ-shifted graded tangent bundle. The frame selects $H_{\ell-1} = \{H^i_{\ell-1}\}_{i\leqslant \ell-1} = \mathrm{Im}(\varphi_\ell)$.
- *Vertical trivialization* $\mathrm{Ker}(d\pi_\ell) \cong F_\ell \times \mathfrak{g}_\ell$ via $\gamma_\ell : \mathrm{Ker}(d\pi_\ell) \to \mathfrak{g}_\ell$
  Soldering form $\vartheta_\ell = \{\vartheta^i_\ell\}_{i<\ell} : \mathrm{gr}_{<\ell}(TF_\ell) \xrightarrow{\cong} \mathfrak{g}_{<\ell}$
- Via pullback, the filtration on $TF_{\ell-1}$ induces a filtration on TF_ℓ:

$$\begin{array}{cccccccc}
TF_\ell = & T^{-\mu}_\ell & \supset & \dots & \supset & T^{\ell-1}_\ell & \supset & T^\ell_\ell = \mathrm{Ker}(d\pi_\ell) \\
& \downarrow & & & & \downarrow & & \\
TF_{\ell-1} = & T^{-\mu}_{\ell-1} & \supset & \dots & \supset & T^{\ell-1}_{\ell-1} & &
\end{array}$$

- For $i<0$, we have pair of exact sequences

$$\begin{tikzcd} 0 \arrow[r] & T_\ell^{i+\ell+1}/T_\ell^{i+\ell+2} \arrow[r] & T_\ell^i/T_\ell^{i+\ell+2} \arrow[r] \arrow[d, "b_\ell^i"] & T_\ell^i/T_\ell^{i+\ell+1} \arrow[r] \arrow[dl, "\cong"'] & 0 \\ 0 \arrow[r] & T_{\ell-1}^{i+\ell}/T_{\ell-1}^{i+\ell+1} \arrow[r] & T_{\ell-1}^i/T_{\ell-1}^{i+\ell+1} \arrow[r] & T_{\ell-1}^i/T_{\ell-1}^{i+\ell} \arrow[r] & 0 \end{tikzcd}$$

and take H_ℓ^i s.t. $T_\ell^i/T_\ell^{i+\ell+2} \supset (b_\ell^i)^{-1}\left(H_{\ell-1}^i\right) = H_\ell^i \oplus T_\ell^{i+\ell+1}/T_\ell^{i+\ell+2}$

- For $0 \leqslant i \leqslant \ell-1$, we have

$$\begin{tikzcd} 0 \arrow[r] & T_\ell^\ell \arrow[r] & T_\ell^i \arrow[r] \arrow[d, "b_\ell^i"] & T_\ell^i/T_\ell^\ell \arrow[r] \arrow[dl, "\cong"'] & 0 \\ 0 \arrow[r] & T_{\ell-1}^{\ell-1} \arrow[r] & T_{\ell-1}^i \arrow[r] & T_{\ell-1}^i/T_{\ell-1}^{\ell-1} \arrow[r] & 0 \end{tikzcd}$$

and take H_ℓ^i s.t. $T_\ell^i \supset (b_\ell^i)^{-1}\left(H_{\ell-1}^i\right) = H_\ell^i \oplus T_\ell^\ell$. Set also $H_\ell^\ell = T_\ell^\ell$

Tanaka-Weisfeiler method revisited

- In all cases $H^i_\ell \xrightarrow{\cong} H^i_{\ell-1}$ via b^i_ℓ and we may define an *$(\ell+1)$-frame* $\varphi_{\ell+1} : \mathfrak{g}_{\leqslant \ell} \to \mathrm{gr}^{[\ell+1]}(TF_\ell)$ as usual:

$$\varphi_{\ell+1}|_{\mathfrak{g}_{\leqslant \ell-1}} = \left(b^i_\ell\right)^{-1} \circ \varphi_\ell$$
$$\varphi_{\ell+1}|_{\mathfrak{g}_\ell} = \gamma_\ell^{-1}$$

  where $\mathrm{gr}^{[\ell+1]}(TF_\ell)$ is the $(\ell+1)$-shifted graded tangent bundle

- Construction of $(\ell+1)$-th structure function is similar but requires additional care since Lie bracket is compatible with filtration on TF_ℓ only for non-positive filtration indices. It splits into *horizontal* $c^-_{H_\ell} \in C^{\ell+1,2}(\mathfrak{m},\mathfrak{g})$ and *vertical* $c^+_{H_\ell} \in (\mathfrak{g}_{0\leqslant\cdots\leqslant\ell-1})^* \otimes C^{\ell,1}(\mathfrak{m},\mathfrak{g})$
- Normalization is choice of subspaces $N^-_{\ell+1} \subset C^{\ell+1,2}(\mathfrak{m},\mathfrak{g})$ and $N^+_{\ell+1} \subset C^{\ell,1}(\mathfrak{m},\mathfrak{g})$ complementary to image of Spencer map δ

Tanaka-Weisfeiler method revisited

Thm[Tanaka '70s, Zelenko '09, Kruglikov-S.-The '21] Assume $(M, \mathcal{D}, F_0)$ is strongly regular and of finite type, i.e., $\mathfrak{g} = \mathfrak{g}_{-\mu} \oplus \cdots \oplus \mathfrak{g}_0 \oplus \cdots \oplus \mathfrak{g}_d$. Then:

1. $\exists$ fiber bundle $\pi : P \to M$ of $\dim P = \dim \mathfrak{g}$ with parallelism Φ s.t. any equivalence of $(M, \mathcal{D}, F_0)$ lifts to equivalence of Φ
2. $\dim \mathrm{Aut}\,(M, \mathcal{D}, F_0) \leqslant \dim(\mathfrak{g})$, if $\dim(\mathrm{Aut}\,(M, \mathcal{D}, F_0)) = \dim(\mathfrak{g})$ then $(M, \mathcal{D}, F_0) \underset{\substack{\text{locally,} \\ \text{up to deform.}}}{\cong} (G/P, \mathcal{D}^{\text{flat}}, F_0^{\text{flat}})$

