Global optimization methods in inverese convex programming problems

O. V. Khamisov

Melentiev Energy Systems Institute, Irkutsk

"Application of Numerical Optimization Methods for Solving Inverse Problems",
April 14, 2021

- Convex parametric optimization

$$x(p) \in \operatorname{Arg}\min_x \{f(x,p)\colon g(x,p) \leqslant 0,\ x \in X\},$$

$f\colon X \times P \to \mathbb{R}$, $g\colon X \times P \to \mathbb{R}$, $X \subset \mathbb{R}^n$, $p \in P \subset \mathbb{R}^q$, ($g$ can be a vector-function),

$f(\cdot, p)$ is convex, $g(\cdot, p)$ is convex, X and P are convex compact sets.

- First inverse problem: a point $\hat{x}$ is given, find $\hat{p} \in P$: $x(\hat{p}) = \hat{x}$.
- Second inverse problem: a set $\hat{X}$ is given, find $\hat{p} \in P$: $x(\hat{p}) \in \hat{X}$.
- Third inverse problem: a point-to-set mapping $p \rightrightarrows \hat{X}(p)$ is given, find $\hat{p} \in P$: $x(\hat{p}) \in \hat{X}(\hat{p})$.
- Parametric description: $\hat{X}(p) = \{x \in X : h(x,p) \leqslant 0\}$. ($h$ also can be a vector-function)

- General statement (in the sense of A.S. Antipin). Find (x^*, p^*):

$$x^* \in \operatorname{Arg}\min_y \{f(y,p^*) \colon g(y,p^*) \leqslant 0, y \in X\}, \qquad (1)$$

$$h(x^*, p^*) \leqslant 0, \;\; p^* \in P. \qquad (2)$$

- Optimal value function: $v(p) = \min_y \{f(y,p) : g(y,p) \leqslant 0, y \in X\}.$

- Substitution of the extremal-type inclusion (1):

$$f(x,p) \leqslant v(p), \;\; g(x,p) \leqslant 0, \;\; x \in X.$$

- Reduced inverse problem. Find (x^*, p^*) :

$$f(x^*, p^*) \leqslant v(p^*), \;\; g(x^*, p^*) \leqslant 0, \;\; x^* \in X. \qquad (3)$$

$$h(x^*, p^*) \leqslant 0, \;\; p^* \in P. \qquad (4)$$

- Auxiliary optimization problem:

$$F(x,p) = v(p) - f(x,p) \to \max_{x,p}, \tag{5}$$

$$g(x,p) \leqslant 0,\ x \in X. \tag{6}$$

$$h(x,p) \leqslant 0,\ p \in P. \tag{7}$$

- Objective function property: $F(x,p) \leqslant 0$ for feasible (x,p).

- Case I: feasible set in parametric problem does not depend on p: $g(x,p) \equiv 0$.

$$v(p) = \min_{y \in X} f(y,p).$$

For a given $\tilde{p}$ we compute $\tilde{y} = \operatorname{Arg}\min_{y \in X} f(y,\tilde{p})$ obtaining an explicit support function in p:

$$v(\tilde{p}) = f(\tilde{y},\tilde{p}), \quad v(p) \leqslant f(\tilde{y},p). \tag{8}$$

- Example 1.

$$\begin{aligned} f(x,p) =& p^2x^2 - px \to \min_x, && \Rightarrow && f'_x(x,p) = 2p^2x - p = 0 && \Rightarrow && x(p) = \frac{1}{2p},\\ x \in X =& [-1,1], && \Rightarrow && -1 \leqslant x(p) = \frac{1}{2p} \leqslant 1 && \Rightarrow && |p| \geqslant \frac{1}{2}.\\ p \in P =& [-1,1]. \end{aligned}$$

$$\Downarrow$$

$$x^*(p) = \begin{cases} -1, & -\frac{1}{2} \leqslant p < 0,\\ \frac{1}{2p}, & \frac{1}{2} \leqslant |p|,\\ 1, & 0 < p \leqslant \frac{1}{2}. \end{cases} \quad v(p) = f(x^*(p),p) = \begin{cases} p^2 + p, & -\frac{1}{2} \leqslant p \leqslant 0,\\ -\frac{1}{4}, & \frac{1}{2} \leqslant |p|,\\ p^2 - p, & 0 \leqslant p \leqslant \frac{1}{2}. \end{cases}$$

- $x^*(0)$ is any point from $[-1,1]$.

