Stopping rules for accelerated gradient methods with additive noise in gradient and influence of relative inexactness.

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Basic assumptions and problem description

We consider convex optimization problem on a convex (not necessarily bounded) set $Q \subset \mathbb{R}^n$

$$\min_{x \in Q} f(x)$$

We will use such designations:

$$(\forall x \in Q)\ \|\nabla f(x) - \tilde{\nabla} f(x)\|_2 \leqslant \delta,$$
$$R = \|x_{start} - x^*\|_2,$$
$$(\forall x, y \in Q)\ \|\nabla f(x) - \nabla f(y)\|_2 \leqslant L\|x - y\|_2$$

We will also consider the strongly convex case:

$$(\forall x, y \in Q)\ f(x) + \langle \tilde{\nabla} f(x), y - x \rangle + \frac{\mu}{2}\|x - y\|_2^2 \leqslant f(y)$$

Also will study the effect of relative imprecision:

$$\|\nabla f(x) - \tilde{\nabla} f(x)\|_2 \leqslant \alpha \|\nabla f(x)\|_2$$

Algorithm description

Algorithm 1 $STM(L, \mu, \tau, x_{start}), \quad Q \subseteq \mathbb{R}^n$

Input: Starting point x_{start}, number of steps N
Set $\tilde{x}_0 = x_{start}$,
Set $A_0 = \frac{1}{L}$,
Set $\alpha_0 = \frac{1}{L}$,
$\psi_0(x) = \frac{1}{2}\|x - \tilde{x}_0\|_2^2 + \alpha_0 \left(f(\tilde{x}_0) + \langle \tilde{\nabla} f(\tilde{x}_0), x - \tilde{x}_0 \rangle + \frac{\mu}{2}\|x - \tilde{x}_0\|_2^2 \right)$,
Set $z_0 = \mathrm{argmin}_{y \in Q}\, \psi_0(y)$,
Set $x_0 = z_0$.
for $k = 1 \ldots N$ **do**

$\alpha_k = \frac{1+\mu_\tau A_{k-1}}{2L} + \sqrt{\frac{1+\mu_\tau A_{k-1}}{4L^2} + \frac{A_{k-1}}{1+\mu_\tau A_{k-1}}}$,

$A_k = A_{k-1} + \alpha_k$,

$\tilde{x}_k = \frac{A_{k-1}x_{k-1} + \alpha_k z_{k-1}}{A_k}$,

$\psi_k(x) = \psi_{k-1}(x) + \alpha_k \left((f(\tilde{x}_k) + \langle \tilde{\nabla} f(\tilde{x}_k), x - \tilde{x}_k \rangle + \frac{\mu}{2}\|x - \tilde{x}_k\|_2^2 \right)$,

$z_k = \mathrm{argmin}_{y \in Q}\, \psi_k(y)$,

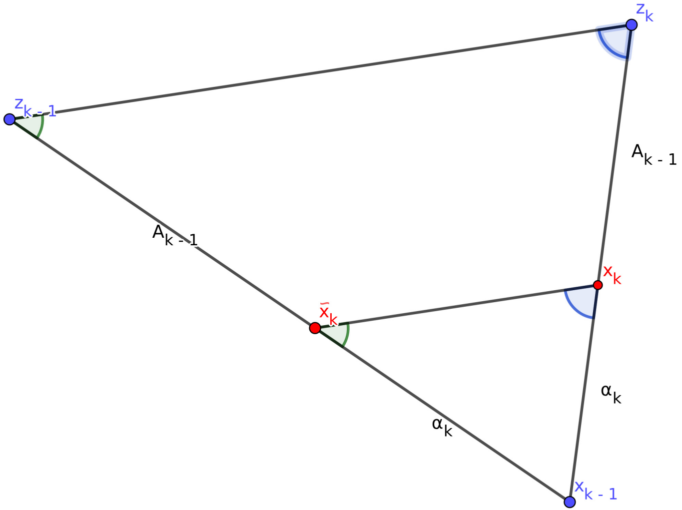
$x_k = \frac{A_{k-1}x_{k-1} + \alpha_k z_k}{A_k}$.

end for
Output: x_N.

