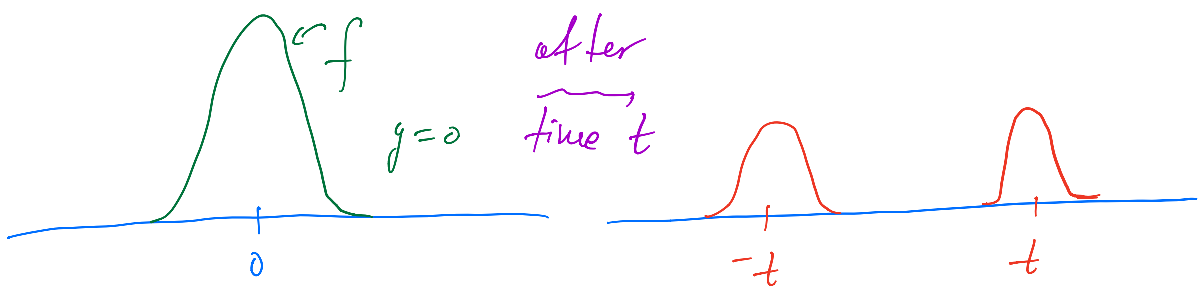
One-dimensional wave equation

$$\begin{cases} u_{tt} - u_{xx} = 0 \\ u\big|_{t=0} = f, \quad \partial_t u\big|_{t=0} = g \end{cases}$$

$\xi = t+x$
$\eta = t-x$
$u_{\xi\eta} = 0$

$$u(t,x) = \frac{1}{2}\big(f(x+t) + f(x-t)\big) + \frac{1}{2}\int_{x-t}^{x+t} g(s)\,ds$$

d'Alembert formula



Consider a nonlinearity

(NLKG$_p$) $$\boxed{\begin{cases} u_{tt} - u_{xx} + u + \underbrace{\lambda |u|^{p-1} u}_{} = 0 \\ u\big|_{t=0} = f, \quad \partial_t u\big|_{t=0} = g \end{cases}}$$

$\lambda = +1$ defocusing

$\lambda = -1$ focusing

Conserved energy is

$$\mathcal{E}(u,u_t) = \int_{\mathbb{R}} \left[\frac{1}{2}\left(u_x^2 + u^2 + u_t^2\right) + \frac{\lambda |u|^{p+1}}{p+1}\right] dx$$

Hamiltonian formulation

$$\partial_t \underbrace{\begin{pmatrix} u \\ u_t \end{pmatrix}}_{U} = \begin{pmatrix} 0 & 1 \\ \partial_x^2 - 1 & 0 \end{pmatrix}\begin{pmatrix} u \\ u_t \end{pmatrix} - \lambda \begin{pmatrix} 0 \\ |u|^{p-1}u \end{pmatrix}$$

$$\dot{U} = J\mathcal{H}U - \lambda F(U)$$

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \mathcal{H} = \begin{pmatrix} -\partial_x^2 + 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Then

$$\frac{d}{dt}\underbrace{\langle \mathcal{H}U, U\rangle}_{\text{free energy}} = 0, \quad \lambda = 0$$

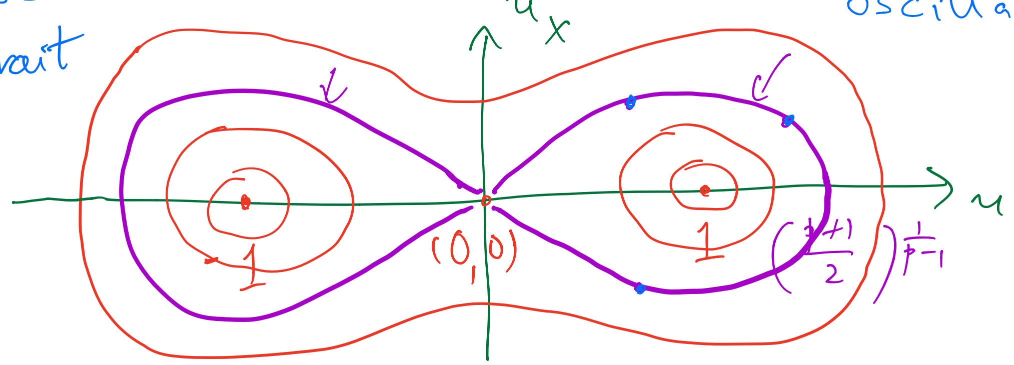
$$\langle \mathcal{H}\dot{U}, U\rangle + \langle \mathcal{H}U, \dot{U}\rangle = \langle \mathcal{H}J\mathcal{H}U, U\rangle + \langle \mathcal{H}U, J\mathcal{H}U\rangle = 0$$

$\boxed{\lambda = -1}$ stationary solutions

$$-u_{xx} + u - |u|^{p-1}u = 0$$

$$\partial_x\Big(\underbrace{-\frac{u_x^2}{2} + \frac{u^2}{2} - \frac{|u|^{p+1}}{p+1}}_{\text{const. energy, nonlinear oscillator}}\Big) = 0$$

phase portrait



two homoclinic orbits $u_\pm$ corresponding to solutions

$$\begin{cases} Q(x+x_0) \\ -Q(x+x_1) \end{cases}, \quad x_0, x_1 \in \mathbb{R}$$

These are the unique stationary solutions of finite energy.

In fact, they decay like $\text{const.}\, e^{\pm|x|}$ as $|x| \to \infty$.

Basic questions about our nonlinear wave equation: Q1: short time existence of smooth solutions for smooth compactly supported data?

Q2 long time existence of smooth solutions?

Q3 if solutions exist for all times $t \geq 0$, what is their asymptotic shape as $t \to \infty$?

