Cloud approximations in optimization algorithms

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Applications areas

- reachable set of controllable system
- trajectory of dynamical system
- estimation of the local extrema regions of the function
- multicriteria optimization
- data analysis
- …

Problems

- phase estimation
- nonlocal optimal control methods
- search for a global function extremum
- variational inequalities
- impacts of normalization
- …

“Cloud approximation” Term

- irregular grid
- randomized mesh
- stochastic grid
- set of points
- ...

The optimal control problem
with box constraints

$$\dot{x} = f(x, u, t)$$

$$x(t_0) = x^0, \quad t \in T = [t_0, t_1]$$

$$u(t) \in U = \{u \in R^r : \underline{u}_i \le u_i \le \overline{u}_i\}$$

$$I(u) = \varphi(x(t_1)) \to \min$$

The approximation problem of the reachable set

Stabilization of a nonlinear pendulum

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = u_1 - \sin x_1$$

$$| u_1(t) | \le 1$$

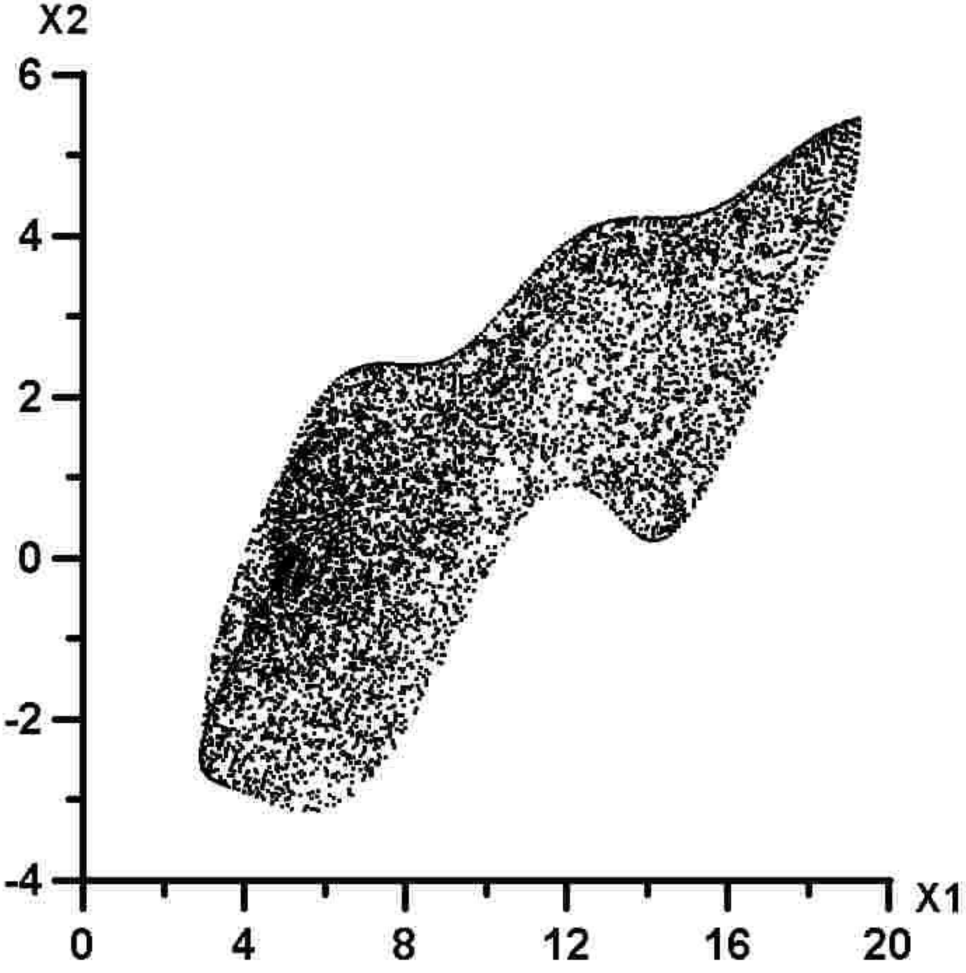
$$t \in [0, 5]$$

$$x(0) = (5, 0)$$

$$I(u) = x_1^2(5) + x_2^2(5) \to \min$$

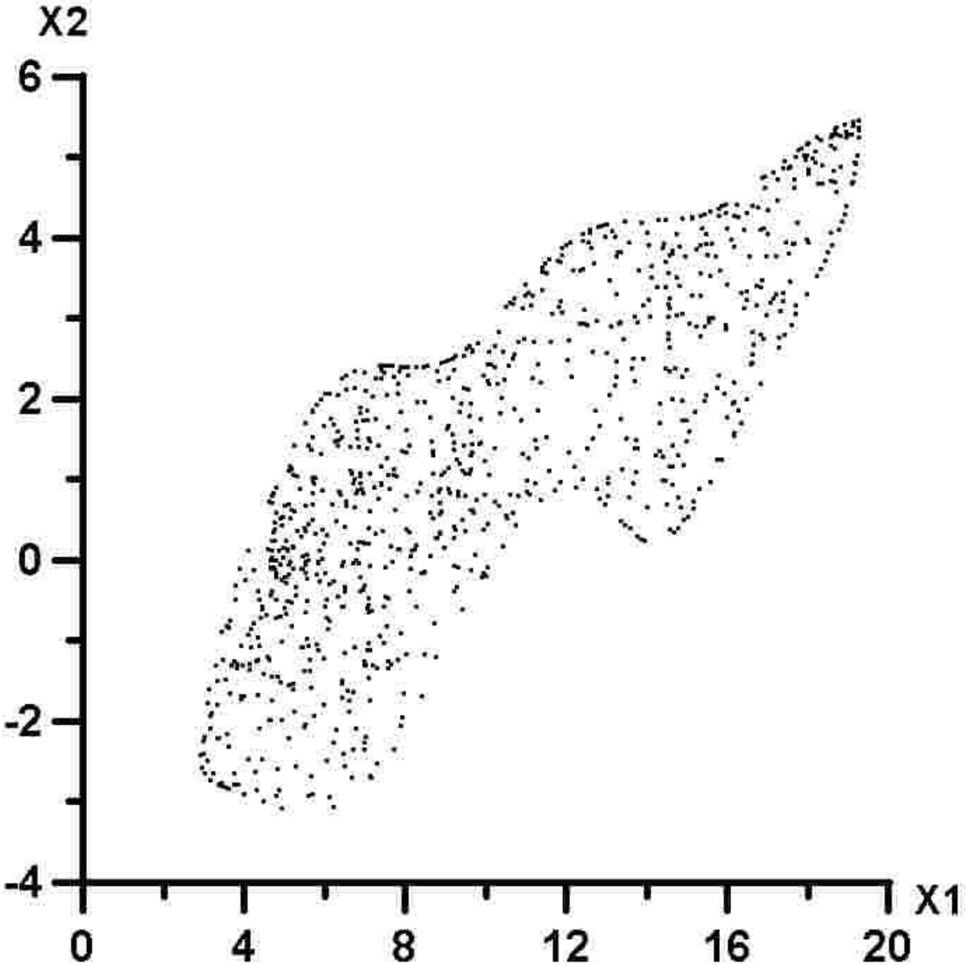
The approximation problem of the reachable set