Figure 5 describes the position of the vertices. On the sides, not their lengths are marked, but the relationships in the corresponding sides in the similarity of triangles. In the case $Q = \mathbb{R}^n$, we can simplify the step of the algorithm by replacing it with:

$$z_k = z_{k-1} - \frac{\alpha_k}{1 + A_k \mu_\tau} \left(\tilde{\nabla} f(\tilde{x}_k) + \mu_\tau (z_{k-1} - \tilde{x}_k) \right).$$

Alogrithm description



Convergence comparison

We will use the following functions for experiments.

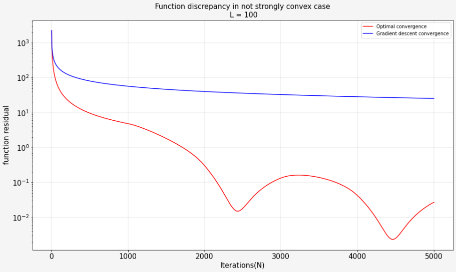
$$n = 2m + 1,$$
$$\left(\forall k \leqslant \frac{1}{2}(n-1)\right)$$
$$f(x_k) - f(x^*) \geqslant \frac{3}{32}\frac{LR^2}{(k+1)^2}$$
$$f(x) = \frac{L}{8}\left(x_1^2 + \sum_{k=1}^{2m}(x_k - x_{k+1})^2 + x_n^2\right) - \frac{L}{4}x_1,$$

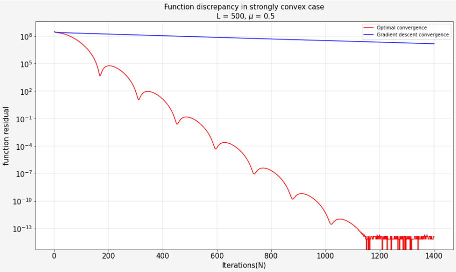
And for strongly convex case:

$$f(x) = \frac{\mu\left(\frac{L}{\mu} - 1\right)}{8}\left(x_1^2 + \sum_{k=1}^{\infty}(x_k - x_{k+1})^2 - 2x_1\right) + \frac{\mu}{2}\|x\|_2^2,$$
$$\chi = \frac{L}{\mu},$$
$$f(x_N) - f(x^*) \geqslant \frac{\mu}{2}\left(\frac{\sqrt{\chi} - 1}{\sqrt{\chi} + 1}\right)^{2N},\ N \geqslant 1,$$

Convergence comparison

Let us compare the convergence of gradient descent and the accelerated STM method.





$$\mu = 0 \Rightarrow f(x_k) - f(x^*) \leqslant \frac{4LR^2}{N^2},$$

$$\mu > 0 \Rightarrow f(x_k) - f(x^*) \leqslant LR^2 \exp\left(-\frac{1}{2}\sqrt{\frac{\mu}{L}}N\right)$$

Convergence with additive noise

Remind the conception:

$$\|\nabla f(x) - \tilde{\nabla} f(x)\|_2 \leqslant \delta$$

Also we will use:

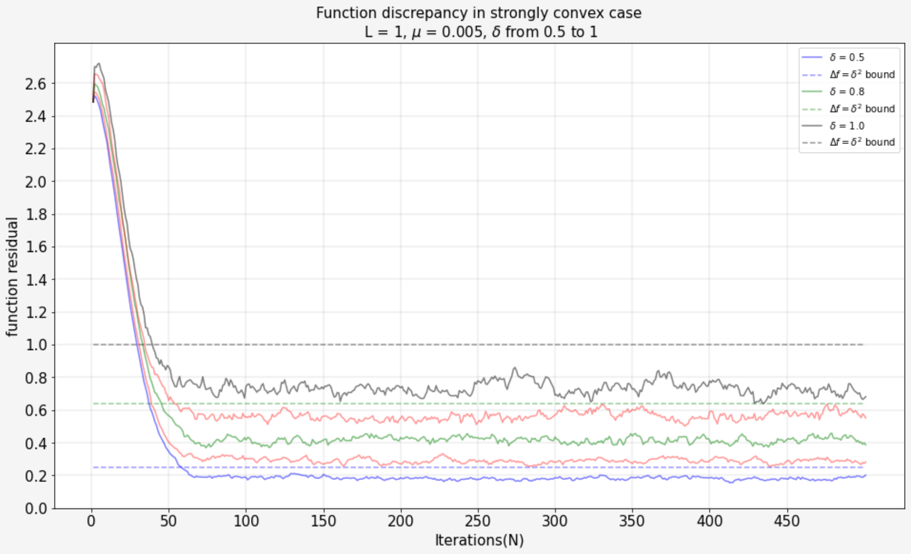
$$\tilde{R}_N = \max_{k \leqslant N}\{\|x_k - x^*\|_2, \|z_k - x^*\|_2, \|\tilde{x}_k - x^*\|_2\}$$

