Calogero-Sutherland systems and Dunkl operators at infinity

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Classical Calogero-Moser-Sutherland system

1970s

$$H^{CMS} = \frac{1}{2}\sum_{i=1}^{N} p_i^2 - \kappa^2 \sum_{i<j} \frac{1}{(q_i - q_j)^2}$$

The equations of motion

$$\begin{cases} \dot{p}_i = -\frac{\partial H^{CMS}}{\partial q_i} \\ \dot{q}_i = \frac{\partial H^{CMS}}{\partial p_i} \end{cases}$$

can be presented in the Lax form: $\dot{L} = [L, M]$.

Moser 1975

$$L_{jk} = p_j \delta_{jk} + \kappa(1 - \delta_{jk}) \frac{1}{q_j - q_k}$$

$$M_{jk} = \kappa \delta_{jk} \sum_{s \neq j} \frac{1}{(q_j - q_s)^2} - \kappa(1 - \delta_{jk}) \frac{1}{(q_j - q_k)^2}$$

The CMS system is classically integrable. The traces $I_n = \mathrm{tr} L^n$, $n = 1, \ldots, N$ are the integrals of motion in involution $\{I_n, I_k\} = 0$.

Quantum Calogero-Sutherland system

$$H^{CS} = -\sum_{i=1}^{N}\left(\frac{\partial}{\partial q_i}\right)^2 + 2\left(\frac{\pi}{L}\right)^2 \sum_{i<j}^{N} \frac{\beta(\beta - K_{ij})}{\sin^2\left(\frac{\pi}{L}(q_i - q_j)\right)}$$

$\prod_{i<j} |\sin(\frac{\pi}{L}(q_i - q_j))|^{\beta}$ represents the vacuum state with eigenenergy $E_0 = (\pi\beta/L)^2\, N(N^2 - 1)/3$. After conjugating by the vacuum state and passing to the exponential variables $x_i = e^{\frac{2\pi i q_i}{L}}$:

$$H = \sum_{i=1}^{N}\left(x_i \frac{\partial}{\partial x_i}\right)^2 + \beta \sum_{i<j} \frac{x_i + x_j}{x_i - x_j}\left(x_i \frac{\partial}{\partial x_i} - x_j \frac{\partial}{\partial x_j}\right) - 2\beta \sum_{i<j} \frac{x_i x_j}{(x_i - x_j)^2}\,(1 - K_{ij})\,.$$

K_{ij} is the coordinate exchange operator of particles i and j.

Dunkl operators

The Heckman–Dunkl operators $\mathcal{D}_i$ in the form suggested by Polychronakos:

$$\mathcal{D}_i = x_i \frac{\partial}{\partial x_i} + \beta \sum_{j \neq i} \frac{x_i}{x_i - x_j} (1 - K_{ij})$$

These operators satisfy the relations

$$K_{ij}\mathcal{D}_i = \mathcal{D}_j K_{ij},$$
$$[\mathcal{D}_i, \mathcal{D}_j] = \beta(\mathcal{D}_j - \mathcal{D}_i)K_{ij}.$$

which coincide with the relations of the degenerate affine Hecke algebra after the renormalization $\mathcal{D}_i \to \frac{1}{\beta}\mathcal{D}_i$.

Operators $H_k = \mathrm{Res}_\pm \left(\sum_i (\mathcal{D}_i)^k\right)$ commute and represent integrals of motion of the quantum Calogero-Sutherland model.