Stabilization of a nonlinear pendulum



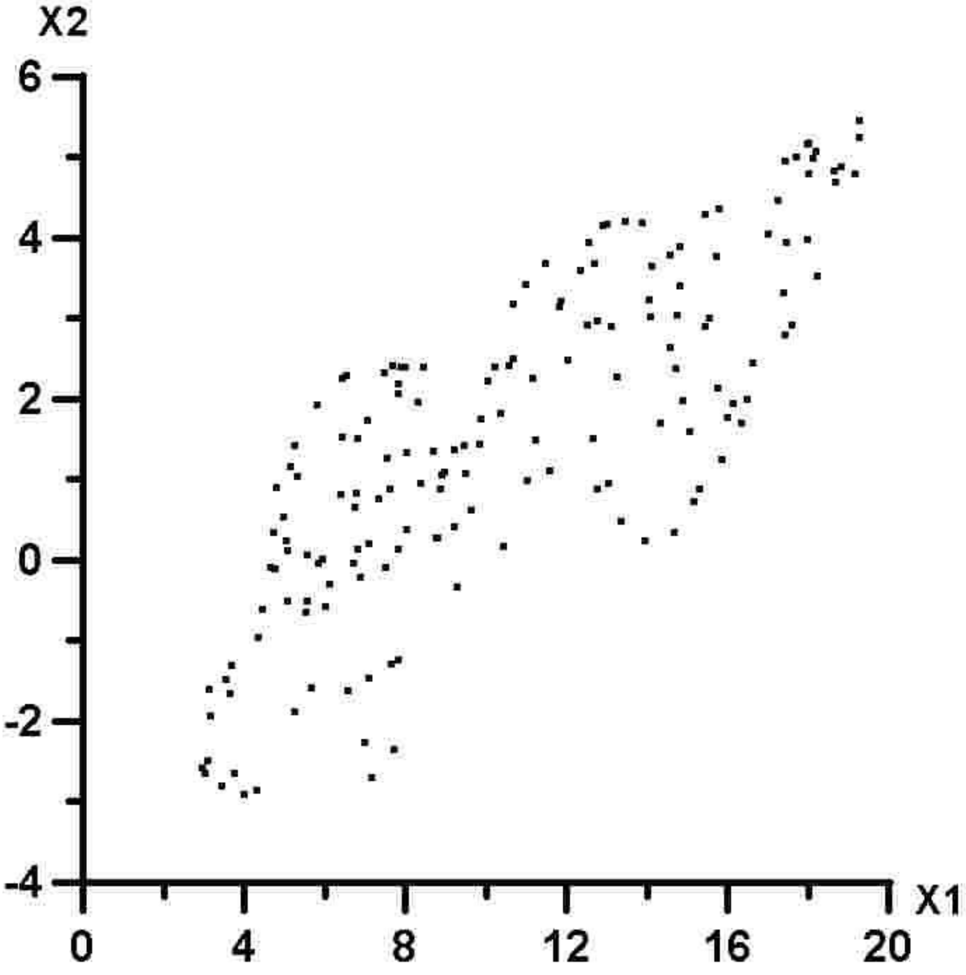
The approximation problem of the reachable set

Stabilization of a nonlinear pendulum



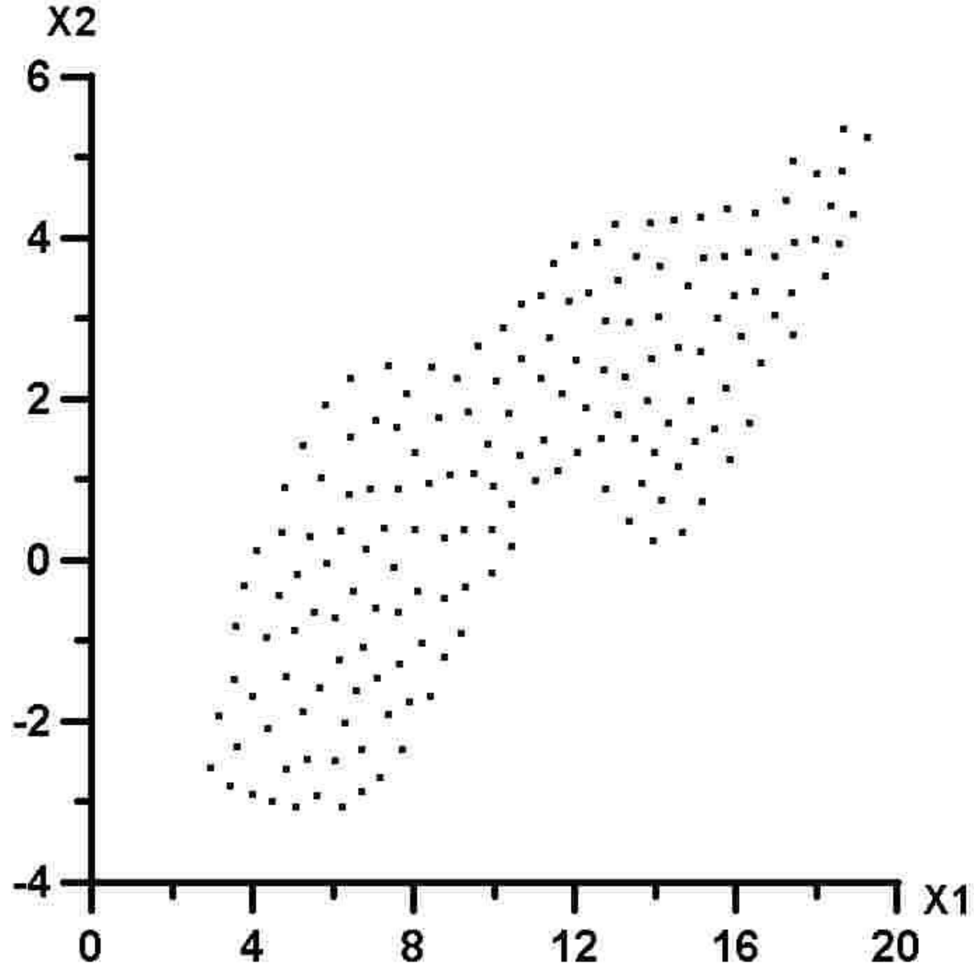
The approximation problem of the reachable set

Stabilization of a nonlinear pendulum



The approximation problem of the reachable set

Stabilization of a nonlinear pendulum



A common approach
input-output system

- uniform approximation of input parameters
- processing of multiple output parameters

Approach to the construction of algorithms

- only scalable basic operations (Euclidean norms, L_1-norms, ...)
- no more than subquadratic computing schemes
- visualization
- …