– G is Lie group with $Lie(G) = \mathfrak{g}$, P subgroup with $Lie(P) = \sum_{i \geqslant 0} \mathfrak{g}_i$

– $\mathcal{D}^{\text{flat}}$ and F_0^{flat} are uniquely determined G-invariant geometric data

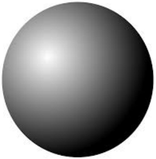
Second Part

Strongly-regular Levi nondegenerate CR mnfds

Thm[Tanaka '70s] Let $(M, \mathcal{D}, \mathcal{J})$ be a strongly regular CR mnfd of type $(\mathfrak{m}, J)$ and set $\mathfrak{g}_0 = \mathrm{der}(\mathfrak{m}, J)$. Then:

1. $\dim \mathfrak{g} < \infty$ if and only if for any nonzero $v \in \mathfrak{m}_{-1}$, there is $w \in \mathfrak{m}_{-1}$ s.t. $[v, w] \neq 0$ (i.e, the Levi form is nondegenerate); in this case
2. $\exists\ \pi : P \to M$ with parallelism Φ and $\dim \mathrm{Aut}\,(M, \mathcal{D}, \mathcal{J}) \leqslant \dim(\mathfrak{g})$

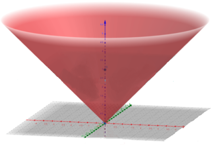
Strictly-pseudoconvex hypersurface case:



- $S^{2n+1} = G/P$
- $G = SU(n+1, 1)$, $P =$ stabilizer of isotropic line

Levi degenerate CR mnfds

$$M = \{z \in \mathbb{C}^3 : (\operatorname{Re} z^1)^2 + (\operatorname{Re} z^2)^2 - (\operatorname{Re} z^3)^2 = 0\,,\ \operatorname{Re} z^3 > 0\} \subset \mathbb{C}^3$$



- It is Levi degenerate at all points (its completion in $\mathbb{C}P^4$ is homogeneous for $G = SO^o(3,2)$), foliated by complex leaves and yet it admits no CR straightening
- This can be seen by checking the necessary condition for local CR straightenings found by Freeman (1977)

Freeman sequence

- The *Freeman sequence* is a sequence

$$\underline{\mathcal{F}}_{-1} \supset \underline{\mathcal{F}}_{0} \supset \underline{\mathcal{F}}_{1} \supset \cdots \supset \underline{\mathcal{F}}_{p-1} \supset \underline{\mathcal{F}}_{p} \supset \underline{\mathcal{F}}_{p+1} \supset \cdots$$

of complex vector fields given for any $p \geqslant -1$ by

$$\underline{\mathcal{F}}_{p} = \underline{\mathcal{F}}^{10}_{p} \oplus \overline{\underline{\mathcal{F}}^{10}_{p}} \qquad \text{where} \qquad \underline{\mathcal{F}}^{10}_{-1} := \underline{\mathcal{D}}^{10} \qquad \text{and}$$

$$\underline{\mathcal{F}}^{10}_{p} := \left\{ X \in \underline{\mathcal{F}}^{10}_{p-1} : [X, \underline{\mathcal{D}}^{01}] = 0 \mod \underline{\mathcal{F}}^{10}_{p-1} \oplus \underline{\mathcal{D}}^{01} \right\}$$

- $\underline{\mathcal{F}}_{-1} = \underline{\mathcal{D}}^{\mathbb{C}}_{-1} = \underline{\mathcal{D}}^{\mathbb{C}}$ while $\underline{\mathcal{F}}_{p}$ is subalgebra of $\mathfrak{X}(M)^{\mathbb{C}}$ if $p \geqslant 0$
- $\underline{\mathcal{F}}^{10}_{p}$ for $p \geqslant 0$ is the left kernel of the higher order Levi form

$$\mathcal{L}^{p+1} : \underline{\mathcal{F}}^{10}_{p-1} \otimes \underline{\mathcal{D}}^{01} \longrightarrow \mathfrak{X}(M)^{\mathbb{C}} / (\underline{\mathcal{F}}^{10}_{p-1} \oplus \underline{\mathcal{D}}^{01})$$
$$(X, Y) \longrightarrow [X, Y] \mod \underline{\mathcal{F}}^{10}_{p-1} \oplus \underline{\mathcal{D}}^{01}$$

Freeman sequence

- $(M, \mathcal{D}, \mathcal{J})$ is *regular* if we have flags of distributions

$$
\begin{aligned}
TM &= \mathcal{D}_{-\mu} \supset \cdots \supset \mathcal{D}_{-1} = \mathcal{D} \\
\mathcal{D}^{\mathbb{C}} &= F_{-1} \supset F_0 \supset F_1 \supset \cdots \supset F_{p-1} \supset F_p \supset F_{p+1}
\end{aligned}
$$

- Cauchy characteristic distribution $\mathrm{Re}(\underline{\mathcal{F}_0}) = \{X \in \underline{\mathcal{D}} \mid [X, \underline{\mathcal{D}}] \subset \underline{\mathcal{D}}\}$ is $\mathcal{J}$-stable and involutive. Its maximal leaves are complex mnfds, however, finite nondegeneracy tells $\nexists$ no local CR straightenings:

Def. $(M, \mathcal{D}, \mathcal{J})$ is *k-nondegenerate* if $F_p \neq 0$ for all $-1 \leqslant p \leqslant k-2$ and $F_{k-1} = 0$. Otherwise, we say it is *holomorphically degenerate*.

Abstract cores

Def. A *core* is finite dimensional real v.s. $\mathfrak{m} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{m}^p$ with:

(i) complex structure $J : \bigoplus_{p\geqslant -1} \mathfrak{m}^p \longrightarrow \bigoplus_{p\geqslant -1} \mathfrak{m}^p$ compatible with the grading

(ii) a bracket of graded Lie algebras on $\mathfrak{m}_- = \bigoplus_{p<0} \mathfrak{m}^p$ that satisfies $[Jv, Jw] = [v, w]$ for $v, w \in \mathfrak{m}^{-1}$ and which is nondegenerate and fundamental in Tanaka's sense

For any $p \geqslant -1$, we set $\hat{\mathfrak{m}}^p = \mathfrak{m}^p \otimes \mathbb{C}$, $\hat{\mathfrak{m}}^p = \mathfrak{m}^{p(10)} \oplus \mathfrak{m}^{p(01)}$, and we understand $\mathfrak{m}^{-1(10)}$ as space of maps from $\mathfrak{m}^{-1(01)}$ to $\hat{\mathfrak{m}}^{-2}$ using (ii).