- scattering $\|(u, u_t) - (v, v_t)\|_{H^1 \times L^2} \to 0$ as $t \to \infty$ with v a free KG solution of finite energy

$v_{tt} - v_{xx} + v = 0$

- or does u decompose into a superposition of Lorentz transformed ground states $\pm Q$

moving at different speeds, plus radiation?

(Q4) if (u, u_t) has a finite maximal time of existence $t_* > 0$ then we know that $\|(u, u_t)(t)\|_{H^1 \times L^2} \to \infty$ as $t \to t_*^-$. But how does "blow up" happen?

(Q1) elementary, contraction mapping in $C([0,t), (H^1 \times L^2)(\mathbb{R}))$ using Sobolev embedding $H^1(\mathbb{R}) \hookrightarrow L^q(\mathbb{R})$, $2 \le q < \infty$ gives local well posedness.

(Q2) defocusing: global existence via conserved energy

focusing: no, we can have that $\|(u, u_t)\|_{H^1 \times L^2} \to \infty$ in finite time $t \to t_* > 0$.

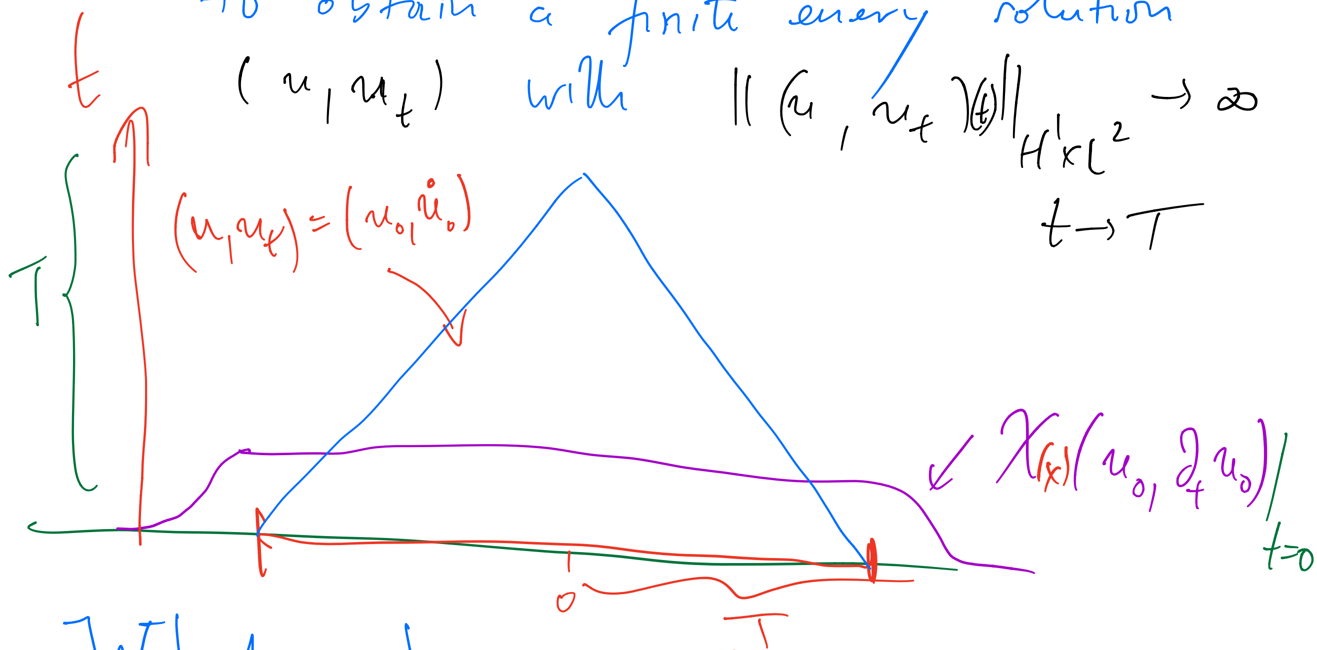
Look for $u = u_0(t)$, so

$$\ddot{u}_0 + u_0 - |u_0|^{p-1} u_0 = 0$$

$$u_0(t) = c_\alpha (T - t)^{-\frac{2}{p-1}} (1 + o(1))$$

$t \to T > 0$

Use finite propagation speed to obtain a finite energy solution (u, u_t) with $\|(u, u_t)(t)\|_{H^1 \times L^2} \to \infty$, $t \to T$



What is known: • $\lambda > 0$ defocusing

global existence and scattering for $p > 5$.

Cannot hold for $p = 3$ due to long-range phase corrections.

Small data: DELORT (2000) ....
HAYASHI - NAUMKIN

$\boxed{\text{Open problem } 2 \le p \le 5 \text{ for large data}}$

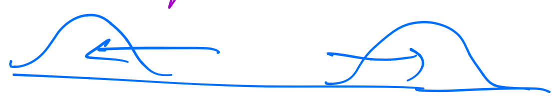
• $\lambda < 0$ focusing

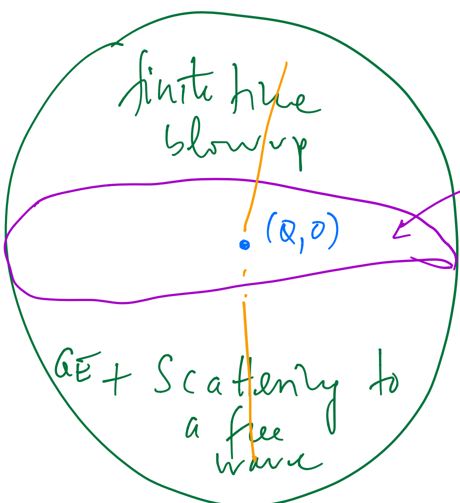
limited understanding, much remains to be done!

for *even* data and energies

$$\mathcal{E}(f,g) < \mathcal{E}(Q,0) + \delta$$

we have a descrip of the dynamics (Payne-Sattinger ~1975, Krieger-Nakanishi-S, 2013)





co-dimension 1 center-stable manifold at $(Q,0)$ in $H^1 \times L^2$

GE + Scattering to $(Q,0)$.