Test problems 01

$$\dot{x}_1 = x_2$$

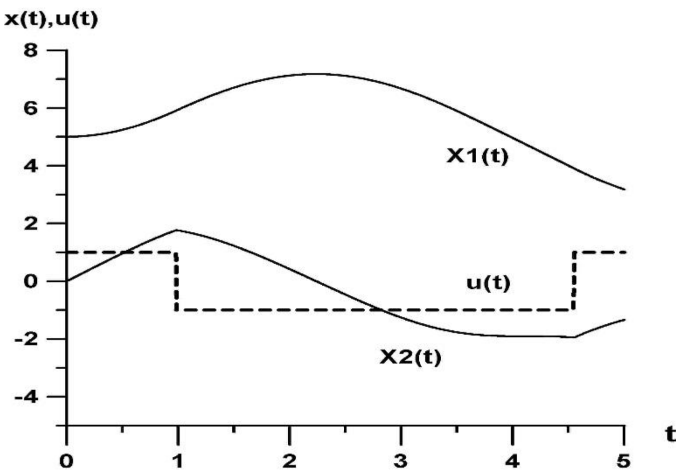
$$\dot{x}_2 = u_1 - \sin x_1$$

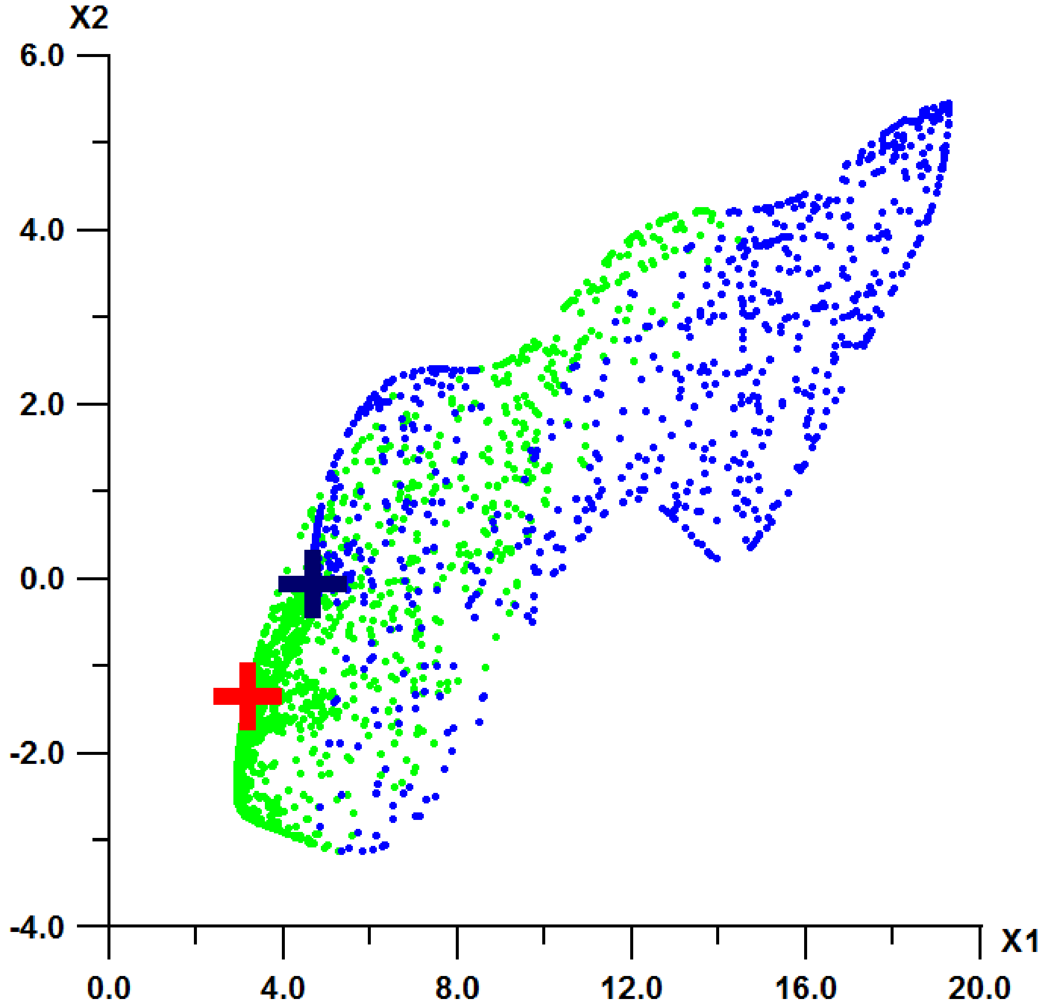
$$x(0) = (5, 0) \quad t \in [0, 5] \quad |u_1(t)| \le 1$$

$$I(u) = x_1^2(5) + x_2^2(5) \to \min$$

Iteration – 1000,
CPU time is 7591 sec.,
Number of solved Cauchy problems – 186015

N	Functional	Value
1	1.190817e+01	0.457
2	2.182900e+01	0.543





The global optimization problem Approach

- Identification and evaluation of local extremum areas
- decomposition of the original problem into subtasks with the "small" reachable sets