(iii) an injective $\mathbb{C}$-linear map for any $p \geqslant 0$

$$L^{p+2} : \mathfrak{m}^{p(10)} \longrightarrow \mathfrak{m}^{p-1(10)} \otimes (\mathfrak{m}^{-1(01)})^* \bigcap \hat{\mathfrak{m}}^{-2} \otimes S^{p+2}(\mathfrak{m}^{-1(01)})^*$$

Abstract cores

Lemma Let $(M, \mathcal{D}, \mathcal{J})$ be k-nondegenerate CR mnfd with $\mathcal{D}_{-\mu} = TM$. For every $x \in M$, the $\mathbb{Z}$-graded vector space $\mathfrak{m}_x = \bigoplus_{p \in \mathbb{Z}} \mathfrak{m}_x^p$ defined by

$$\mathfrak{m}_x^p = \frac{\mathcal{D}_p|_x}{\mathcal{D}_{p+1}|_x} \qquad \text{for all } p \leqslant -2\,,$$

$$\mathfrak{m}_x^p = \frac{\mathrm{Re}(F_p)|_x}{\mathrm{Re}(F_{p+1})|_x} \qquad \text{for all } p \geqslant -1\,,$$

has the natural structure of an abstract core of depth μ and height $k-2$.

Proof of (iii) For every $p \geqslant 0$ we have $[\underline{\mathcal{F}}_p^{10}, \underline{\mathcal{D}}^{01}] \subset \underline{\mathcal{F}}_{p-1}^{10} \oplus \underline{\mathcal{D}}^{01}$ and $[\underline{\mathcal{F}}_p^{10}, \underline{\mathcal{F}}_0^{01}] \subset \underline{\mathcal{F}}_p^{10} \oplus \underline{\mathcal{F}}_0^{01}$ and the higher order Levi form

$$\mathcal{L}^{p+2} : \underline{\mathcal{F}}_p^{10} \otimes \underline{\mathcal{D}}^{01} \longrightarrow \underline{\mathcal{F}}_{p-1}^{10} \oplus \underline{\mathcal{D}}^{01} / (\underline{\mathcal{F}}_p^{10} \oplus \underline{\mathcal{D}}^{01})$$

induces desired $L^{p+2} : \mathfrak{m}_x^{p(10)} \times \mathfrak{m}_x^{-1(01)} \longrightarrow \mathfrak{m}_x^{p-1(10)}$ at $x \in M$. $\square$

Abstract cores

For $k = 1$, the core reduces to the usual notion of Levi-Tanaka algebra $\mathfrak{m} = \mathfrak{m}^{-\mu} + \cdots + \mathfrak{m}^{-1}$ for nondegenerate CR mnfds. In contrast with this case, it does not posses a structure of Lie algebra and even constructing a "standard model" is involved. I restricted to CR mnfds $(M, \mathcal{D}, \mathcal{J})$ of hypersurface type, i.e., cores $\mathfrak{m}$ s.t. $\mathfrak{m}_{-} = \mathfrak{m}^{-2} \oplus \mathfrak{m}^{-1}$ is the Heisenberg algebra. Two main ingredients:

- *Weisfeiler infinite-dimensional contact algebra*
- *CR algebras (aka locally homogeneous CR mnfds)*

Lie algebras of infinitesimal symmetries

Strictly pseudo-convex CR manifolds

- $(M, \mathcal{D}, \mathcal{J})$ of dimension $2n+1$
- $\mathfrak{m} = \mathfrak{m}^{-2} \oplus \mathfrak{m}^{-1}$ Heisenberg algebra of signature $\mathrm{sgn}(J) = (n, 0)$
- $\mathfrak{g} = \mathfrak{su}(n+1, 1)$ with

$$\mathfrak{g}^p = \begin{cases} \mathbb{R}^* & \text{for } p = 2 \,, \\ (\mathbb{C}^n)^* & \text{for } p = 1 \,, \\ \mathfrak{u}(n) \oplus \mathbb{R}Z & \text{for } p = 0 \,, \\ \mathbb{C}^n & \text{for } p = -1 \,, \\ \mathbb{R} & \text{for } p = -2 \,. \end{cases}$$

Tube over the future light cone

- $M = \{z \in \mathbb{C}^3 : (\mathrm{Re}\, z^1)^2 + (\mathrm{Re}\, z^2)^2 - (\mathrm{Re}\, z^3)^2 = 0\,,\ \mathrm{Re}\, z^3 > 0\} \subset \mathbb{C}^3$
- $\mathfrak{m}_- = \mathfrak{m}^{-2} \oplus \mathfrak{m}^{-1}$ Heisenberg algebra of signature $\mathrm{sgn}(J) = (1,0)$
- $\mathfrak{g} = \mathfrak{so}(3,2)$ with