- Example 1. Optimal value function $v(p) = \min_{x\in[-1,1]} f(x,p)$.

$$f(x,p) = p^2x^2 - px$$

$$x^*(p) = \begin{cases} -1, & -\frac{1}{2} \leqslant p \leqslant 0, \\ \frac{1}{2p}, & \frac{1}{2} \leqslant |p|, \\ 1, & 0 \leqslant p \leqslant \frac{1}{2}. \end{cases}$$

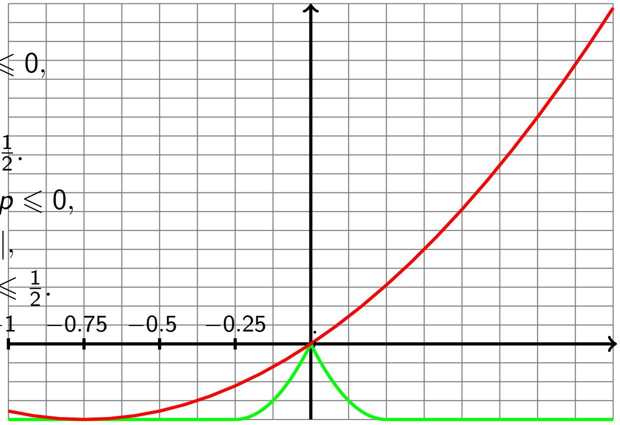
$$v(p) = \begin{cases} p^2 + p, & -\frac{1}{2} \leqslant p \leqslant 0, \\ -\frac{1}{4}, & \frac{1}{2} \leqslant |p|, \\ p^2 - p, & 0 \leqslant p \leqslant \frac{1}{2}. \end{cases}$$

$$\tilde{p} = -0.75$$

$$\tilde{x} = x^*(\tilde{p}) = -\frac{2}{3}$$

$$f(\tilde{x},p) = p^2\tilde{x}^2 - p\tilde{x}$$

$$f(\tilde{x},p) = \frac{4}{9}p^2 + \frac{2}{3}p$$

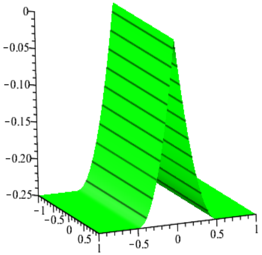


- Example 2. Find (x^*, p^*):

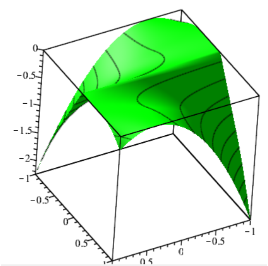
$$x^* \in \operatorname*{Arg\,min}_{y}\{p^2x^2 - px : x \in [-1, 1]\}$$

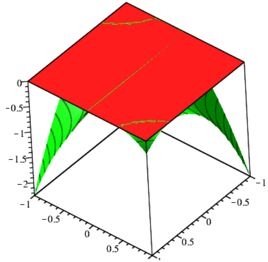
$$h(x^*, p^*) = (x^* - p^*)^2 \leqslant 0, \;\; p^* \in P = [-1, 1].$$

Optimal value function $v(p)$



Objective function $F(y, p) = v(p) - f(y, p)$





- Example 2. Support at $p_0 = -0.75$ function, $y_0 = -\frac{2}{3}$, $f(y_0, p) = \frac{4}{9}p^2 + \frac{2}{3}p$.