In the presence of the additive noise STM convergence:

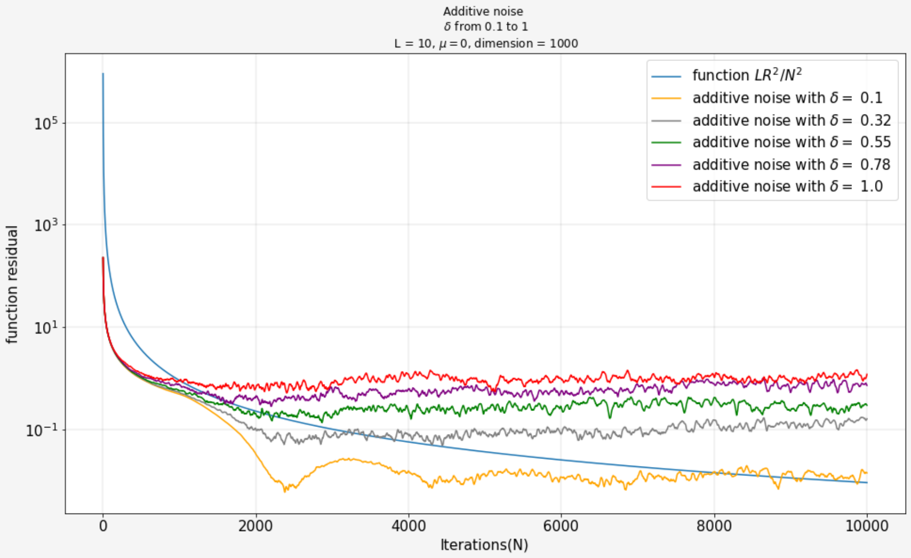
$$f(x_N) - f(x^*) \leqslant \frac{4LR^2}{N^2} + N\frac{\delta^2}{2L} + 3\delta\tilde{R}_N,$$

$$f(x_N) - f(x^*) \leqslant LR^2 \exp\left(-\frac{1}{2}\sqrt{\frac{\mu}{2L}}N\right) + \left(1 + \sqrt{\frac{\mu}{2L}}\right)\left(\frac{\delta^2}{2L} + \frac{\delta}{\mu}\right)$$

Plots of convergence



Plots of convergence



Stopping rule for STM

Theorem

Assume that we know we value of $f(x^*)$ and such bound, R_*, that $\|x^*\| \leqslant R_*$. Using stopping rule $\forall \zeta > 0$

$$f(x_N) - f(x^*) \leqslant \frac{\delta^2}{2L} N + 3R_*\delta + \zeta$$

And it is guaranteed, that the criteria will reached in

$$N_{stop} = O\left(\frac{\sqrt{LR^2}}{\zeta}\right).$$

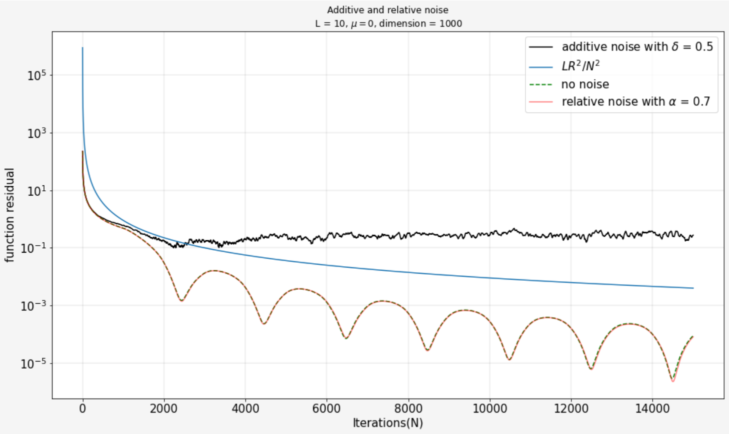
Choosing $\zeta \sim \varepsilon, \delta \sim \frac{\varepsilon}{R_*}$ we get estimation:

$$f(x_{N_{stop}}) - f(x^*) \leqslant \varepsilon,$$

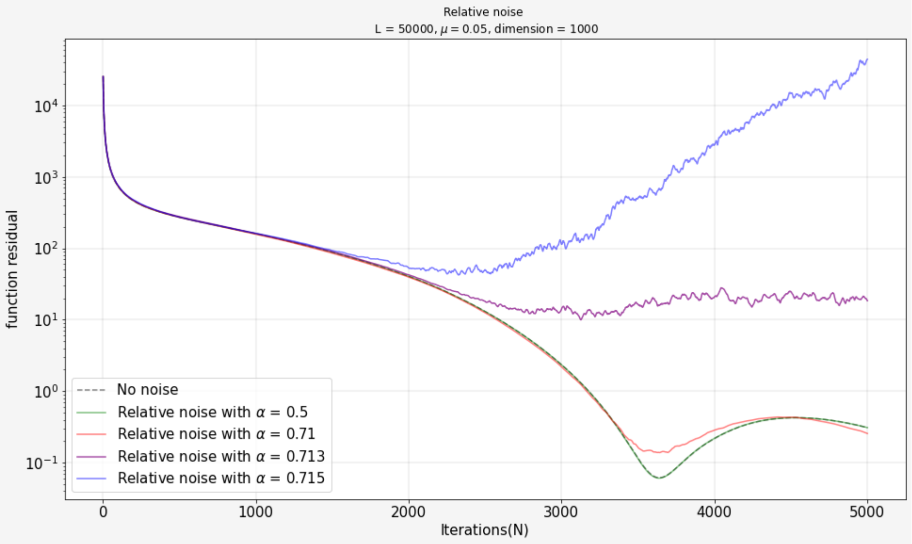
$$N_{stop} = O\left(\frac{\sqrt{LR^2}}{\varepsilon}\right)$$

Remind the conception:

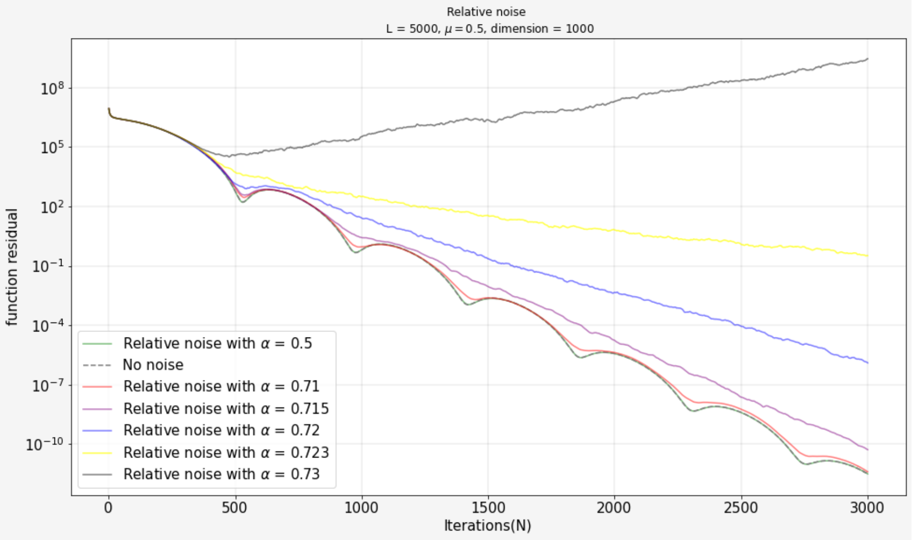
$$\|\nabla f(x) - \tilde{\nabla} f(x)\|_2 \leqslant \alpha \|\nabla f(x)\|_2$$