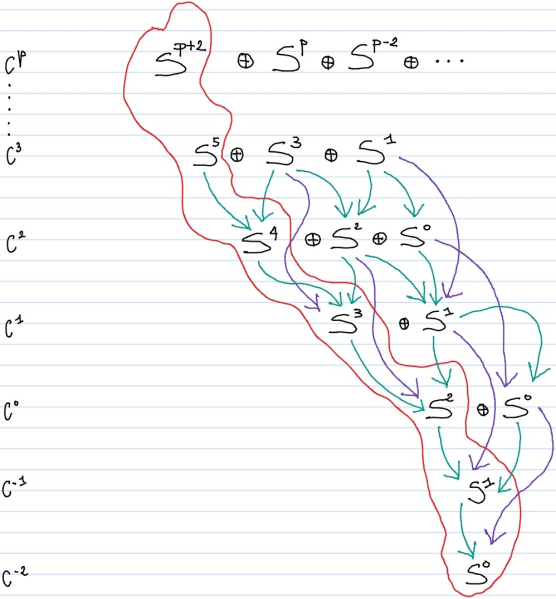
$$\mathfrak{g}^p = \begin{cases} \mathbb{R}^* & \text{for } p = 2\,, \\ \mathbb{C}^* & \text{for } p = 1\,, \\ \mathfrak{sp}_2(\mathbb{R}) \oplus \mathbb{R}Z & \text{for } p = 0\,, \\ \mathbb{C} & \text{for } p = -1\,, \\ \mathbb{R} & \text{for } p = -2\,, \end{cases}$$

where

$$\mathfrak{m} = \mathfrak{m}^{-2} \oplus \mathfrak{m}^{-1} \oplus \mathfrak{m}^0 \qquad \text{with}$$

$$\mathfrak{m}^{-2} = \mathbb{R}\,, \quad \mathfrak{m}^{-1} = \mathbb{C}\,, \quad \mathfrak{m}^0 = \mathrm{Re}(S^{2,0} \oplus S^{0,2})$$

WEISFEILER INFINITE DIMENSIONAL CONTACT ALGEBRA $C = C^{-2} + C^{-1} + C^{0} + \dots$



$\dim C^{-2} = 1$, $\dim C^{-1} = 2n$

Notation: $S^k = S^k(C^{-1})$

$\mathrm{ad}\, C^{-2} : C^p \longrightarrow C^{p-2}$

Acting trivially on C^{-2}

Adjoint action of C^{-2}

Adjoint action of C^{-1}

THE UNIVERSAL CR ALGEBRA $(\mathcal{C}, \mathfrak{u})$

$\mathcal{C}^3$	S^5 $+5,+3,+1,-1,-3,-5$	S^3 $+3,+1,-1,-3$	S^1 $+1,-1$	
$\mathcal{C}^2$	S^4 $+4,+2,0,-2,-4$	S^2 $+2,0,-2$		S^0 0
$\mathcal{C}^1$		S^3 $+3,+1,-1,-3$	S^1 $+1,-1$	
$\mathcal{C}^0$		S^2 $+2,0,-2$		S^0 0
$\mathcal{C}^{-1}$			S^1 $+1,-1$	
$\mathcal{C}^{-2}$				S^0 0

Eigenvalues of J

$\mathfrak{u}$ = complex subalgebra of $\hat{\mathcal{C}} = \mathcal{C} \otimes \mathbb{C}$

$\mathfrak{u} = \oplus\, \mathfrak{u}^p$

$\mathfrak{u}^p$ = sum of all J-eigenspaces of non minimal eigenvalue

$\mathfrak{u} \cap \bar{\mathfrak{u}}$ = complexified isotropy algebra

$\mathfrak{u}^p \cap \bar{\mathfrak{u}}^p$ = sum of all J-eigenspaces of non extremal eigenvalue

$\mathfrak{m} = \oplus\, \mathfrak{m}^p$ core

$\mathfrak{m}^p$ = sum of all J-eigenspaces of extremal eigenvalue

Universal CR algebra

Prop. The pair $(\mathfrak{c}, \mathfrak{u})$ is an holomorphically nondegenerate CR algebra. Moreover, any abstract core $\mathfrak{m}$ (of hypersurface type) admits a natural immersion into the core of $(\mathfrak{c}, \mathfrak{u})$ s.t. $[\mathfrak{m}^{p(10)}, \mathfrak{c}^{-1(01)}] \subset \mathfrak{m}^{p-1(10)}$ if $p \geqslant 1$.

Sketch of the proof

Holomorphic nondegeneracy The p-th term of Freeman sequence of $(\mathfrak{c}, \mathfrak{u})$ is $\bigoplus_{0 \leqslant j \leqslant p-1} (\mathfrak{u}^j \cap \overline{\mathfrak{u}}^j) \oplus \bigoplus_{j \geqslant p} \mathfrak{u}^j$ so their intersections is isotropy $\mathfrak{u} \cap \overline{\mathfrak{u}}$.
Universal property In this case the injective $\mathbb{C}$-linear maps in the defining property of a core are isomorphisms

$$L^{p+2} : S^{p+2}\mathbb{C}^n \longrightarrow S^{p+1}\mathbb{C}^n \otimes \overline{\mathbb{C}^n}^* \bigcap \mathbb{C} \otimes S^{p+2}(\overline{\mathbb{C}^n})^*$$

Rescaling the eigenvalues of the previous slide to get a complex structure, one directly constructs the immersion. $\square$

STRICTLY PSEUDO-CONVEX CR MANIFOLDS

C^2	S^4 $+4, +2, 0, -2, -4$		S^2 $+2, 0, -2$		S^0 0
C^1		S^3 $+3, +1, -1, -3$		S^1 $+1, -1$	
C^0			S^2 $+2, 0, -2$		S^0 0
C^{-1}				S^1 $+1, -1$	
C^{-2}					S^0 0

$\mathfrak{g} = \mathfrak{su}(m+1, 1)$

TUBE OVER THE FUTURE LIGHT CONE

C^2	S^4 $+4,+2,0,-2,-4$		S^2 $+2,0,-2$		S^0 0
C^1		S^3 $+3,+1,-1,-3$		S^1 $+1,-1$	
C^0			S^2 $+2,0,-2$		S^0 0
C^{-1}				S^1 $+1,-1$	
C^{-2}					S^0 0

$\mathfrak{g} = so(3,2)$

Models as prolongations of type $\mathfrak{m}$

Def. A *model* of type $\mathfrak{m}$ is a $\mathbb{Z}$-graded Lie subalgebra $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}^p$ of $\mathfrak{c}$ satisfying

(i) $\mathfrak{g}^p = \mathfrak{c}^p$ for $p < 0$;

(ii) the grading element Z is in $\mathfrak{g}^0$;

(iii) $\hat{\mathfrak{g}}^p = (\hat{\mathfrak{g}}^p \cap \mathfrak{u}^p) + (\hat{\mathfrak{g}}^p \cap \bar{\mathfrak{u}}^p)$ for all $p \geqslant 0$;

(iv) the natural projection π of $\hat{\mathfrak{c}}^p$ onto the J-eigenspace of maximum eigenvalue satisfies
$$\pi(\hat{\mathfrak{g}}^p \cap \mathfrak{u}^p) = \mathfrak{m}^{p(10)}$$
for all $p \geqslant 0$.