$$F(x,p) = v(p) - f(x,p) \leqslant f(y_0,p) - f(x,p).$$

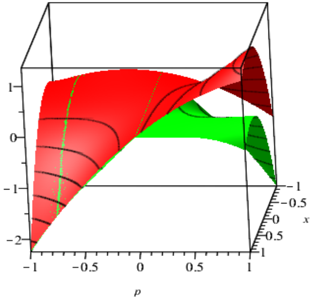
$$(x_1, p_1) = \arg\min\{F(x,p) : h(x,p) \leqslant 0,\ x \in X, p \in P\} = (0.5276, 0.9476).$$

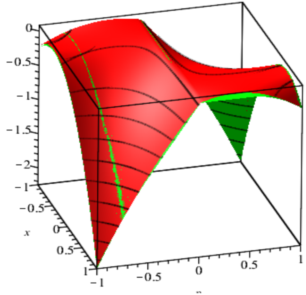
$$F_k(x,p) = \max_{i=\overline{0,k}} f(y_k,p) - f(x,p)$$

$$(x_k, p_k) = \arg\min\{F_k(x,p) : \ldots\}$$

$$y_k = \arg\min\{f(y,p_k) : y \in Y\}$$

If $|h(x_k,p_k)| \leqslant \varepsilon$: stop.





$$x_7 = 0.7071,\ p_7 = 0.7075,\ x^* = 0.7071, p^* = 0.7071$$

PROBLEMS WITH PARAMETERS IN CONSTRAINTS

$$G(p) = \{x \in X \subset \mathbb{R}^n : a^i(p)^T x - b_i \leq 0, i = 1, \dots, m\},$$

X – convex compact set.

Parameter $p \in P \subset \mathbb{R}^d$, P – convex compact set.

Feasibility function

$$w(p) = \min_{x \in X} \max_i \{a^i(p)^T x - b_i\}.$$

$$G(p) \neq \emptyset \Leftrightarrow w(p) \leq 0.$$

$$w(p) = \min_{x \in X} \max_i \{a^i(p)^T x - b_i\}.$$

Support majorant at $\tilde{p} \in P$

$$z \to \min,$$

$$z \geq a^i(\tilde{p})^T x - b_i, i = 1, \ldots, m$$

$$x \in X.$$

$(\tilde{x}, \tilde{z})$ – solution.

$$w(\tilde{p}) = \max_i \{a^i(\tilde{p})^T \tilde{x} - b_i\},$$

$$w(p) \leq \max_i \{a^i(p)^T \tilde{x} - b_i\}.$$

$\varphi_w(p, \tilde{p}) = \max_i \{a^i(p)^T \tilde{x} - b_i\}$ – support majorant.

Support minorant $S_u = \{u \in \mathbb{R}^m_+ : \sum_{i=1}^m = 1\}$.

$$w(p) = \min_{x \in X} \max_i \{a^i(p)^T x - b_i\} =$$

$$= \min_{x \in X} \max_{u \in S_u} \sum_{i=1}^m \{u_i[a^i(p)^T x - b_i]\} =$$

$$= \max_{u \in S_u} \min_{x \in X} \sum_{i=1}^m \{u_i[a^i(p)^T x - b_i]\} \geq$$

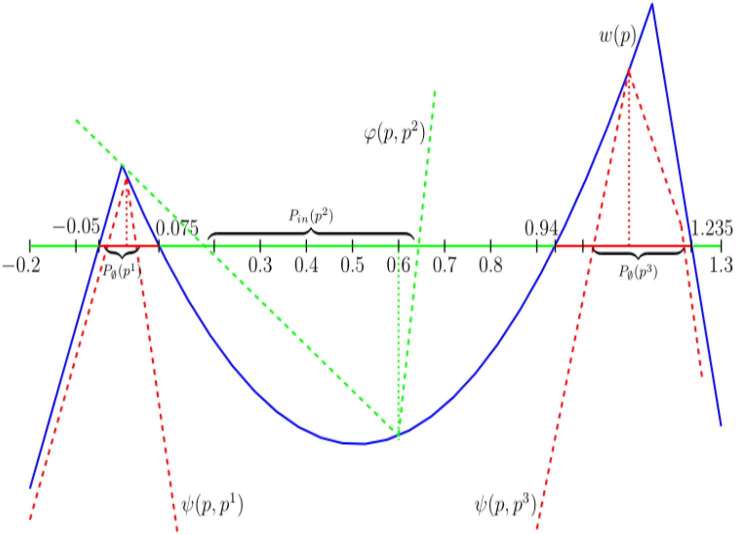
$$\geq \min_{x \in X} \sum_{i=1}^m \{\tilde{u}_i[a^i(p)^T x - b_i]\}.$$