δ-ball in $H^1 \times L^2(\mathbb{R})$

Essential here that linearized operator

$$L = -\partial_x^2 + 1 - pQ^{p-1}$$

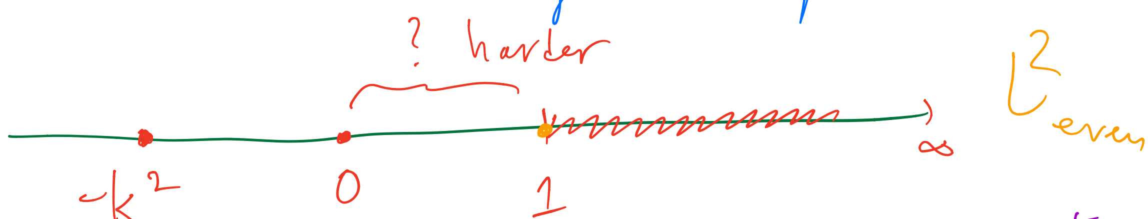
has a <u>unique</u> negative eigenvalue

note:

$$\langle LQ, Q\rangle = \langle (\overbrace{-Q''+Q-Q^p}^{=0}) - (p-1)Q^p, Q\rangle$$

$$= -(p-1)\int_{\mathbb{R}} Q^{p+1}(x)\,dx < 0$$

So L has negative spectrum.



$L(Q')=0$ translational symmetry

Use min-max to show $\sigma(L)\cap(-\infty,0) = \{-k^2\}$ simple eigenvalue.

So we need to exhibit some $\tilde{Q}$ such that $f \perp \tilde{Q} \Rightarrow \langle Lf, f\rangle \geq 0$

To find $\tilde{Q}$ requires a variational characterization of the ground state Q.

In higher dimensions, we are lead to

a variational characterization even for the problem of EXISTENCE of the ground state.

In general dimensions how to solve

$$(\mathcal{E}LL) \qquad -\Delta \varphi + \varphi - |\varphi|^{p-1}\varphi = 0 .$$

Define the stationary energy

$$J(\varphi) = \int_{\mathbb{R}^d} \left[\frac{1}{2}\left(|\nabla \varphi|^2 + \varphi^2 \right) - \frac{|\varphi|^{p+1}}{p+1} \right] dx$$

continuous functional on $H^1(\mathbb{R}^d)$

provided $p+1 \leq 2^* = \dfrac{2d}{d-2}$, $d \geq 3$

$$\dot{H}^1(\mathbb{R}^d) \hookrightarrow L^{2^*}(\mathbb{R}^d)$$

In fact, a solution to $(\mathcal{E}LL)$ can only exist if $p < 2^* - 1$ (integrate by parts against $x \cdot \nabla$)

Cannot minimize $J[\varphi]$ directly but note

$$K(\varphi) = \int_{\mathbb{R}^d} \left(-\Delta\varphi + \varphi - |\varphi|^{p-1}\varphi\right)\varphi \, dx$$

$$= \int_{\mathbb{R}^d} |\nabla\varphi|^2 + \varphi^2 - |\varphi|^{p+1} \, dx$$

$$= 0$$

Nehari functional

Lemma:

$$\inf\{J(\varphi) \mid K(\varphi) = 0, \ \varphi \in H^1_{rad}(\mathbb{R}^d)\setminus\{0\}\}$$

$$= J(Q) \quad \text{and } \pm Q \text{ unique}$$

minimizers

Point is the Euler-Lagrange equation

$\varphi_* \geq 0$ minimizer

$$J'(\varphi_*) + \mu K'(\varphi_*) = 0$$

$$\underbrace{\langle J'(\varphi_*), \varphi_*\rangle}_{K(\varphi_*)=0} + \mu \underbrace{\langle K'(\varphi_*), \varphi_*\rangle}_{-(p-1)\|\varphi_*\|_{p+1}^{p+1}} = 0$$

$\|\varphi\|_{H^1}$

$= \|\varphi\|_{H^1}$ So $\boxed{\mu = 0}$ and

$$\boxed{\begin{array}{c} J'(\varphi_*) = 0 \\ \varphi_* \geq 0 \\ \varphi_* \in H^1_{rad}(\mathbb{R}^d) \end{array}} \xrightarrow{\text{UNIQUENESS}} \boxed{\varphi_* = Q}$$

$\varphi_* > 0$, φ_* smooth by elliptic regularity and maximum principle

HOW TO PROVE UNIQUENESS OF Q?

Before discussing uniqueness, we turn to more applications of uniqueness

i) $$-\Delta Q + Q - Q^3 = 0 \quad \text{in } \mathbb{R}^d, \ 1 \leq d \leq 3$$

$Q > 0$ ground state, radial

$$L = -\Delta + 1 - 3Q^2$$

$$J[Q+v] = \int \frac{1}{2}|\nabla Q + \nabla v|^2 + \frac{1}{2}(Q+v)^2 - \frac{1}{4}(Q+v)^4$$

$$= J[Q] + \frac{1}{2}\langle Lv, v\rangle + O(\|v\|_{H^1}^3)$$

$$K[Q+v] = -2\langle Q^3, v\rangle + \langle (L-3Q^2)v, v\rangle + O(\|v\|^3_{H^1})$$

CLAIM: $v \perp Q^3 \Rightarrow \langle Lv, v\rangle \geq 0$

By min-max therefore, L has a unique (and simple) negative eigenvalue

Assume FALSE. Then let $f \perp Q^3$, $f \in H^1(\mathbb{R}^d)$, $\langle Lf, f\rangle = -1$.