Cherednik-Dunkl operators

One can use another set of commuting elements d_i of the degenerate affine Hecke algebra

$$d_i = x_i \frac{\partial}{\partial x_i} + \beta \sum_{j<i} \frac{x_i}{x_i - x_j} (1 - K_{ij}) + \beta \sum_{i<j} \frac{x_j}{x_i - x_j} (1 - K_{ij}) + (N - i) \, .$$

The elements $d_i = \mathcal{D}_i + \beta \sum_{j<i} K_{ij}$ satisfy relations

$$K_{i,i+1} d_i = d_{i+1} K_{i,i+1} + 1$$

$$[d_i, d_j] = 0.$$

The Hamiltonian above is expressed by the formula

$$H = \sum_i \left(d_i^2 - \beta d_i \right) .$$

Bosonic limit for scalar CS system. Observing Nazarov-Sklyanin and Veselov-Sergeev construction

The Dunkl operators produce the equivariant family of N functions $f_i(x_1,\ldots;x_i;\ldots x_N)\in\mathbb{C}[x_i]\otimes\Lambda^{+}[x_1,\ldots x_{i-1},x_{i+1},\ldots x_N]$ such that $K_{ij}f_i=f_j$.
The symmetrization is the sum:

$$(E_N f)(x_1,...,x_N)=f_1(x_1;x_2,...,x_N)+f_2(x_1;x_2;x_3,...,x_N)+\ldots\\ +f_N(x_1,...,x_{N-1};x_N).$$

Integrals of motion $H_k=\sum_i\mathcal{D}_i^k$

$$\begin{array}{ccc}\Lambda^{+}[x_1,\ldots,x_N] & \xrightarrow{\iota_{i,N}} & \mathbb{C}[x_i]\otimes\Lambda^{+}[x_1,\ldots x_{i-1},x_{i+1},\ldots x_N]\\ & & \downarrow {\mathcal{D}_i}^k\\ \Lambda^{+}[x_1,\ldots,x_N] & \xleftarrow[E_N]{} & \mathbb{C}[x_i]\otimes\Lambda^{+}[x_1,\ldots x_{i-1},x_{i+1},\ldots x_N]\end{array},$$

For the equivariant family $K_{ij}f_i = f_j$ the action of Dunkl operator can be rewritten as

$$\mathcal{D}_i^{(N)} f_i = x_i \frac{\partial}{\partial x_i} f_i + \beta \sum_{j \neq i} \frac{x_j}{x_i - x_j}(f_i - f_j).$$

The procedure can be illustrated by the following matrix formula:

$$H_k = (1, 1, \dots) \begin{pmatrix} x_1 \frac{\partial}{\partial x_1} + \beta \sum_{i=2}^{N} \frac{x_1}{x_1 - x_i} & -\beta \frac{x_1}{x_1 - x_2} & \dots \\ -\beta \frac{x_2}{x_2 - x_1} & x_2 \frac{\partial}{\partial x_2} + \beta \sum_{i \neq 2} \frac{x_2}{x_2 - x_i} & \vdots \\ \vdots & \dots & \ddots \end{pmatrix}^k \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_N \end{pmatrix}$$

Here $f_k \equiv f$.

The main idea is to regard the equivariant Heckman-Dunkl operators as a quantum L-operator acting on the space of polynomial functions of one variable with coefficients being symmetric polynomials of the remaining $N - 1$ variables.

The limit is defined as a projective limit of finite models $\hat{\Lambda} = \varprojlim \Lambda_N^+$

$\hat{\Lambda}$ is a free commutative algebra generated by Newton sums $p_k = \sum_i x_i^k$

$$\begin{array}{ccc} \hat{\Lambda} & \xrightarrow{\mathscr{H}_k} & \hat{\Lambda} \\ \Big\downarrow{\scriptstyle \alpha_N} & & \Big\downarrow{\scriptstyle \alpha_N} \\ \Lambda_N^+ & \xrightarrow[H_k^{(N)}]{} & \Lambda_N^+ \end{array}$$

where $\alpha_N : \hat{\Lambda} \to \Lambda_N^+$ the canonical projection $p_k \to p_k^{(N)} = \sum_{i=1}^N x_i^k$.