The global optimization problem formulation

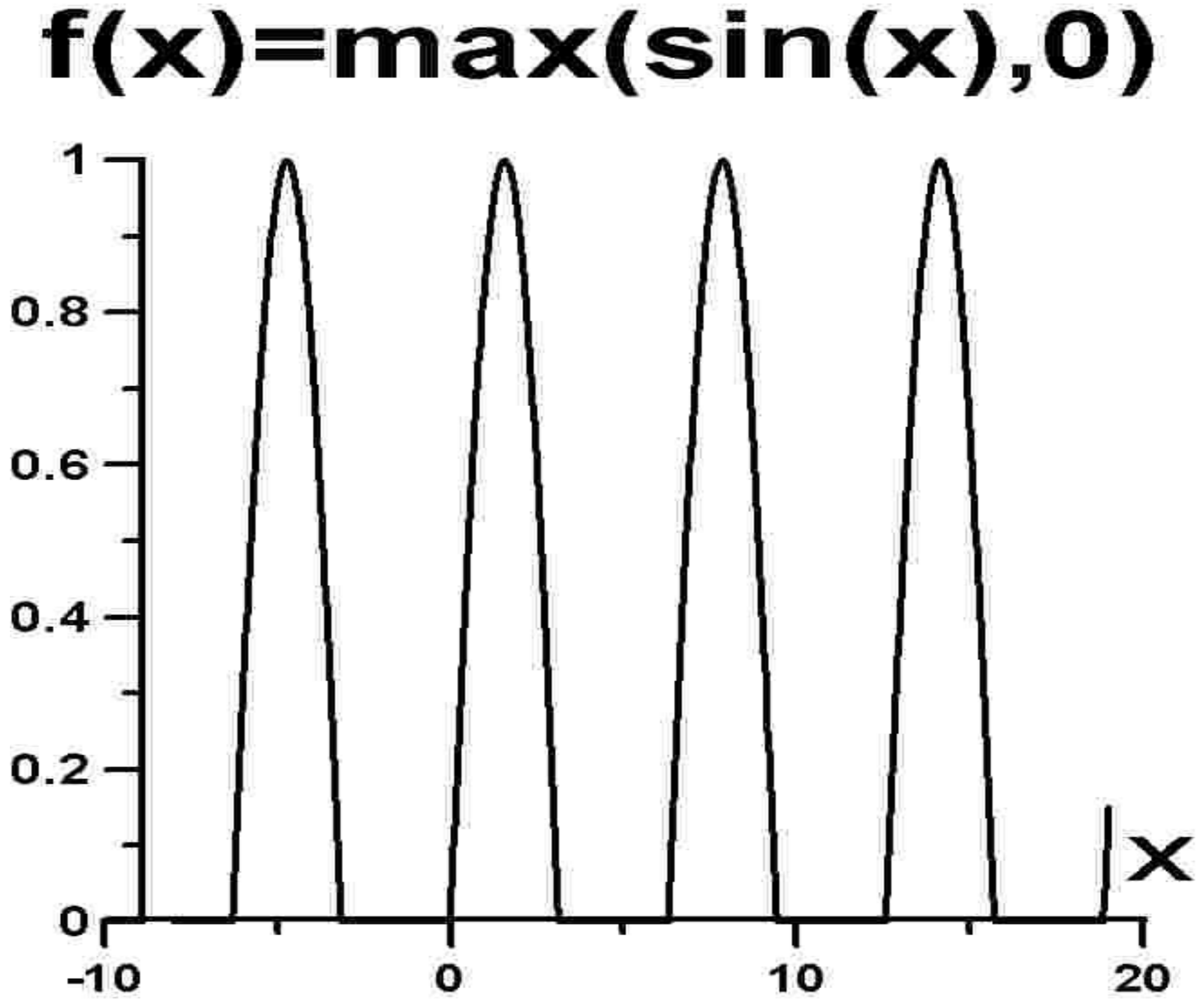
$$f(x) -> \min, \; x \in X$$

$$f(x) -> \min, \; x_l \leq x \leq x_g$$

$$x \in X = \{ x_l \leq x \leq x_g \} : f(x) = f^*$$

$$X^* \in X : \forall x \in X^* \; f(x) = f^*$$

Example



Classification of optimization problems by the number of extrema

- “Low extremes” – 2-5 extrema
- “Medium-extreme” – 5-30 extrema
- “Multiextremal” – 30-"many extrema“
- with a multivalued solution – infinitely many extrema

Example

Optimization of Atomic Molecular Potentials
Morse Potential

- the number of variables = 3 * number of atoms
- the number of local extrema grows as an exponent of the atoms number
- with the number of atoms = 147, the estimate of the number of local extrema is 10**60
- officially registered record is 240 atoms. The number of extrema > 10**100?

Cambridge Cluster Database

Classification of optimization problems by structure of extrema

- several extrema with different values of the function
- several extrema with the same value of the function
- a set of solutions with the same value of the function
- sets of solutions with different values of the function

Global optimization
problems and hopes

- the problem of the volume ratio of the search set and the possibilities of searching, the resource of probes

- *the problem dimension is* 100*,*
  box [0,1]*, volume* 1

- *the problem dimension is* 100*,*
  box [0,2], *volume* 2**100=(1024)**10 = 1.26*10**30

Global optimization
problems and hopes

- in order to solve the global optimization problem, one must be able to solve **only two** problems:

1) find one point in the region of the global extremum attraction;
2) find the minimum of unimodal function

Open problems
Control of trajectory beams

- Control in conditions of indeterminacy (in the system both control and perturbation)

- The problem of formulating the "nuclear problem“

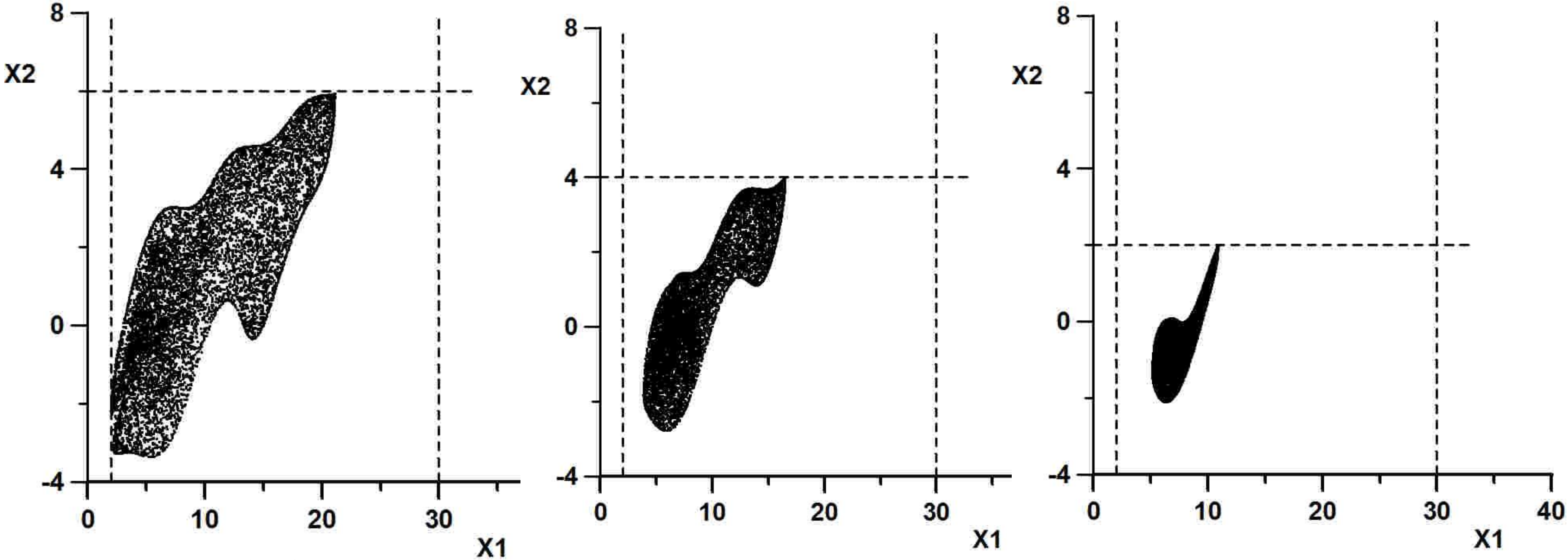
- Example: impacts of normalization

$$u^k(t) + \delta u^k(t), \|\delta u^k(t)\| \le \Delta$$

Test problem 02

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = u_1 - \sin(x_1)$$

$$x_1(t_0) = 5,\ x_2(t_0) = 0, \quad |u(t)| \le \alpha, \ \ t \in [0,5]$$



$2 \le x_1(t_1) \le 30$ $\quad$ $2 \le x_1(t_1) \le 30$ $\quad$ $2 \le x_1(t_1) \le 30$