Define $v = \varepsilon Q + \delta f$. Then

$$K_0(Q+v) = -2\varepsilon\|Q\|_4^4 - \delta^2(1+3\langle Q^2 f, f\rangle) + O(\varepsilon^2) + O(\varepsilon\delta) + O(\delta^3)$$

$= 0$ WITH $\varepsilon \simeq \delta^2$

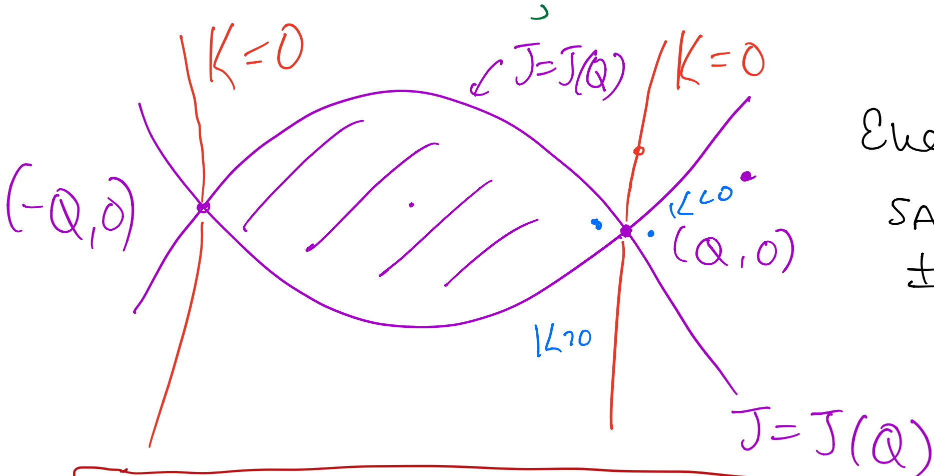
BUT $J(Q+v) = J(Q) + \frac{1}{2}\langle Lv, v\rangle + O(\|v\|^3_{H^1})$

$$= J(Q) - \frac{\delta^2}{2} + O(\delta^3)$$

$< J(Q)$

CONTRADICTION TO VARIATIONAL CHARACTERIZATION OF Q.

ii) GRAPHIC DEPICTION OF FUNCTIONALS J and K, HINGES ON UNIQUENESS



Energy is a SADDLE near $\pm Q$.

DYNAMICAL THEORY OF PAYNE–SATTINGER '75: $\mathcal{E}(u|_0, \partial_t u|_0) < \mathcal{E}(Q,0) = J(Q)$

$\Rightarrow$ GE (~75) + SCATTERING (~2010, IBRAHIM, MASMOUDI, NAKANISHI)

$\mathcal{E} > J(Q) \Rightarrow$ FTB in $t>0$ and $t<0$.

A Closer look at the uniqueness problem in $\mathbb{R}^3$

$$\varphi(x) = y(r), \qquad r = |x|$$

$$-\Delta\varphi + \varphi - \varphi^3 = 0, \quad \varphi \in H^1(\mathbb{R}^3)$$

$$(*) \begin{cases} -y''(r) - \frac{2}{r}y'(r) + y(r) - y^3(r) = 0 \\ y(0) = b > 0, \quad y'(0) = 0 \end{cases}$$