$$\mathscr{H}_2 = \sum_{k>0,n>0} knp_{k+n} \frac{\partial}{\partial p_k} \frac{\partial}{\partial p_n} + \sum_{n>0} n^2 p_n \frac{\partial}{\partial p_n} -$$

$$-\beta \sum_{n>0} n^2 p_n \frac{\partial}{\partial p_n} + \beta \sum_{k>0,n\geqslant 0} (k+n) p_k p_n \frac{\partial}{\partial p_{k+n}}$$

The space $\hat{\Lambda}$ is an irreducible representation of the Heisenberg algebra, generated by the elements p_n and $\frac{\partial}{\partial p_n}$ and can be regarded as a polynomial version of the Fock space. It contains the vacuum vector $|0\rangle_+$ and dual vacuum vector ${}_+\langle 0|$, such that

$$\frac{\partial}{\partial p_n}|0\rangle_+ = 0, \qquad {}_+\langle 0|p_n = 0, \qquad n = 0, 1, \ldots .$$

Introduce an operator $\Phi(z) : \hat{\Lambda} \otimes \mathbb{C}[z] \to \hat{\Lambda} \otimes \mathbb{C}[z]$

$$\Phi(z) = \exp\left(\sum_{n \geqslant 0} z^n \frac{\partial}{\partial p_n}\right).$$

The canonical projection $\alpha_N : \hat{\Lambda} \to \Lambda_N^+$ can be defined for an element $|v\rangle_+ = F(p_0, p_1, p_2, \ldots)|0\rangle_+ \in \hat{\Lambda}$:

$$\alpha_N|v\rangle_+ = {}_+\langle 0|\Phi(x_N)\ldots\Phi(x_2)\Phi(x_1)|v\rangle_+.$$

Indeed,

$$\Phi(x_N)\ldots\Phi(x_2)\Phi(x_1)|v\rangle_+ = F(p_0 + N, p_1 + \sum_{i=1}^N x_i, p_2 + \sum_{i=1}^N x_i^2, \ldots)|0\rangle_+.$$

We have the following equality of linear maps $\hat{\Lambda} \to \mathbb{C}[x_i] \otimes \Lambda^+_{N-1}$:

$$(1 \otimes \alpha_{N-1})\Phi(x_i) = \iota_{i,N}\alpha_N.$$

Applying both sides to an element $|v\rangle_+ \in \hat{\Lambda}$ we get

$$_+\langle 0|\Phi(x_N)\dots\Phi(x_2)\Phi(x_1)|v\rangle_+.$$

Introduce an operator $\Phi^*(z) : \hat{\Lambda} \otimes \mathbb{C}[z] \to \hat{\Lambda} \otimes \mathbb{C}[z]$:

$$\Phi^*(z) = \varphi^-(z) \exp\left(-\sum_{n\geqslant 0} z^n \frac{\partial}{\partial p_n}\right), \quad \text{where} \qquad \varphi^-(z) = \sum_{n=0}^{\infty} \frac{p_n}{z^n}.$$

Define a linear map $\mathcal{S} : \hat{\Lambda} \otimes \mathbb{C}[z] \to \hat{\Lambda}$ as

$$\mathcal{S}F(\{p_n\}) = \oint \frac{d\xi}{\xi}\Phi^*(\xi)F(\xi, \{p_n\}).$$

The map $\mathcal{S}$ is the pullback of the finite symmetrization:

$$E_N(\alpha_{N-1} \otimes 1)F(z, \{p_n\}) = \alpha_N \mathcal{S}(F(z, \{p_n\})).$$

Dunkl operator in the limit

Define an operator $\mathcal{D} : \hat{\Lambda} \otimes \mathbb{C}[z] \to \hat{\Lambda} \otimes \mathbb{C}[z]$:

$$\mathcal{D}(F(z, \{p_n\})) = z \frac{\partial}{\partial z} F(z, \{p_n\}) + \beta z \oint \frac{d\xi}{\xi^2} \frac{1}{1 - \frac{z}{\xi}} \Phi^*(\xi) \Phi(z) F(\xi, \{p_n\}).$$