Remark We do not require J to be in $\mathfrak{g}^0$. In that case $\mathfrak{g}$ decomposes into J-eigenspaces and $\mathfrak{m} \subset \mathfrak{g}$, however, this is too strong requirement.

Models as prolongations of type $\mathfrak{m}$

Thm (S.'15) Let $\mathfrak{g} = \bigoplus \mathfrak{g}^p$ be a model of type $\mathfrak{m}$. Then:

(i) $\mathfrak{q} = \hat{\mathfrak{g}} \cap \mathfrak{u}$ is $\mathbb{Z}$-graded complex Lie subalgebra of $\hat{\mathfrak{g}}$;

(ii) the pair $(\mathfrak{g}, \mathfrak{q})$ *is a CR algebra* with real *isotropy algebra* $\mathfrak{g}_o = \mathfrak{g} \cap \mathfrak{u}$, which is $\mathbb{Z}$-graded in nonnegative degrees;

(iii) $\dim \mathfrak{g} < \infty$ and the associated locally homogeneous CR mnfd with CR algebra of infinitesimal automorphisms $(\mathfrak{g}, \mathfrak{q})$ is of hypersurface type and strongly regular of *type* $\mathfrak{m}$ (in particular k-nondegenerate, where $k = \mathrm{height}(\mathfrak{m}) + 2$).

Third Part

Classification of abstract cores in dimension 7

Three main *classes of abstract cores* associated with strongly regular 7-dimensional CR mnfds:

Class	$\mathfrak{m}^{-2}$	$\mathfrak{m}^{-1}$	$\mathfrak{m}^{0}$	$\mathfrak{m}^{1}$	$\mathfrak{m}^{p}$ $(p>1)$
(A)	$\mathbb{R}$	$\mathbb{C}^3$	0	0	0
(B)	$\mathbb{R}$	$\mathbb{C}^2$	$\mathbb{C}$	0	0
(C)	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{C}$	$\mathbb{C}$	0

The case $\mathfrak{m}^{-2}=\mathbb{R}$, $\mathfrak{m}^{-1}=\mathbb{C}$, $\mathfrak{m}^{0}=\mathbb{C}^2$ and $\mathfrak{m}^{p}=0$ for all $p>0$ is not permissible. Class (A) is Levi-nondegenerate and there exists exactly one core in the 3-nondegenerate class (C). The description of equivalence classes of cores in (B) splits into $\mathrm{sgn}(J)=(2,0)$ and $\mathrm{sgn}(J)=(1,1)$.

Classification of abstract cores of class (B)

Any core $\mathfrak{m} = \mathfrak{m}^{-2} \oplus \mathfrak{m}^{-1} \oplus \mathfrak{m}^{0}$ of class (B) is determined by complex line $\mathfrak{m}^{0(10)} \subset S^2\mathbb{C}^2$. We are interested in action on $S^2\mathbb{C}^2$ of Lie group $K_\sharp = \mathbb{C}^\times \cdot K$, where

$$
\begin{aligned}
K &= \mathrm{SO}_3(\mathbb{R}) \cong \mathrm{SU}(2)/\mathbb{Z}_2 && \text{if } \operatorname{sgn}(J) = (2,0)\ , \\
K &= \mathrm{SO}^+(2,1) \cong \mathrm{SU}(1,1)/\mathbb{Z}_2 && \text{if } \operatorname{sgn}(J) = (1,1)\ .
\end{aligned}
$$

In other words we want a description of representatives for the *orbits of K acting on $\mathbb{CP}^2$*. We say that an orbit is *admissible* if its stabilizer in K is not discrete. (Equivalently, the Lie algebra $\mathfrak{n}_\sharp$ of the stabilizer $N_\sharp \subset K_\sharp$ is bigger than $\mathbb{C}$.)

Classification of cores of class (B) with $\mathrm{sgn}(J) = (2,0)$

Thm (S.'15) Every 7-dimensional core $\mathfrak{m}$ of type (B) and $\mathrm{sgn}(J) = (2,0)$ is isomorphic with one and only one of the canonical forms $\mathfrak{m}_t$ in the following Table, $t \in [0,1]$.

<table>
<tr><td>$\mathfrak{m}_t = \bigoplus \mathfrak{m}_t^p$
$t \in [0,1]$</td><td>$\mathfrak{m}_t^{0(10)}$</td><td>$\mathfrak{n}_\sharp$</td></tr>
<tr><td>$t = 0, 1$</td><td rowspan="2">$(1+t)e_1^{10} \odot e_1^{10} + (1-t)e_2^{10} \odot e_2^{10}$</td><td>$\mathbb{C} \oplus \mathfrak{so}_2(\mathbb{R})$</td></tr>
<tr><td>$t \in (0,1)$</td><td>$\mathbb{C}$</td></tr>
</table>

Classification of cores of class (B) with $\mathrm{sgn}(J) = (1, 1)$

Thm (S.'15) Every 7-dimensional core $\mathfrak{m}$ of type (B) and $\mathrm{sgn}(J) = (1, 1)$ is isomorphic with one and only one of canonical forms $\mathfrak{m}_t$, $\widetilde{\mathfrak{m}}_t$, $\mathfrak{m}_\pm$, $\mathfrak{m}_{<0}$ and $\mathfrak{m}_{\mathrm{null}}$ in the following Tables.

Rem. This is not exhaustive of the topological structure of moduli space of cores. For instance, one can show that $\mathfrak{m}_{<0}$ is a point at infinity of $\mathfrak{m}_t$'s whereas $\mathfrak{m}_{\mathrm{null}}$ is arbitrarily close to $\widetilde{\mathfrak{m}}_0$.