$r = t =$ time

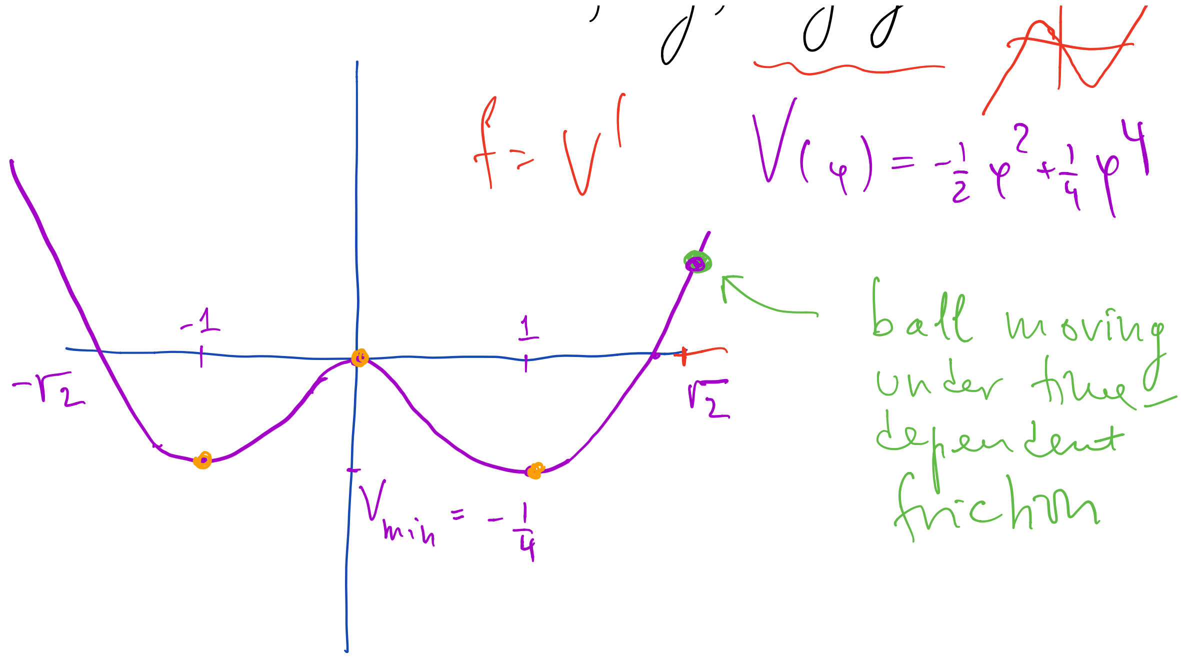
Then $(*)$ is a damped oscillator

Energy

$$E_b(t) = \frac{1}{2}\dot{y}^2(t) - \frac{1}{2}y^2(t) + \frac{1}{4}y^4(t)$$

$$= \frac{1}{2}\dot{y}^2(t) + V(y(t)) > 0$$

$$\dot{E}_b(t) = \dot{y}(\ddot{y} + \underbrace{V'(y)}_{f(u) = -u + u^3}) = -\frac{2\dot{y}^2(t)}{t} \leq 0$$



$E_b(t)$ decreasing, strictly unless $\varphi = \text{const} = 0$ or ± 1,

$$\lim_{t \to \infty} E_b(t) = E_b(\infty) \geq V_{min} = -\frac{1}{4}$$

$$E_b(0) - E_b(\infty) = 2 \int_0^\infty \frac{\dot{\varphi}(t)^2}{t} dt < \infty$$

$\boxed{\text{Corollary: } \exists \text{ sequence } t_j \nearrow \infty \text{ so that } \lim_{j \to \infty} \left(y_b(t_j), \dot{y}_b(t_j) \right) = \begin{cases} (0,0) \\ (\pm 1, 0) \end{cases}}$

Proof: $\displaystyle\sum_{n=1}^\infty \frac{1}{n} \int_{n-1}^n \dot{y}_b(t)^2 dt < \infty$

So $\liminf\limits_{n\to\infty} \int_{n-1}^{n} \dot{y}_b(t)^2\,dt = 0$

$$\sup_{t\geq 0}\left(|y_b(t)| + |\dot{y}_b(t)|\right) < \infty$$

from bounded energy.

From ODE $\sup\limits_{t\geq 1} |\ddot{y}_b(t)| < \infty$

$$\liminf_{n\to\infty} \max_{n-1\leq t\leq n} |\dot{y}_b(t)| = 0$$

$$|\dot{y}_b(t_j)| + |\ddot{y}_b(t_j)| \to 0$$

$$\Rightarrow \qquad |f(y_b(t_j))| \to 0$$

so $\quad y_b(t_j) \to 0 \text{ or } \pm 1 \quad \blacksquare$

So the ω-limit set of every orbit $(y_b(t), \dot{y}_b(t))$ contains at least one of $(0,0)$, $(-1,0)$, $(1,0)$

THEOREM: ω-limit set of every orbit $(y_b, \dot{y}_b)(t)$ equals one of these stationary solutions

Proof: Easy for the wells: Haraux Jendoubi

$E_b(t)$ decreasing and

$$E_b(t_j) \to V_{min} = -\frac{1}{4}$$

so

$$E_b(\infty) = -\frac{1}{4},$$

and the entire orbit eventually falls into the wells.

Harder for the saddle $(0,0)$:

We know $(y_b(t_j), \dot{y}_b(t_j)) \to (0,0)$

so

$$E_b(t) > 0 \quad \forall\ t \geq 0$$

(assuming $y_b \not\equiv 0$) and $E_b(\infty) = 0$.

If the claim fails, then

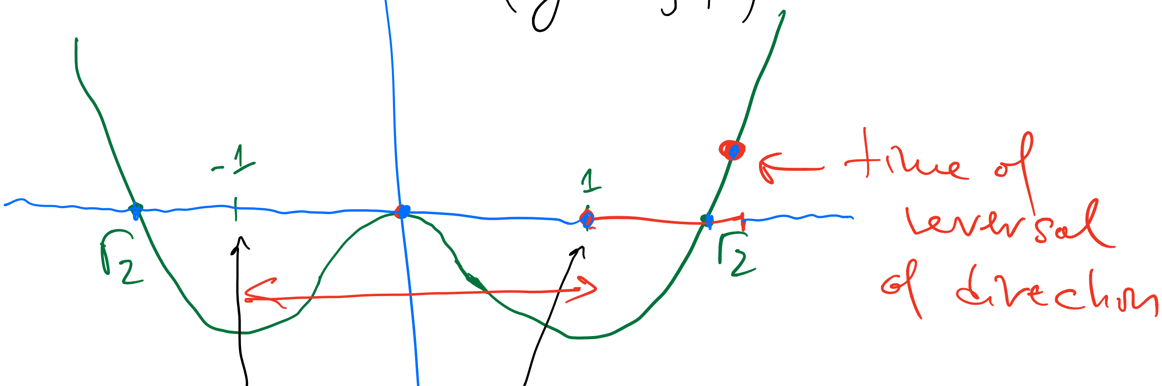
$\exists \quad \tau_j \nearrow \infty \quad$ with

$\dot{y}(\tau_j) = 0$ points of reversal of direction.