Support point $\tilde{p} \in P$

$$z \to \min,$$
$$z \geq a^i(\tilde{p})^T x - b_i, i = 1, \ldots, m$$
$$x \in X.$$

$(\tilde{x}, \tilde{z})$ – solution,
$\tilde{u}_i$ – optimal dual variables.

$\psi_w(p, \tilde{p}) = \min_{x \in X} \sum_{i=1}^{m} \{\tilde{u}_i[a^i(p)^T x - b_i]\}$ – support minorant.

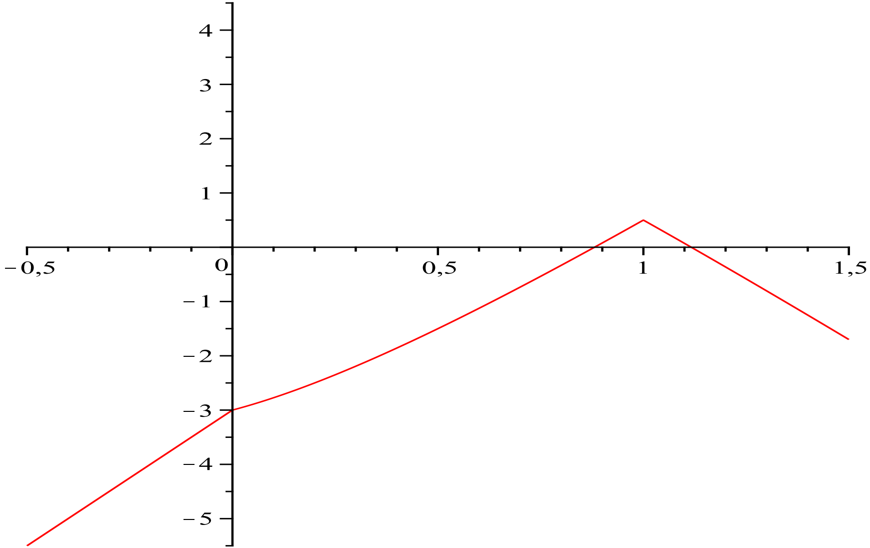


$w(p)$
$\varphi(p,p^2)$
$P_{in}(p^2)$
-0.05
0.075
0.94
1.235
-0.2
0.3
0.4
0.5
0.6
0.7
0.8
1.3
$P_\emptyset(p^1)$
$P_\emptyset(p^3)$
$\psi(p,p^1)$
$\psi(p,p^3)$