$\mathfrak{m}_t = \bigoplus \mathfrak{m}_t^p$ $t \in [-1, 1]$	$\mathfrak{m}_t^{0(10)}$	$\mathfrak{n}_\sharp$
$t = \pm 1$	$(1+t)e_1^{10} \odot e_1^{10} + (t-1)e_2^{10} \odot e_2^{10}$	$\mathbb{C} \oplus \mathfrak{so}_2(\mathbb{R})$
$t = 0$		$\mathbb{C} \oplus \mathfrak{so}(1, 1)$
$t \neq \pm 1, 0$		$\mathbb{C}$

$\widetilde{\mathfrak{m}}_t = \bigoplus \widetilde{\mathfrak{m}}_t^p$ $t \in \mathbb{R}$	$\widetilde{\mathfrak{m}}_t^{0(10)}$	$\mathfrak{n}_\sharp$
$t \in \mathbb{R}$	$e_1^{10} \odot e_1^{10} - e_2^{10} \odot e_2^{10} + 2(t-1)e_1^{10} \odot e_2^{10}$ $+i((1+t)e_1^{10} \odot e_1^{10} - (1+t)e_2^{10} \odot e_2^{10} - 2e_1^{10} \odot e_2^{10})$	$\mathbb{C}$
$\mathfrak{m}_\pm = \bigoplus \mathfrak{m}_\pm^p$	$\mathfrak{m}_\pm^{0(10)}$	$\mathfrak{n}_\sharp$
$t = \pm 1$	$(1+t)e_1^{10} \odot e_1^{10} + (-1+t)e_2^{10} \odot e_2^{10} - 2ie_1^{10} \odot e_2^{10}$	$\mathbb{C}$

$\mathfrak{m}_{<0} = \bigoplus \mathfrak{m}^p_{<0}$	$\mathfrak{m}^{0(10)}_{<0}$	$\mathfrak{n}_\sharp$
	$e_1^{10} \odot e_2^{10}$	$\mathbb{C} \oplus \mathfrak{so}_2(\mathbb{R})$

$\mathfrak{m}_{\mathrm{null}} = \bigoplus \mathfrak{m}^p_{\mathrm{null}}$	$\mathfrak{m}^{0(10)}_{\mathrm{null}}$	$\mathfrak{n}_\sharp$
	$e_1^{10} \odot e_1^{10} - e_2^{10} \odot e_2^{10} - 2ie_1^{10} \odot e_2^{10}$	$\mathbb{C} \oplus (\mathbb{R} \subsetplus \mathbb{R})$

Models for 7-dimensional 2-nondegenerate CR manifolds

Thm (S.'15) For each of the *seven admissible cores* $\mathfrak{m} = \bigoplus \mathfrak{m}^p$, there is an associated model $\mathfrak{g} = \bigoplus \mathfrak{g}^p$ and a 7-dimensional 2-nondegenerate CR mnfd $M = G/G_o$ which is globally defined.

To state this result, we first need to recall the description of $\mathbb{Z}$-gradings of semisimple real Lie algebras.

Gradings of complex simple Lie algebras

Let $\mathfrak{g} = \mathfrak{h} + \sum_{\alpha\in\Delta} \mathfrak{g}^\alpha$ be a root space decomposition of a complex simple Lie algebra $\mathfrak{g}$ w.r.t. a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ and denote by $\mathfrak{h}_\mathbb{R} \subset \mathfrak{h}$ the real subspace where all roots are real valued. Any *integral weight* $\lambda \in \mathfrak{h}_\mathbb{R}$ (i.e. $\lambda(\alpha) \in \mathbb{Z}$ for any $\alpha \in \Delta$) defines a grading $\mathfrak{g} = \sum_{p\in\mathbb{Z}} \mathfrak{g}_p$ by

$$\mathfrak{g}_0 = \mathfrak{h} + \sum_{\substack{\alpha\in\Delta \\ \lambda(\alpha)=0}} \mathfrak{g}^\alpha \;, \qquad \mathfrak{g}_p = \sum_{\substack{\alpha\in\Delta \\ \lambda(\alpha)=p}} \mathfrak{g}^\alpha \quad \forall p \in \mathbb{Z}\backslash\{0\} \;,$$

and all possible gradings on $\mathfrak{g}$ are of this form, for some choice of $\mathfrak{h}$ and λ. There exists set of positive roots $\Delta^+ \subset \Delta$ s.t. λ is dominant, i.e. $\lambda(\alpha) \geqslant 0$ for all $\alpha \in \Delta^+$. Let Π be the set of positive simple roots, which we identify with nodes of the Dynkin diagram. A grading is s.t. $\mathfrak{g}_{-1}$ generates $\sum_{p<0} \mathfrak{g}_p$ if and only if $\lambda(\alpha) \in \{0, 1\}$ for all $\alpha \in \Pi$. We denote a grading by marking with a cross the nodes of the Dynkin diagram corresponding to simple roots α with $\lambda(\alpha) = 1$. The Lie subalgebra $\mathfrak{g}_0$ is reductive and its Dynkin diagram is obtained from the Dynkin diagram of $\mathfrak{g}$ by removing all crossed nodes.