By energy, must have

$$|y(\tau_j)| > \sqrt{2}$$

$$E_b(\tau_s) = V(y(\tau_j)) > 0$$



$$\tau_j < t_1 < t_2 < \tau_{j+1}$$

$$y_b(t_1) = -1, \quad y_b(t_2) = +1$$

$$t_2 - t_1 = \int_{t_1}^{t_2} 1\, dt = \int_{t_1}^{t_2} \frac{dy}{\dot{y}(t)}$$

$$= \int_{-1}^{1} \frac{dy}{\sqrt{2(E(y) - V(y))}}$$

$$\leq 2 \int_0^1 \frac{dy}{\sqrt{2E(t_2) + y^2/2}} \lesssim -\log E(t_2)$$

Moreover

$$E(t_2) = E(\tau_{j+1}) + 2\int_{t_2}^{\tau_{j+1}} \frac{\dot{y}(t)^2}{t}\, dt$$

$$\geq \frac{2}{\tau_{j+1}} \int_1^{\sqrt{2}} \sqrt{2(E(y) - V(y))}\, dy$$

$$\gtrsim \frac{1}{\tau_{j+1}}$$

and

$$\tau_{j+1} - \tau_j \lesssim \log \tau_{j+1}$$

$$\Longrightarrow \quad \tau_j \lesssim j \log j$$

BUT

$$\infty > \int_0^\infty \frac{\dot{y}(t)^2}{t}\, dt \gtrsim \sum_{l=l_0}^{\infty} \frac{1}{\tau_{l+1}} \underbrace{\int_{\tau_l}^{\tau_{l+1}} \dot{y}^2(t)\, dt}_{\gtrsim 1}$$

$$\gtrsim \sum_j \frac{1}{j \log j} \qquad \blacksquare$$

Towards uniqueness of bound states:

i) $b \in (0, \sqrt{2}] \Rightarrow E_b(0) \leq 0$

and $E_b(t) < 0$ if $t > 0$

so $(y_b, \dot{y}_b)(t) \to (1, 0)$

ii) define

$\mathcal{B}_{\pm} = \{ b > 0 \mid (y_b, \dot{y}_b)(t) \to (\pm 1, 0)$ as $t \to \infty \}$

Then $\mathcal{B}_{\pm}$ are OPEN SETS

$$(0, \infty) = \mathcal{B}_+ \cup \mathcal{B}_- \cup \underbrace{\mathcal{B}_0}_{\text{bound states}}$$

and $\mathcal{B}_0 \neq \emptyset$

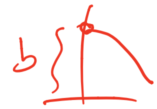
iii) Existence of the ground state implies that

Open sets

$$S_+ := \{\, b>0 \mid \inf_{t\geq 0} y_b(t) > 0 \,\} \neq (0,\infty)$$

$$S_0 := \{\, b>0 \mid \inf_{t\geq 0} y_b(t) = 0 \,\}$$

$$S_- = \{\, b>0 \mid \inf y_b(t) < 0 \,\}$$



$$b_0 := \sup\{\, b>0 \mid (0,b) \subset S_+ \,\}.$$

$$\boxed{b_0 \in S_0}$$

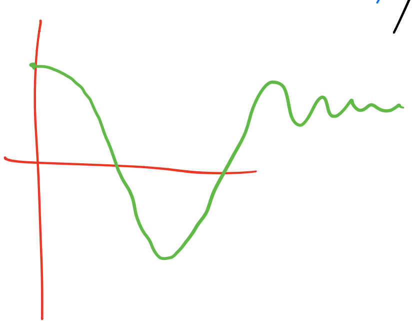
UNIQUENESS MEANS $(b_0,\infty) \subset S_-$

iv) Equation of variation

$$\delta_b = \frac{\partial y_b}{\partial b}$$ solves

$$(\mathrm{Var}) \quad \begin{cases} \ddot{\delta}_b(t) + \dfrac{2}{t}\dot{\delta}_b(t) - \delta_b(t) + 2 y_b^2 \delta_b(t) = 0 \\ \delta_b(0) = 1, \quad \dot{\delta}_b(0) = 0 \end{cases}$$