$-4 \le x_2(t_1) \le 6$ $\quad$ $-4 \le x_2(t_1) \le 4$ $\quad$ $-4 \le x_2(t_1) \le 2$

$|u(t)| \le \alpha^* = 1.145529$ $\quad$ $|u(t)| \le \alpha^* = 0.783166$ $\quad$ $|u(t)| \le \alpha^* = 0.406571$

Popular approximative constructions

- “boxes”
- spheres
- ellipsoids
- meshes ("cloud")
- parallelotopes
- simplexes
- ovaloids
- …

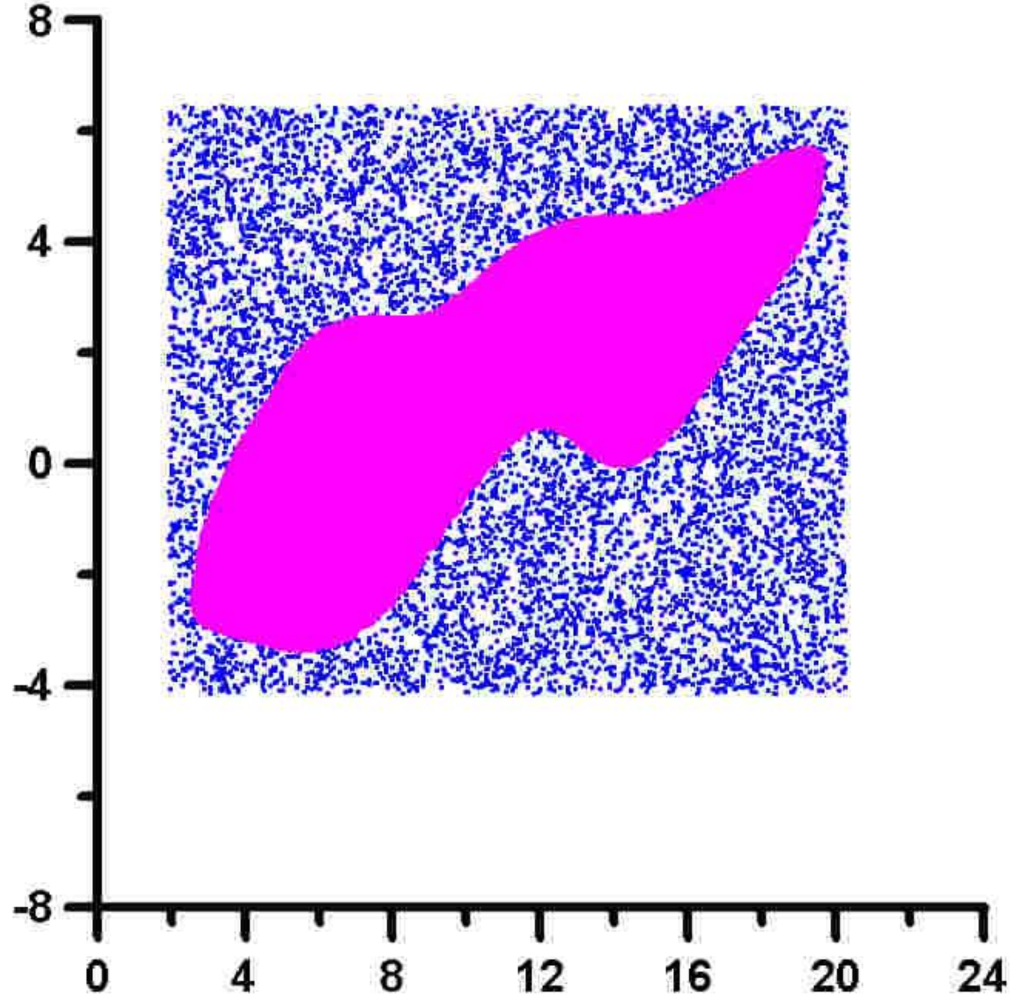
“Cloud approximation”
Algorithms for realization of "set-theoretic" operations

- association
- intersection
- addition
- convexication
- delineation
- boundary approximation
- evaluation of the "diameter“
- estimation of "spread“
- estimate the volume of the set

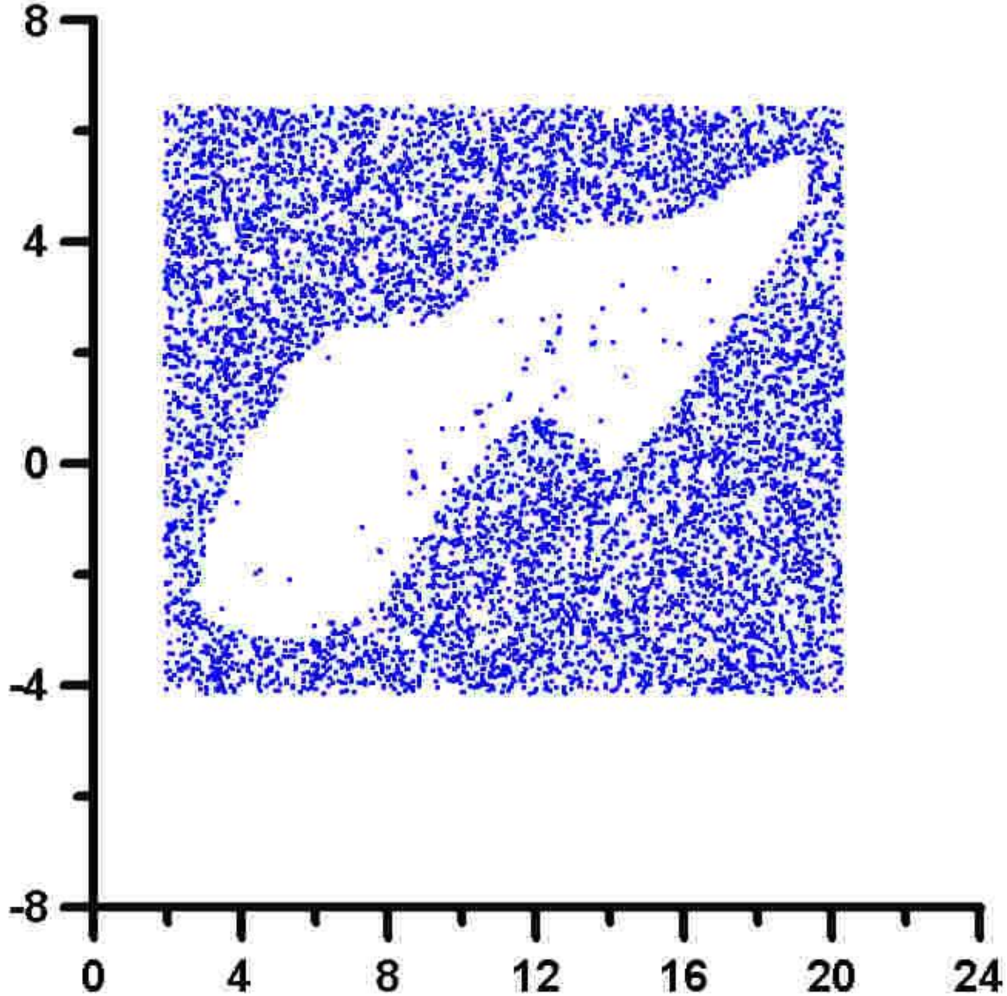
“Cloud approximation”
Algorithm for addition constructing