Gradings of a real simple Lie algebras

Let $\mathfrak{g}$ be real simple Lie algebra. Fix Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$, a maximal abelian subspace $\mathfrak{a} \subset \mathfrak{p}$ and a maximal torus $\mathfrak{t}$ in $Z_{\mathfrak{k}}(\mathfrak{a})$. Then $\mathfrak{h} = \mathfrak{a}^{\mathbb{C}} + \mathfrak{t}^{\mathbb{C}}$ is Cartan subalgebra of $\mathfrak{g}^{\mathbb{C}}$, conjugation of $\mathfrak{g}^{\mathbb{C}}$ w.r.t. $\mathfrak{g}$ induces an isometric involution $\alpha \mapsto \bar{\alpha}$ on $\mathfrak{h}^*_{\mathbb{R}}$, sending roots into roots. We say that a root α is compact if $\bar{\alpha} = -\alpha$ and denote by $\Delta_\bullet$ the set of compact roots. There exists an involutive automorphism $\varepsilon\colon \Pi\backslash\Delta_\bullet \to \Pi\backslash\Delta_\bullet$ s.t.

$$\bar{\alpha} = \varepsilon(\alpha) + \sum_{\beta \in \Pi \cap \Delta_\bullet} b_{\alpha,\beta}\beta \text{ for all } \alpha \in \Pi\backslash\Delta_\bullet \ .$$

The *Satake diagram* of $\mathfrak{g}$ is the Dynkin diagram of $\mathfrak{g}^{\mathbb{C}}$ with additional data:

– nodes in $\Pi \cap \Delta_\bullet$ are painted black;

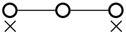
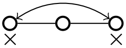
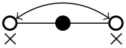
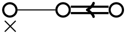
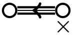
– if $\alpha \in \Pi\backslash\Delta_\bullet$ and $\varepsilon(\alpha) \neq \alpha$ then α and $\varepsilon(\alpha)$ are joined by an arrow.

The grading on $\mathfrak{g}^{\mathbb{C}}$ determined by a $\lambda \in \mathfrak{h}_{\mathbb{R}}$ induces grading on $\mathfrak{g}$ iff

1. $\{\beta \in \Pi \mid \lambda(\beta) = 1\} \cap \Delta_\bullet = \varnothing$;
2. if $\alpha \in \{\beta \in \Pi \mid \lambda(\beta) = 1\}$ then $\varepsilon(\alpha) \in \{\beta \in \Pi \mid \lambda(\beta) = 1\}$.

Models for 7-dimensional 2-nondegenerate CR manifolds

Real graded semisimple Lie algebras $\mathfrak{g} = \bigoplus \mathfrak{g}^p$ such that $\mathfrak{g}_- = \mathfrak{g}_{-2} + \mathfrak{g}_{-1}$ is the *Heisenberg algebra with* $\dim \mathfrak{g}_- = 5$:

$\mathfrak{g}$	Grading	$\mathfrak{g}^{-2}$	$\mathfrak{g}^{-1}$	$\mathfrak{g}^{0}$
$\mathfrak{sl}_4(\mathbb{R})$		$\mathbb{R}$	$\mathbb{R}^2 \oplus (\mathbb{R}^2)^*$	$\mathfrak{gl}_2(\mathbb{R}) \oplus \mathbb{R}$
$\mathfrak{su}(2,2)$		$\mathbb{R}$	$\mathbb{R}^2 \oplus (\mathbb{R}^2)^*$	$\mathfrak{gl}_2(\mathbb{R}) \oplus \mathbb{R}$
$\mathfrak{su}(1,3)$		$\mathbb{R}$	$\mathbb{R}^2 \oplus (\mathbb{R}^2)^*$	$\mathfrak{u}(2) \oplus \mathbb{R}$
$\mathfrak{sp}_6(\mathbb{R})$		$\mathbb{R}$	$\mathbb{R}^4$	$\mathfrak{sp}_4(\mathbb{R}) \oplus \mathbb{R}$
G_2		$\mathbb{R}$	$S^3\mathbb{R}^2$	$\mathfrak{sl}_2(\mathbb{R}) \oplus \mathbb{R}$

Models for 7-dimensional 2-nondegenerate CR manifolds

Assume $J \in \mathfrak{g}^0$ and consider a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ that contains the grading element Z and J. Such Cartan subalgebras always exist as Z and J are commuting semisimple elements. Now $\mathfrak{h} \subset \mathfrak{g}^0$ and it decomposes in $\mathfrak{h} = \mathfrak{h}_\bullet \oplus \mathfrak{h}_\circ$ where

$$\mathfrak{h}_\bullet = \left\{ H \in \mathfrak{h} \mid \text{all eigenvalues of } H \text{ are purely imaginary} \right\} ,$$
$$\mathfrak{h}_\circ = \left\{ H \in \mathfrak{h} \mid \text{all eigenvalues of } H \text{ are real} \right\} .$$

Clearly $J \in \mathfrak{h}_\bullet$ and $Z \in \mathfrak{h}_\circ$. A careful analysis of all the Cartan subalgebras of Lie algebras in previous Table together with existence of J acting with the desired properties on the root spaces leads to:

Models for 7-dimensional 2-nondegenerate CR manifolds

Thm (S.'15) Let $\mathfrak{g}$ be a model of type $\mathfrak{m}$, $\dim(\mathfrak{m}) = 7$, $\mathrm{height}(\mathfrak{m}) = 0$, which is a semisimple real Lie algebra with $J \in \mathfrak{g}^0$. Then:

(i) $\mathfrak{g}$ is isomorphic to $\mathfrak{sl}_4(\mathbb{R})$, $\mathfrak{su}(1,3)$ or $\mathfrak{su}(2,2)$;

(iii) there exists always an associated 7-dimensional and 2-nondegenerate homogeneous CR mnfd $M = G/G_o$ which is globally defined. It is of type $\mathfrak{m}$ where signature $\mathrm{sgn}(J)$ and equivalence class of $\mathfrak{m}^{0(10)}$ are:

$\mathfrak{g}$	$\mathrm{sgn}(J)$	$\mathfrak{m}^{0(10)}$	$\mathfrak{h} = \mathfrak{n}_\sharp$
$\mathfrak{sl}_4(\mathbb{R})$	$(1,1)$	$\mathfrak{m}^{0(10)}_{t=0} : e^{10}_1 \odot e^{10}_1 - e^{10}_2 \odot e^{10}_2$	$\mathbb{C} \oplus \mathfrak{so}(1,1)$
$\mathfrak{su}(1,3)$	$(1,1)$	$\mathfrak{m}_{<0} : e^{10}_1 \odot e^{10}_2$	$\mathbb{C} \oplus \mathfrak{so}_2(\mathbb{R})$
$\mathfrak{su}(2,2)$	$(2,0)$	$\mathfrak{m}^{0(10)}_{t=0} : e^{10}_1 \odot e^{10}_1 + e^{10}_2 \odot e^{10}_2$	$\mathbb{C} \oplus \mathfrak{so}_2(\mathbb{R})$