note: • (a) $y_b(t) \longrightarrow \pm 1$ implies



$$\ddot{\delta}_b + \delta_b \approx 0$$

harmonic oscillator

$\sin t, \ \cos t$

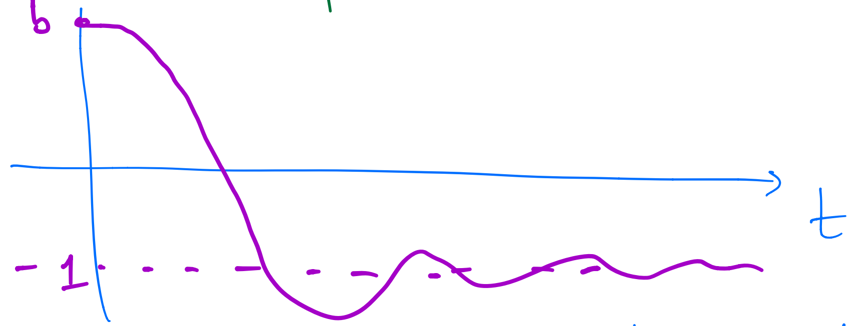
• (b) $y_b(t) \to 0$ implies

$$\ddot{\delta}_b(t) + \frac{2}{t}\dot{\delta}_b(t) - \delta_b(t) \approx 0$$

with solutions $\sim \pm \frac{e^{t}}{t}$ / $\frac{e^{-t}}{t}$

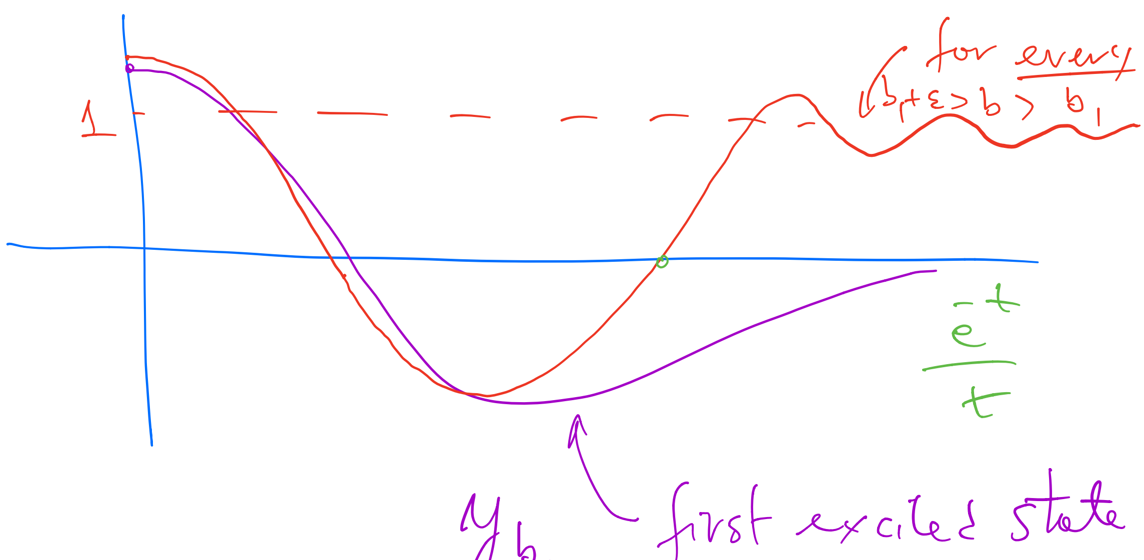
These two types of behaviors are natural:

(a)



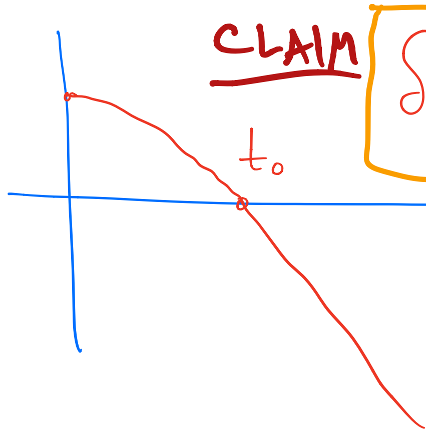
as we vary b we will see the same oscillations

(b)



here expect $\delta_{b_1}(t)$ positive and large

v) Key step in analytic uniqueness proof of the ground state y_{b_0}:



CLAIM $\boxed{\delta_{b_0}(t) \text{ has a } \underline{\text{UNIQUE zero}} \text{ and } \delta_{b_0}(t) \to -\infty \text{ as } t \to +\infty.}$

Proof hinges on two Wronskians

1st $W(t) = t^2 \left(y_0 \dot{\delta}_0 - \dot{y}_0 \delta_0 \right)(t)$

where $y_0 = y_{b_0}, \quad \delta_0 = \delta_{b_0}$.

Then $\dot{W}(t) = -2 t^2 y_0^2 \, \delta_0(t)$.

If $\delta_0(t) > 0 \quad \forall \; t \geq 0$,

then

$$t^{-2} W(t) = -\frac{2}{t^2} \int_0^t s^2 y_0^2(s) \, \delta_0(s) \, ds < 0$$

$$\left(\frac{\delta_0}{y_0} \right)^{\cdot}(t) < 0 \Rightarrow \frac{\delta_0(t)}{y_0(t)} < \frac{1}{y_0(0)}$$

$$0 < \delta_0(t) < \frac{y_0(t)}{b_0} \longrightarrow 0 \quad \text{as } t \to \infty$$

But then $\delta_0(t) \sim \frac{c}{t} e^{-t}$ as $t \to \infty$, $c > 0$

and we also have

$$y_0(t) \sim c' \frac{e^{-t}}{t} \quad \text{as } t \to \infty$$

So $W(t) \longrightarrow 0$ as $t \to \infty$

CONTRADICTION

By topological argument exclude a second zero.

2nd Next, we need to prove that

$$\delta_0(t) \longrightarrow -\infty \quad \text{as } t \to +\infty.$$

Define McLeod's function

$$\boxed{v(t) = y_0(t) + \lambda t \dot{y}_0(t)}$$

$\boxed{v(t_0) = 0}$ $\qquad \lambda = -\frac{y_0(t_0)}{t_0 \dot{y}_0(t_0)} > 0$

$(\delta_0(t_0) = 0)$

Consider

$$\widetilde{W}(t) = t^2(\delta_0 \dot{v} - \dot{\delta}_0 v)(t)$$

Then $\widetilde{W}(0)=0$, $\widetilde{W}(t_0)=0$

If $\delta_0(t) \not\to -\infty$ then $\delta_0(t) \to 0$. This would again lead to a contradiction, see McLeod 1993, TAMS

vi) Existence of excited states has been known since Ryder's 1967 paper. For every $n \geq 0$ he shows the existence of a bound state $y_{b_n}(t)$ with exactly n zero crossings.

Coffman 1972 proved uniqueness of y_{b_0} for cubic non linearity.

1983 Peletier, Serrin
McLeod, Serrin
Zhang
Kwong
McLeod
1993 Clemons, Jones

prove uniqueness for more general nonlinearities

nonuniqueness construction by Peletier, Serrin

No analytic proof of uniqueness of ALL excited states known (y_b with one or more crossings and $y_b(t) \to 0$ as $t \to \infty$). Remains an open problem

2012 AMS GTM book by Hastings, McLeod lists this for the cubic nonlinearity as one of three open problems in nonlinear ODEs.

Nonuniqueness examples introduce "plateaus of friction" slowing y down

THEOREM (COHEN, LI, S., 2021) The first 20 excited states (radial) for

$$-\Delta\varphi + \varphi - \underbrace{\varphi^3}_{|\varphi|^{p-1}\varphi \quad p=3.3} = 0, \quad \mathbb{R}^3$$

are unique.