- outlining the original "cloud" with a box
- generation of test points from a box
- fixation in the "cloud"-addition of test points far from the points of the original "cloud"

“Cloud approximation”
Algorithm for addition constructing



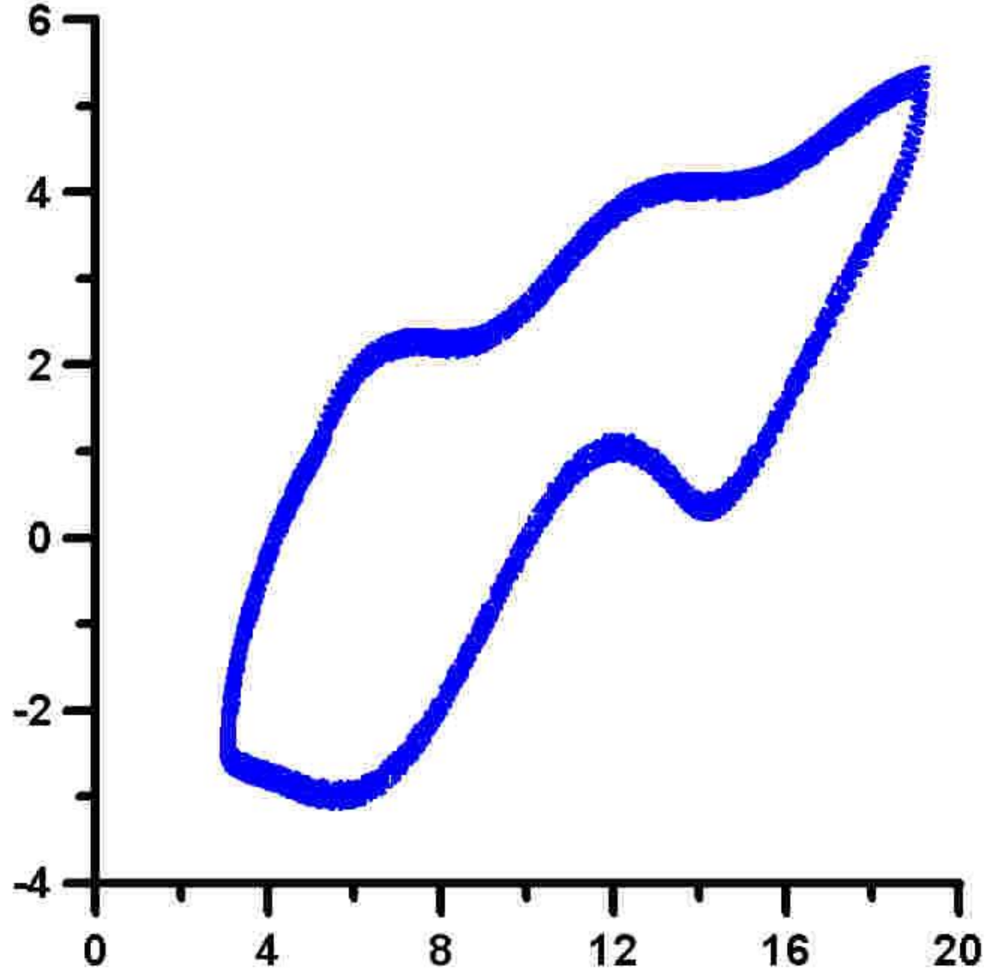
“Cloud approximation”
Algorithm for addition constructing



“Cloud approximation”
Algorithm for the boundary approximation

- the construction of a "cloud"-addition
- removal of points close to the points of the original "cloud"

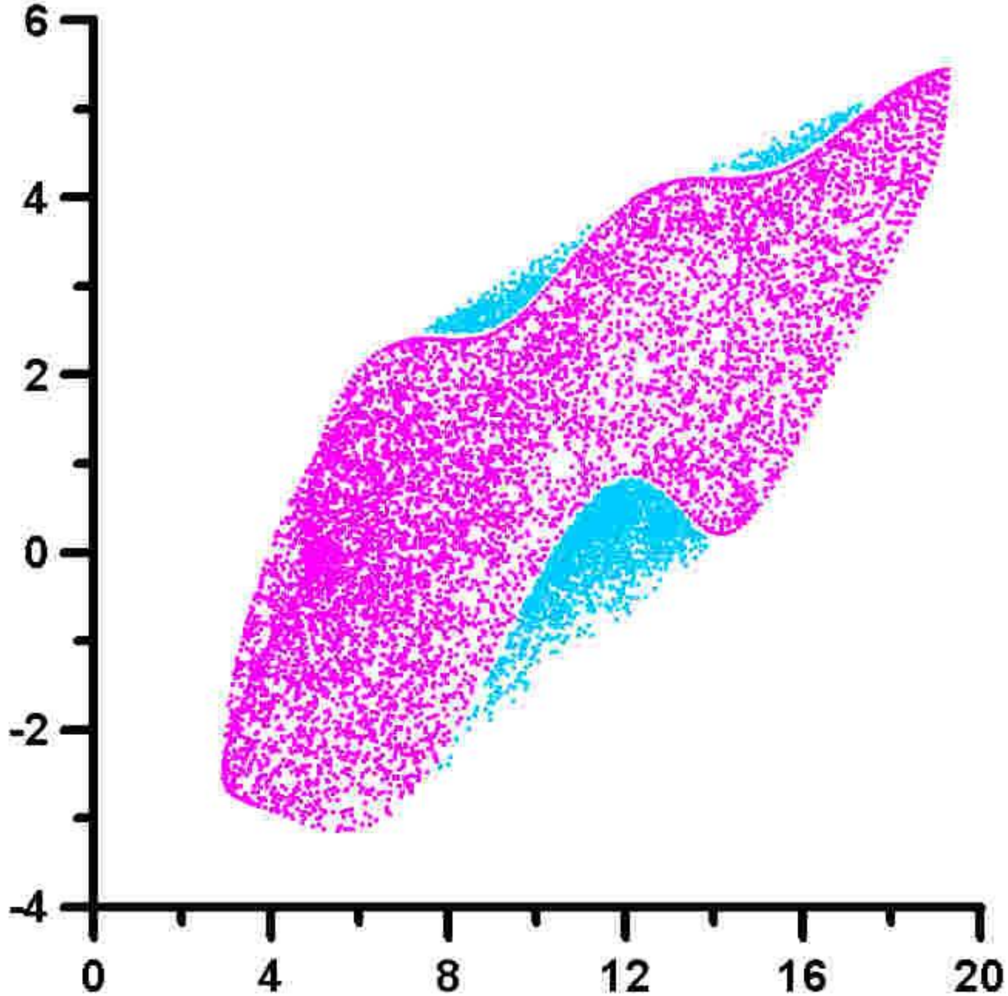
“Cloud approximation”
Algorithm for the boundary approximation