The operator $\mathcal{D}$ is the pullback of the equivariant family of Heckman operators $\mathcal{D}_i^{(N)}$.

$$\begin{array}{ccc} \hat{\Lambda} \otimes \mathbb{C}[z] & \xrightarrow{\alpha_{N-1} \otimes 1} & \Lambda_{N-1}^{+} \otimes \mathbb{C}[z] \\ \Big\downarrow \mathcal{D} & & \Big\downarrow \mathcal{D}_i^{(N)} \\ \hat{\Lambda} \otimes \mathbb{C}[z] & \xrightarrow[\alpha_{N-1} \otimes 1]{} & \Lambda_{N-1}^{+} \otimes \mathbb{C}[z] \end{array}$$

The operators $\tilde{\mathcal{H}}_k : \hat{\Lambda} \xrightarrow{\Phi(z)} \hat{\Lambda} \otimes \mathbb{C}[z] \xrightarrow{\mathcal{D}^k} \hat{\Lambda} \otimes \mathbb{C}[z] \xrightarrow{\mathcal{S}} \hat{\Lambda}$ generate a commutative family of Hamiltonians.

Antisymmetric construction

The Hamiltonian H_k can be presented as

$$\begin{array}{ccc} \Lambda^-[x_1,\dots,x_N] & \xrightarrow{\bar{\iota}_i} & \mathbb{C}[x_i]\otimes\Lambda^-[x_1,\dots x_{i-1},x_{i+1},\dots x_N] \\ & & \Big\downarrow {D_i}^k \\ \Lambda^-[x_1,\dots,x_N] & \xleftarrow[\bar{\mathcal{E}}]{} & \mathbb{C}[x_i]\otimes\Lambda^-[x_1,\dots x_{i-1},x_{i+1},\dots x_N] \end{array},$$

where $\bar{\iota}_i$ is the natural embedding and $\bar{\mathcal{E}}$ is the total antisymmetrization: for any polynomial $\bar{f}(x_i;x_1,\dots,x_N)\in\mathbb{C}[x_i]\otimes\Lambda^-[x_1,\dots x_{i-1},x_{i+1},\dots x_N]$

$$(\bar{\mathcal{E}}\bar{f})(x_1,...,x_N)=\bar{f}(x_1;x_2,...,x_N)-\bar{f}(x_2;x_1,x_3,...,x_N)-\dots$$
$$-\bar{f}(x_N;x_1,...,x_{N-1}).$$

Bosonic Fock space

The bosonic Fock space $\mathcal{F}$ is usually realized as irreducible representation of the Heisenberg algebra $\mathcal{H}$ with generators a_n, $n \in \mathbb{Z}$ and q:

$$[a_n, a_m] = n\delta_{m,n}, \qquad [q, a_n] = q\delta_{0,n}.$$

A vacuum vector $|0\rangle$ and a dual vacuum $\langle 0|$ of $\mathcal{F}$ satisfy the following relations

$$a_n|0\rangle = 0, \quad \langle 0|a_{-n} = 0, \quad n \geq 0.$$

Denote by $\langle n|$ and $|n\rangle$ the following vectors:

$$|n\rangle = q^{-\ n}|0\rangle, \qquad \langle n| = \langle 0|q^n,$$

these vectors are biorthogonal $\langle n|m\rangle = \delta_{n,m}$ and have the following properties

$$\langle n|a_0 = n\langle n|, \quad a_0|n\rangle = n|n\rangle.$$

Further we identify generators a_n and their following representation

$$a_{-n} = p_n, \quad a_n = n\frac{\partial}{\partial p_n}, \quad n > 0, \quad a_0 = p_0, \quad q = e^{\frac{\partial}{\partial p_0}}.$$