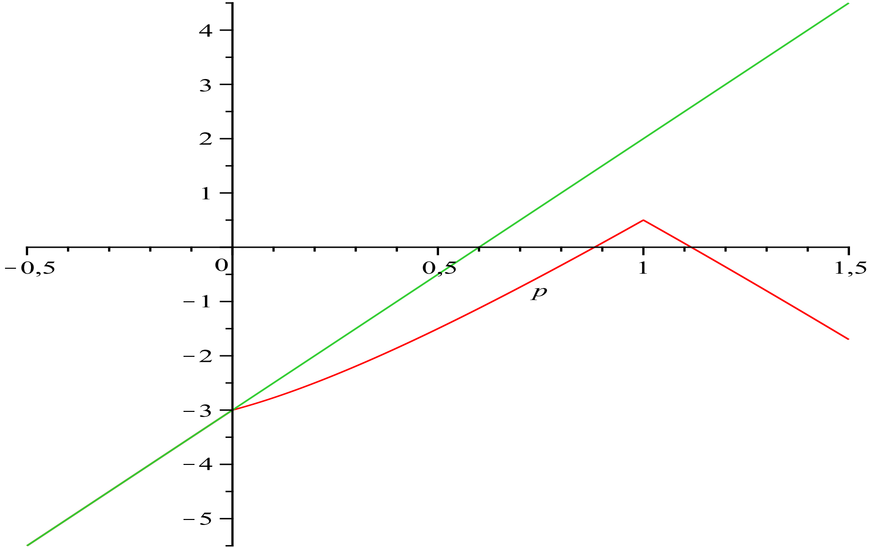
Example. 4

$$
\begin{aligned}
px_1 + x2 &\leq 1, \\
-x_1 - px_2 &\leq -2, \\
-5 \leq x_j \leq 5&, j = 1, 2, \\
p &\in [-0.5, 1.5].
\end{aligned}
$$

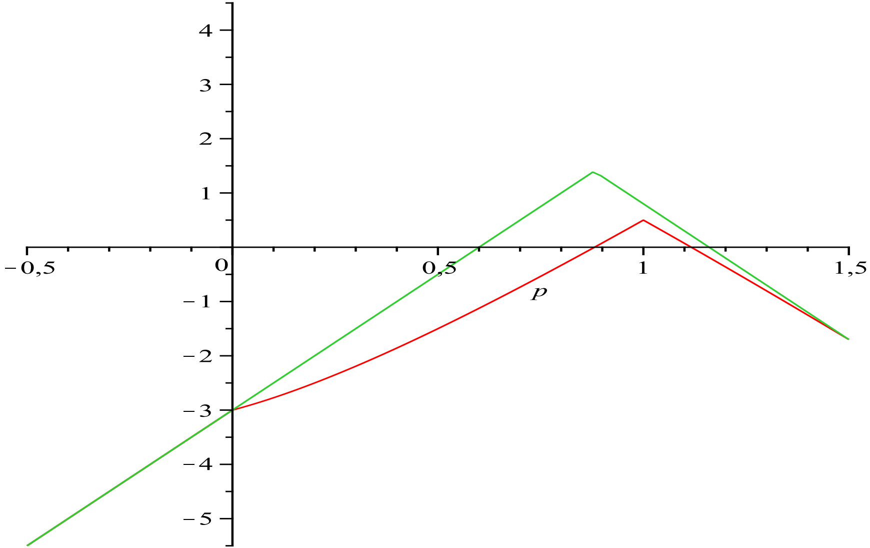
$w(p)$.



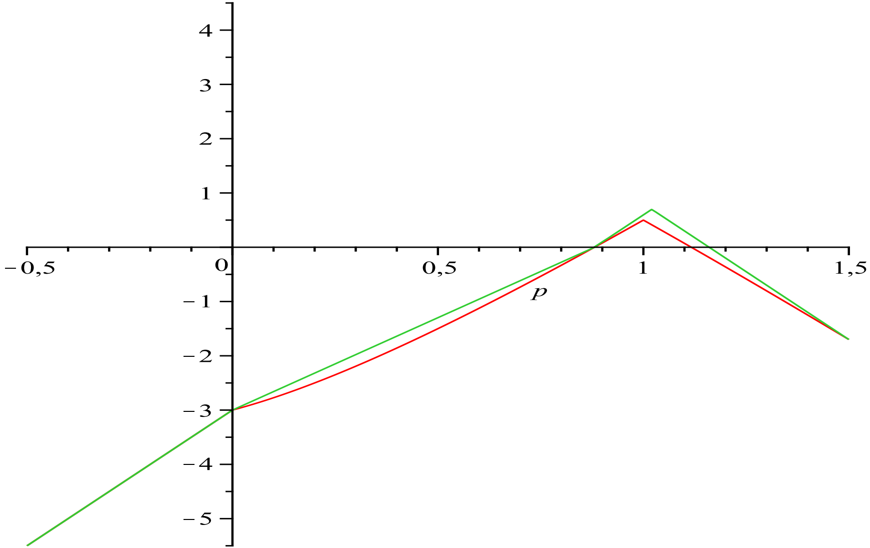
$w(p), \psi_w(p, -0.5)$.