“Cloud approximation”
Algorithm for the convexification

- selecting pairs of points from the original “cloud“
- generation of test points on the connecting segments
- selection of test points, fixation of the points from the original “cloud” that do not fall in the its neighborhoods

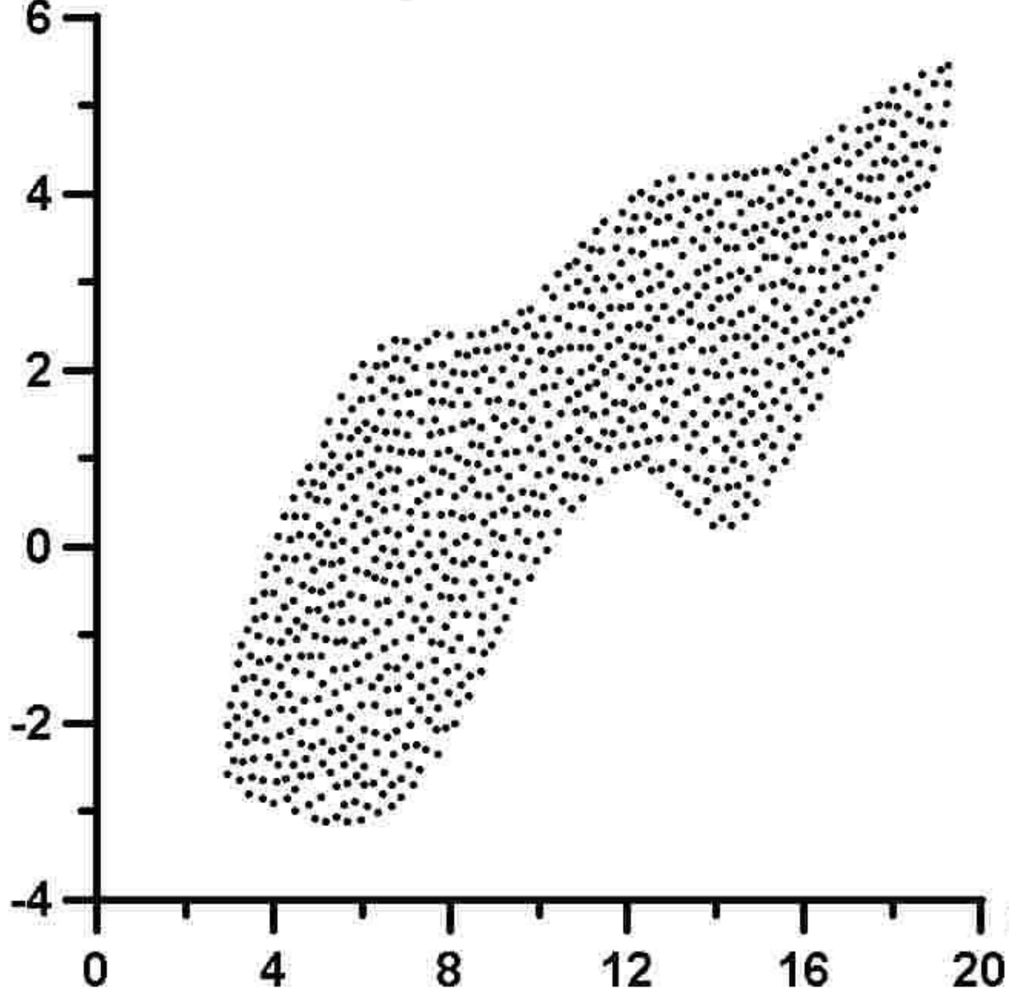
“Cloud approximation”
Algorithm for the convexification



“Cloud approximation”
Algorithm of quasiuniform filling

- generation a start "cloud" from one point
- generation of test points
- selection of test points, fixation of points not including in the neighborhood of a point of the already existing "cloud"

“Cloud approximation”
Algorithm of quasiuniform filling



“Cloud approximation”
The algorithm for estimating the volume of the set ("Archimedes' algorithm")

- outlining the "cloud" with the box
- quasiuniform filling of the "cloud“
- quasiuniform filling of the contouring box
- volume estimation through the ratio of the points number in the set and the box approximations

Estimation of the cluster volume
Archimedes' algorithm

- fixe R – radius of the test sphere
- “the enclosing” box is constructed
- calculate N – the number of cluster elements lying at least R from each other
- calculate M – the number of disjoint spheres of radius R that fill the enclosing box
- cluster volume estimation is equal the box volume * M / N

The FOREL clustering method
"Formal Element"

- convergence is proved in a finite number of steps
- strong dependence on the choice of the first point
- relatively low productivity
- **close to linear computational complexity**

1) Zagoruiko N.G., Yolkina V.N., Lbov G.S. Algorithms for detecting empirical regularities. – Novosibirsk: Science, 1985. – 999 p. (In Russian).

2) Zagoruiko N.G. Applied methods of data and knowledge analysis. – Novosibirsk: IM SB RAS, 1999. – 270 p. – ISBN 5-86134-060-9.

Software
OPTCON-SV (version 0.5)

- algorithms for estimating the record value of the function
- algorithms for generation of "cloud approximation“
- FOREL algorithm for clustering
- tools for research, fixation and visualization of the clusters

Software
OPTCON-SV
Algorithms for "clouds“ generation

- stochastic approximation algorithm
- algorithm of approximation with a search "along the Polak“
- approximation algorithm with Hooke-Jeeves search
- algorithm of deterministic approximation for the function of two variables

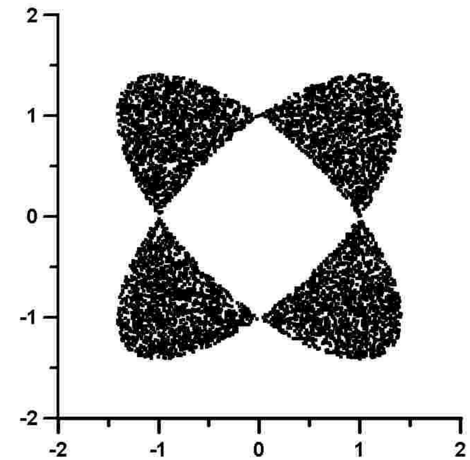
Software OPTCON-SV

Tools

- table of distances between the centers of the clusters
- sphere chart of the cluster
- coordinate cluster chart
- cluster estimation algorithm of Archimedes
- the lower bound of the function value on the cluster