Define a map $\pi_N : \mathcal{F} \to \Lambda^-[x_1, \ldots x_N]$ by the formula

$$\pi_N(v) = \langle 0|\Psi(x_N)\cdots\Psi(x_1)|v\rangle,$$

where we use vertex operators:

$$\Psi(z) = z^{p_0} \exp\left(-\sum_{n>0} \frac{p_n}{nz^n}\right) \exp\left(\sum_{n\geq 0} z^n \frac{\partial}{\partial p_n}\right),$$

$$\Psi^*(z) = z^{-p_0} \exp\left(\sum_{n>0} \frac{p_n}{nz^n}\right) \exp\left(-\sum_{n\geq 0} z^n \frac{\partial}{\partial p_n}\right).$$

Indeed, for $|v\rangle = f(p_1, ..., p_k, ...)|N\rangle$ we have

$$\pi_N|v\rangle = \prod_{i<j}(x_i - x_j) F\left(N, (x_1 + \ldots + x_N), ..., (x_1^k + \cdots + x_N^k), ...\right)$$

due to $\Psi(z)\Psi(y) = (y - z) \vdots\Psi(z)\Psi(y)\vdots$

Fermionic Fock space

The Clifford algebra is generated by fermions ψ_k, ψ_k^* for $k \in \mathbb{Z}$ with anti-commutation relations

$$\psi_i\psi_j + \psi_j\psi_i = \psi_i^*\psi_j^* + \psi_j^*\psi_i^* = 0,$$

$$\psi_i\psi_j^* + \psi_j^*\psi_i = \delta_{ij}.$$

The fermionic Fock space $\mathcal{F}$ can be defined as a representation of the Clifford algebra, where the vacuum vector $|0\rangle$ is defined as follows:

$$\psi_n|0\rangle = 0 \quad n \geq 0, \quad \psi_n^*|0\rangle = 0 \quad n < 0.$$

$\Psi(z)$ and $\Psi^*(z)$ are the following generating functions:

$$\Psi(z) = \sum_{k\in\mathbb{Z}} \psi_k z^k \qquad \Psi^*(z) = \sum_{k\in\mathbb{Z}} \psi_k^* z^{-k-1}$$

Fermionic limit of the CS model in the Fock space $\mathcal{F}$

(joint work with Sergey Khoroshkin)

How to obtain the commutative family of Hamiltonians:

$$\mathscr{H}_k : \mathcal{F} \xrightarrow{\iota} \mathbb{C}[z, z^{-1}]] \otimes \mathcal{F} \xrightarrow{\mathscr{D}^k} \mathbb{C}[z, z^{-1}]] \otimes \mathcal{F} \xrightarrow{\mathcal{A}} \mathcal{F}$$

$$\iota F = \Psi(z) F,$$

$$\mathcal{A} F(z) = \frac{1}{(2\pi i)^2} \int_{z \circlearrowleft 0} dz \int_{u \circlearrowleft z} du \frac{\Psi^*(u) F(z)}{u - z},$$

$$\mathscr{D} F(z) = z \frac{\partial}{\partial z} F(z) + \beta \frac{1}{(2\pi i)^2} \int_{w \circlearrowleft 0} \int_{u \circlearrowleft w} \frac{dw du}{u - w} \frac{\Psi^*(u)}{1 - \frac{w}{z}} \left(\Psi(w) F(z) - \Psi(z) F(w) \right)$$

Here $F \in \mathcal{F}$ and $F(z) \in \mathcal{F} \otimes \mathbb{C}[z, z^{-1}]]$.