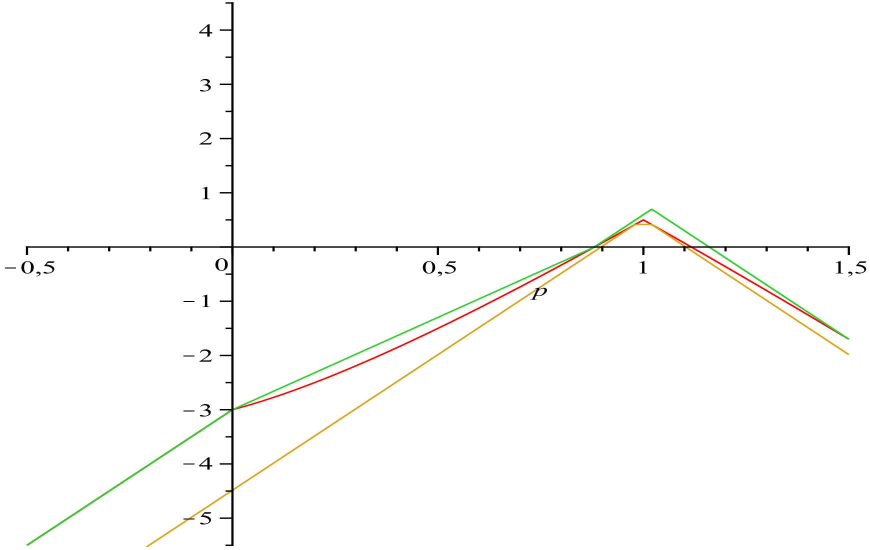
$w(p), \psi_w(p, -0.5), \psi_w(p, 1.5)$.



$w(p), \varphi_w(p, -0.5), \varphi_w(p, 1.5), \varphi_w(p, 0.89).$



$w(p), \varphi_w(p, -0.5), \varphi_w(p, 1.5), \varphi_w(p, 0.89), \psi_w(p, 1.02)$.



LP optimal value function

$$v(p) = \min_x \{c^T x : A(p)x \le b, \underline{x} \le x \le \overline{x}\},$$

$$p \in P \in \mathbb{R}^q,$$

$a_{ij}(p)$ — linear functions.

Find

$$\underline{v} = \min_{p \in P} v(p).$$

Dual description

$$v(p) = \sup_{\lambda \ge 0} \min_{\underline{x} \le x \le \overline{x}} \{c^T x + \lambda^T (A(p)x - b)\}.$$

$\hat{p} \Rightarrow (\hat{\lambda}, \hat{x})$:

$$v(p) \ge \min_{\underline{x} \le x \le \overline{x}} \{c^T x + \hat{\lambda}^T (A(p)x - b)\} = \psi_v(p, \hat{p}).$$

Example 5.