Main ingredients, in addition to those already mentioned above:

- Comparison lemma
- One-pass lemma
- rigorous computer assisted verification of hypotheses of these analytical lemmas via VNODE-LP, based on precise interval arithmetic.
- VNODE-LP is machine verified C++ code, and has been used in proofs before.

- The proof hinges on the fact that the uniqueness problem for the first few bound states can be formulated in terms of inequalities. These finitely many bounds are effective and computable, by interval arithmetic.
- VNODE-LP works well for u^3 but not for more general $|u|^{p-1}u$. So nonlinearity should be smooth which guarantees smooth solutions.
- Purely analytical proof? Nobody has found one in 50 years.
- Investigate consequences of uniqueness.

Crossing lemma: y_{b_*} a bound state

$y_{b_*}(t) \to 0$ from the right,

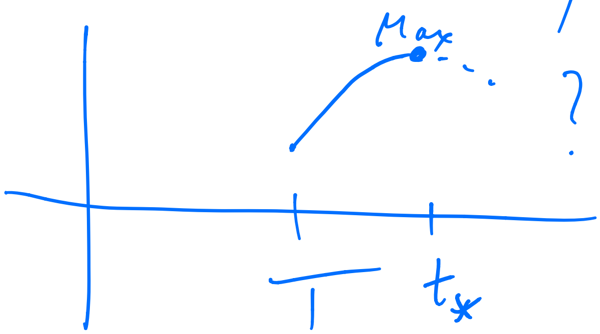
$0 < y_{b_*}(t) \le \frac{1}{\sqrt{3}}$ for $t \ge T$.

If $$0 < y_b(T) < y_{b_*}(T)$$
$$\dot{y}_b(T) < \dot{y}_{b_*}(T)$$

then $y_b(t)$ has a zero crossing after time T.

Proof uses that $f(y) = -y + y^3$ strictly decreasing on $(0, \sqrt{3})$

and $s(t) := y_{b*}(t) - y_b(t)$,



if $y_b(t) > 0 \ \forall t \ge T$

then $S \to 0$ or

$S \to -1$

One pass lemma:

Suppose $y_b(t)$ a solution with $0 < y_b(T) < \frac{1}{2}$

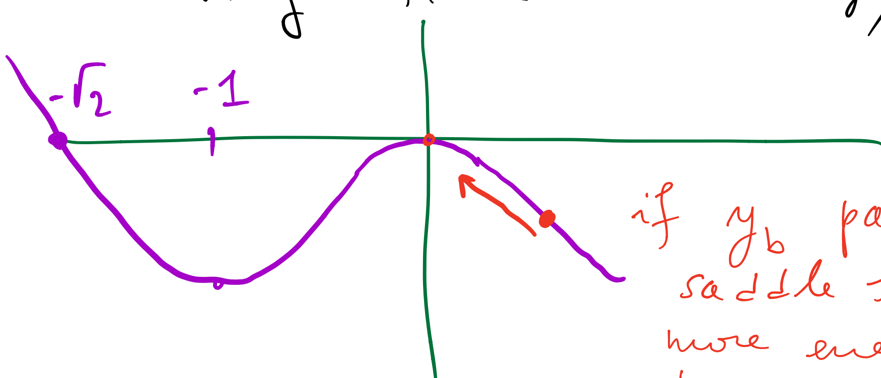
$$\dot{y}_b(T) < 0$$

$$0 < E(T) < \frac{1}{4}$$

$$\to E(T)\left(T - 2\log E(T) + \frac{3}{2}\right) < \frac{3}{8}.$$

If $y_b(t)$ has a zero crossing after time T, then it cannot have another, i.e., it falls into -1.

Proof based on similar considerations involving time and energy as before.



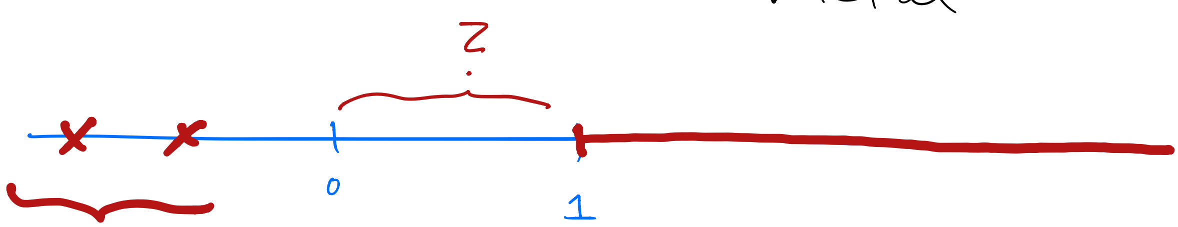
if y_b passes over the saddle then we need more energy than we have.

Possible next steps

$$L_k = -\Delta + 1 - 3\varphi_k^2$$

φ_k = unique bound state with k zeros, $0 \le k \le 20$

$\mathrm{Spec}(L_k)$ in $L^2_{radial}(\mathbb{R}^3)$.



$k+1$ negative eigenvalues

Dynamics of NLKG

$$u_{tt} - \Delta u + u - u^3 = 0$$

$$(u|_0, \partial_t u|_0) = (\varphi_1 + \varepsilon_0, \varepsilon_1)$$

$$\|\varepsilon_0\|_{H^1} + \|\varepsilon_1\|_{L^2} \ll 1$$

Not clear what to expect.

- weakly heteroclinic orbit connecting $\varphi_1 \rightsquigarrow \varphi_0$?
- generic behavior ?
- # of negative eigenvalues of the linearized operator determines dimension of (un)stable manifold