Fermionic limit of the CS model in the Fock space $\mathcal{F}$

How to obtain the commutative family of Hamiltonians:

$$\begin{array}{ccccccc} \mathscr{H}_k : \mathcal{F} & \xrightarrow{\iota} & \mathbb{C}[z,z^{-1}]]\otimes\mathcal{F} & \xrightarrow{\mathscr{D}^k} & \mathbb{C}[z,z^{-1}]]\otimes\mathcal{F} & \xrightarrow{\mathcal{A}} & \mathcal{F} \\ \downarrow \pi_N & & \downarrow 1\otimes\pi_{N-1} & & \downarrow 1\otimes\pi_{N-1} & & \downarrow \pi_N \\ \Lambda_N^- & \xrightarrow{\iota_{N,i}} & \mathbb{C}[x_i]\otimes\Lambda_{N-1}^- & \xrightarrow{\mathcal{D}_i^k} & \mathbb{C}[x_i]\otimes\Lambda_{N-1}^- & \xrightarrow{\bar{\mathcal{E}}} & \Lambda_N^- \end{array}$$

$$\iota F = \Psi(z)F,$$

$$\mathcal{A}F(z) = \frac{1}{(2\pi i)^2}\int_{z\circlearrowleft 0} dz \int_{u\circlearrowleft z} du \frac{\Psi^*(u)F(z)}{u-z},$$

$$\mathscr{D}F(z) = z\frac{\partial}{\partial z}F(z) + \beta\frac{1}{(2\pi i)^2}\int_{w\circlearrowleft 0}\int_{u\circlearrowleft w}\frac{dwdu}{u-w}\frac{\Psi^*(u)}{1-\frac{w}{z}}\left(\Psi(w)F(z)-\Psi(z)F(w)\right)$$

Here $F \in \mathcal{F}$ and $F(z) \in \mathcal{F}\otimes\mathbb{C}[z,z^{-1}]]$.

Hamiltonian

$$\mathscr{H}_2 = \sum_{n,k>0} nkp_{n+k}\frac{\partial}{\partial p_n}\frac{\partial}{\partial p_k} + (1+\beta)\sum_{n>0,k\geq 0}(n+k)p_n p_k \frac{\partial}{\partial p_{n+k}}$$
$$-\beta\sum_{n>0} n^2 p_n \frac{\partial}{\partial p_n} + (p_0 - 1)\sum_{n>0} np_n \frac{\partial}{\partial p_n} +$$
$$+\frac{1}{6}(2p_0^3 - 3p_0^2 + p_0) + \frac{\beta}{6}p_0(p_0^2 - 1).$$

Comparison

Bosonic	Fermionic
space $\hat{\Lambda} = \Lambda[p_0]$	Fock space $\mathcal{F}$
projection $\alpha_N(v) = {}_+\langle 0 \vert \Phi(z_N)\cdots\Phi(z_2)\Phi(z_1) \vert v\rangle_+$	projection $\pi_N(v) = \langle 0 \vert \Psi(z_N)\cdots\Psi(z_2)\Psi(z_1) \vert v\rangle$
$\Phi(z) = \exp\left(\sum_{n\geqslant 0} z^n \frac{\partial}{\partial p_n}\right)$ $\Phi^*(z) = \varphi^-(z)\Phi^{-1}(z)$	$\Psi(z) = z^{p_0}\exp\left(-\sum_{n>0}\frac{p_n}{nz^n}\right)\exp\left(\sum_{n\geq 0} z^n \frac{\partial}{\partial p_n}\right)$ $\Psi^*(z) = z^{-p_0}\exp\left(\sum_{n>0}\frac{p_n}{nz^n}\right)\exp\left(-\sum_{n\geq 0} z^n \frac{\partial}{\partial p_n}\right)$
Symmetrization $\mathcal{S}(\boldsymbol{F}(z))$ $\frac{1}{2\pi i}\oint \Phi(z)\boldsymbol{F}(z)\frac{dz}{z}$	Antisymmetrization $\mathcal{A}(\boldsymbol{F}(z))$ $\frac{1}{(2\pi i)^2}\int_{z\circlearrowleft 0} dz \int_{u\circlearrowleft z} du \frac{\Psi(u)\boldsymbol{F}(z)}{u-z}$

Thank you for your attention!