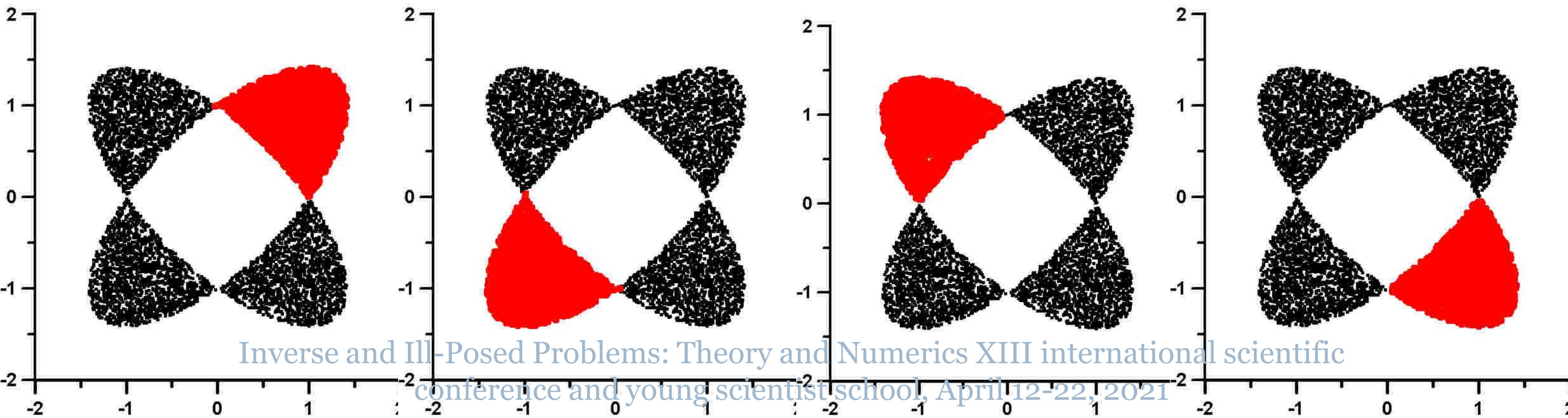
Test problem 03

calculation by the method of stochastic approximation, 1 min. of CPU time, about 600,000 samples, found 6714 points



$$f(x_1, x_2) = (x_1 - 1)^2 (x_1 + 1)^2 + (x_2 - 1)^2 (x_2 + 1)^2$$

$$f(x_1, x_2) \le 1$$



“Cloud approximation”

What do we want and what is obtained with the help of "prequadratic" algorithms

- find an estimate of the record value of the function
- find the estimate of the location region of the global extremum
- reduce the volume of the search box
- exclude unpromising areas
- increase the probability of finding a global extremum
- bypass the hard gullies

Sphere chart of the cluster

The number of cluster points falling into a sphere of increasing radius with center at the center of gravity of the cluster

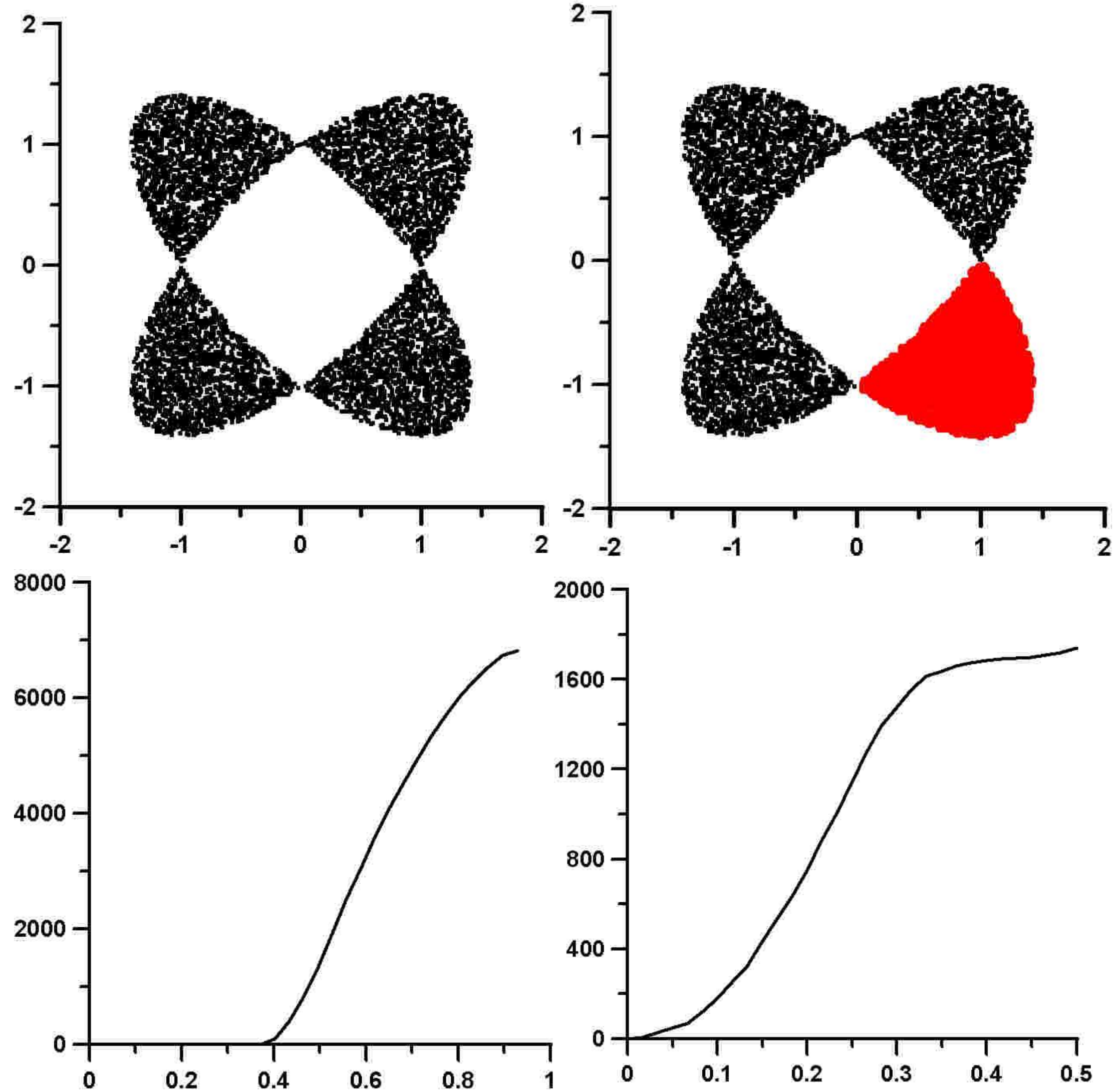


Table of distances between cluster centers

	1	2	3
2	0.975		
3	0.696	0.700	
4	0.695	0.701	1.000

Conclusions

- simple algorithms give a good result
- it is possible to evaluate sets with complex geometry
- there is no strict limitation in dimension
- there is potential for parallelization

- ***"Cloud" approximations can be a useful tool***

Thank you for attention!

Cloud approximations for numerical solution of control theory problems

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