Threshold $\overline{\alpha}$ value



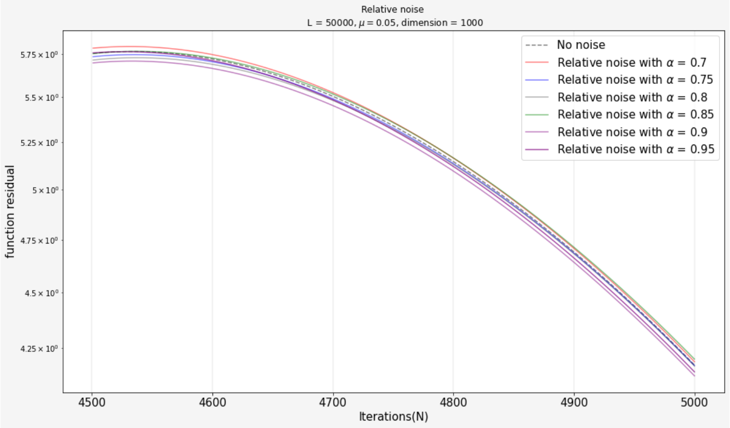
Threshold $\overline{\alpha}$ value



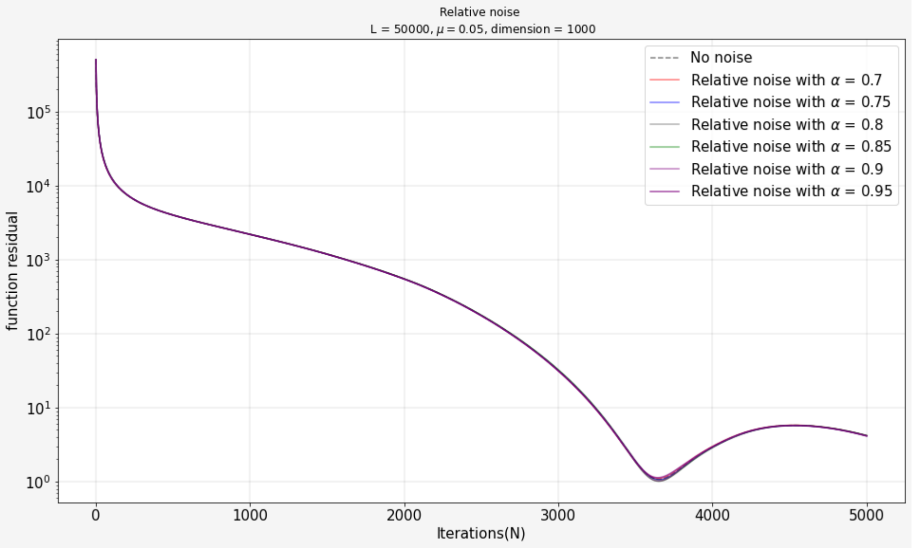
Boosting for relative inexactness

For boosting we can use gradient step in the end of each iteration:

$$x_k := x_k - \frac{1}{L}\tilde{\nabla} f(x_k)$$



Boosting for relative inexactness



Attempts in proofs

Theorem

If $\|\nabla f(\tilde{x}_k)\|_2 = O\left(\|\nabla f(x_k)\|_2\right)$ *and* $\mu > 0$. *Then choosing* $\alpha = O\left(\left(\frac{\mu}{L}\right)^{\frac{3}{4}}\right)$, *we get:*

$$f(x_k) - f(x^*) \leqslant \left(\frac{LR^2}{1-\alpha^2} + \frac{3L\alpha^2}{2\mu(1-\alpha^2)}\left(f(x_0) - f(x^*)\right)\right)\exp\left(-\frac{1}{6\sqrt{2}}\sqrt{\frac{\mu}{L}}k\right)$$

Also we can consider another condition

Theorem

If sequence $\tilde{x}_k$ *satisfies:*

$$f(x_k) - f(x^*) \leqslant O\left(LR^2 exp\left(-\frac{1}{2}\sqrt{\frac{\mu}{2L}}k\right) + \left(1 + \sqrt{\frac{4L}{\mu}}\right)\left(\frac{\delta^2}{\mu} + \frac{\delta^2}{L}\right)\right)$$

We can obtain we same convergence for same α *bound.*