Models for 7-dimensional 2-nondegenerate CR manifolds

Thm (S.'15) Let $\mathfrak{m}$ be core $\mathfrak{m}_{t=1}$ for $\mathrm{sgn}(J) = (2,0)$ and $\mathfrak{m}_{t=\pm 1}$ or $\mathfrak{m}_{\mathrm{null}}$ for $\mathrm{sgn}(J) = (1,1)$. Then the $\mathbb{Z}$-graded subspace $\mathfrak{g} = \mathfrak{c}^{-2} + \mathfrak{c}^{-1} + \mathfrak{g}^0$ of $\mathfrak{c}$ determined by $\mathfrak{g}^0 = \mathfrak{n}_\sharp \oplus \mathfrak{m}^0$ is a model of type $\mathfrak{m}$:

$\dim(\mathfrak{g})$	$\mathrm{sgn}(J)$	$\mathfrak{m}^{0(10)}$	$\mathfrak{n}_\sharp$
10	$(2,0)$	$e_1^{10} \odot e_1^{10}$	$\mathbb{C} \oplus \mathfrak{so}_2(\mathbb{R})$
10	$(1,1)$	$e_1^{10} \odot e_1^{10}$	$\mathbb{C} \oplus \mathfrak{so}_2(\mathbb{R})$
10	$(1,1)$	$e_2^{10} \odot e_2^{10}$	$\mathbb{C} \oplus \mathfrak{so}_2(\mathbb{R})$
11	$(1,1)$	$e_1^{10} \odot e_1^{10} - e_2^{10} \odot e_2^{10} - 2i e_1^{10} \odot e_2^{10}$	$\mathbb{C} \oplus (\mathbb{R} \inplus \mathbb{R})$

In all cases there exists globally defined homogeneous CR mnfd $M = G/G_o$.

THE UNIVERSAL CR ALGEBRA $(\mathcal{C}, \mathfrak{u})$

$\vdots$

$\mathcal{C}^3$	S^5 $+5,+3,+1,-1,-3,-5$		S^3 $+3,+1,-1,-3$		S^1 $+1,-1$	
$\mathcal{C}^2$		S^4 $+4,+2,0,-2,-4$		S^2 $+2,0,-2$		S^0 0
$\mathcal{C}^1$			S^3 $+3,+1,-1,-3$		S^1 $+1,-1$	
$\mathcal{C}^0$				S^2 $+2,0,-2$		S^0 0
$\mathcal{C}^{-1}$					S^1 $+1,-1$	
$\mathcal{C}^{-2}$						S^0 0

Eigenvalues of J

$\mathfrak{u}$ = complex subalgebra of $\hat{\mathcal{C}} = \mathcal{C} \otimes \mathbb{C}$

$\mathfrak{u} = \bigoplus \mathfrak{u}^p$

$\mathfrak{u}^p$ = sum of all J-eigenspaces of non minimal eigenvalue

$\mathfrak{u} \cap \bar{\mathfrak{u}}$ = complexified isotropy algebra

$\mathfrak{u}^p \cap \bar{\mathfrak{u}}^p$ = sum of all J-eigenspaces of non extremal eigenvalue

$\mathfrak{m} = \bigoplus \mathfrak{m}^p$ core

$\mathfrak{m}^p$ = sum of all J-eigenspaces of extremal eigenvalue

Model for 7-dimensional 3-nondegenerate CR manifolds

Class	$\mathfrak{m}^{-2}$	$\mathfrak{m}^{-1}$	$\mathfrak{m}^{0}$	$\mathfrak{m}^{1}$	$\mathfrak{m}^{p}$ $(p>1)$
(C)	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{C}$	$\mathbb{C}$	0

Denote the basis of $\hat{\mathfrak{c}}^{-1}$ by z and $\bar{z}$ and drop symmetric product symbol $\odot$.

Thm (S.'15) There exists a maximal model $\mathfrak{g}$ of type $\mathfrak{m}$ and it is unique up to isomorphism. It is the *8-dimensional Lie subalgebra* $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}^p$ of $\mathfrak{c}$ with components $\mathfrak{g}^{-2} = \mathfrak{c}^{-2}$, $\mathfrak{g}^{-1} = \mathfrak{c}^{-1}$, $\mathfrak{g}^p = 0$ for $p>1$ and

$$\mathfrak{g}^p = \begin{cases} \mathrm{Re}\langle Z, M, \overline{M}\rangle & \text{for } p=0\,, \\ \mathrm{Re}\langle N, \overline{N}\rangle & \text{for } p=1\,, \end{cases}$$

where $M = z^2 + z\bar{z}$ and $N = z^3 + 2z^2\bar{z} + z\bar{z}^2 - 3iz - 3i\bar{z}$. Moreover there is 7-dimensional 3-nondegenerate CR mnfd $M = G/G_o$, $Lie(G) = \mathfrak{g}$, $Lie(G_o) = \mathbb{R}Z$ which is globally defined.

Overview of recent results in 2-nondegenerate 7-dimensional case

- Absolute parallelisms for 2-nondegenerate CR structures in dimension 5 were constructed by Isaev&Zaitsev'13, Medori&Spiro'14, Merker&Pocchiola'20. The question of existence of Cartan connection valued in $\mathfrak{so}(3,2)$ is subtle and is addressed by Gregorovic'19 via the vanishing of certain invariants;
- Construction of an absolute parallelism for 2-nondegenerate CR structures in dimension 7 was obtained by Porter&Zelenko'17 under some algebraic assumptions, using cores and ingenious modification of the prolongation scheme described in part I of this talk. The algebraic assumptions match the models with simple symmetry described in part III;
- Sykes&Zelenko'21 recently ruled out the existence of locally homogeneous CR mnfds associated to many of the generic 2-nondegenerate cores;
- In a very recent paper, Beloshapka showed that upper bound for dimension of symmetry algebra is 17. For 3-nondegenerate 7-dimensional, it is 20.

Thanks!