$$x_1 + x_2 \to \min_{x_1, x_2},$$

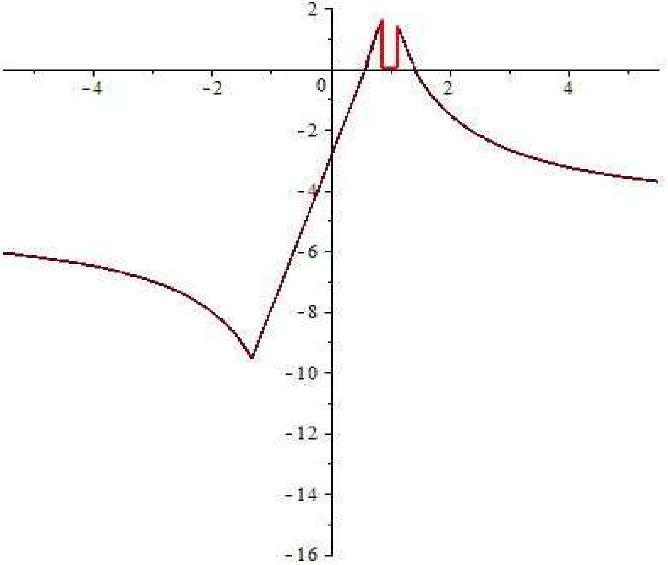
$$p x_1 + x_2 \le 1,$$

$$-x_1 - p x_2 \le -2,$$

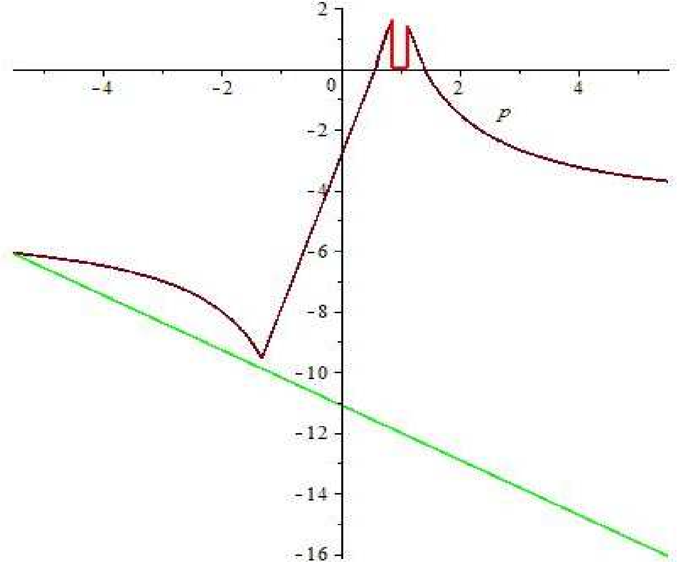
$$-5 \le x_j \le 5, j = 1, 2,$$

$$p \in [-5.5, 5.5].$$

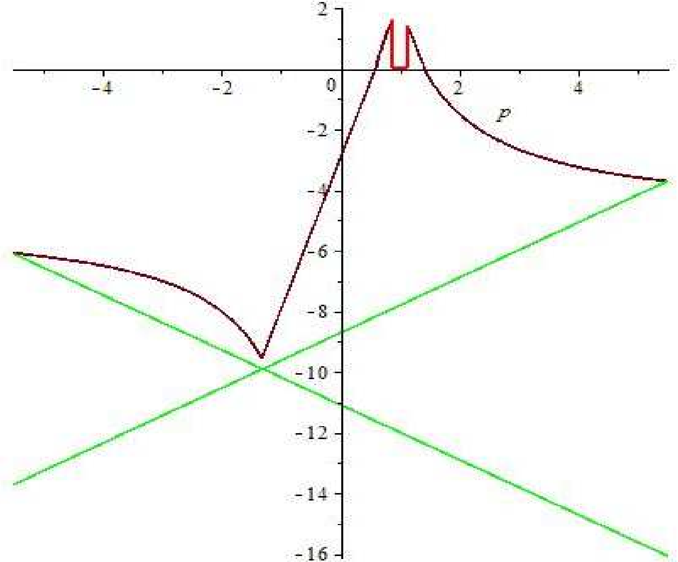
$v(p), p \in [-5.5, 5.5]$



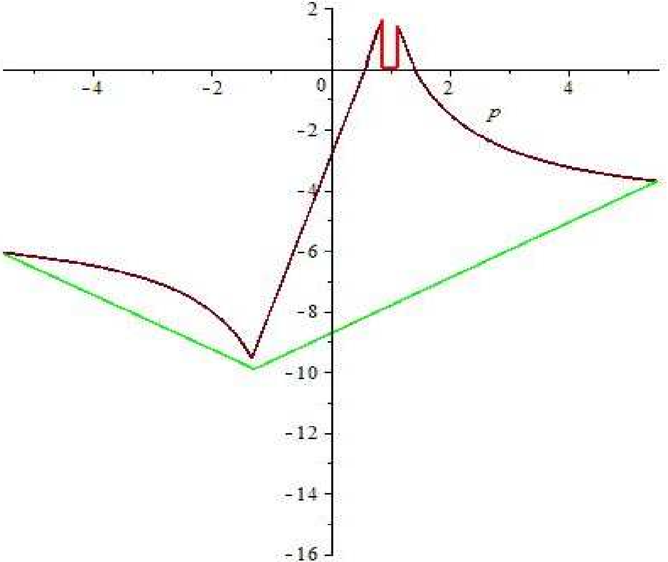
$v(p), p \in [-5.5, 5.5], \psi_v(p, -5.5)$



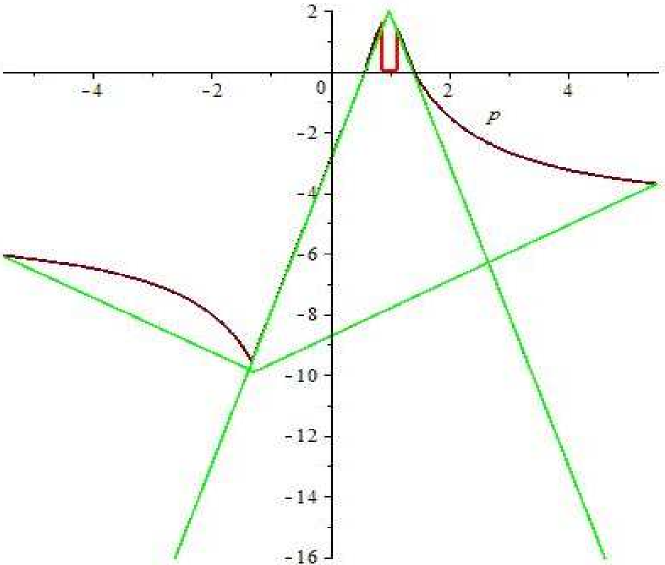
$v(p), p \in [-5.5, 5.5], \psi_v(p, -5.5), \psi_v(p, 5.5)$



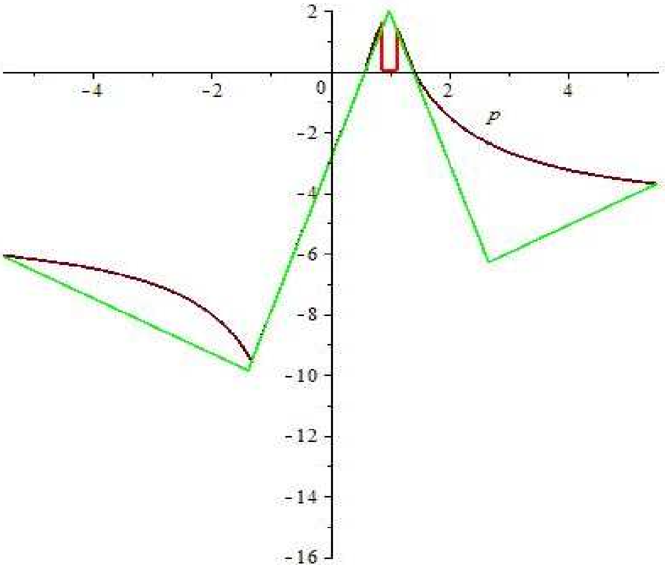
$v(p), p \in [-5.5, 5.5], \psi_v(p, -5.5), \psi_v(p, 5.5)$



$v(p), p \in [-5.5, 5.5], \psi_v(p, -5.5), \psi_v(p, 5.5), \psi(p, -1.3)$



$v(p), p \in [-5.5, 5.5], \psi_v(p, -5.5), \psi_v(p, 5.5), \psi(p, -1.3)$



- Auxiliary problem is equal to minimization of a nonconvex piece-wise linear function.

THANK YOU FOR ATTENTION!