Integrability breaking in extended Hamiltonian systems

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Plan

I - Diffusive transport

II - Slowing down transport: Frequency mismatch

III - Rigorous results

IV - Puzzles at $T = 0$

V - Many-body localization

VI - Technical: Green-Kubo formula, role of the noise

Main collaborator

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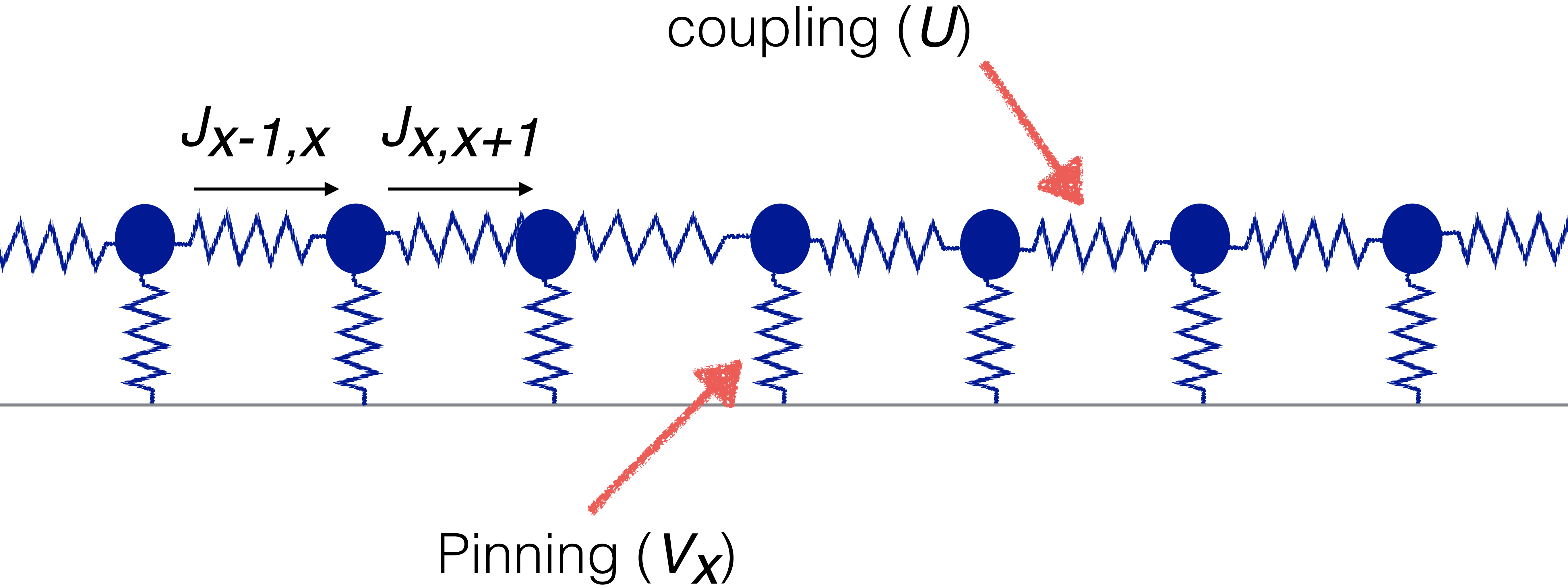
I - Transport

Chain of oscillators



$$H(p,q) = \sum_{x=1}^{L} \frac{p_x^2}{2m} + V_x(q_x) + \sum_{x=1}^{L-1} U(q_x - q_{x+1})$$

$$(p,q) \in \mathbb{R}^{2L}$$

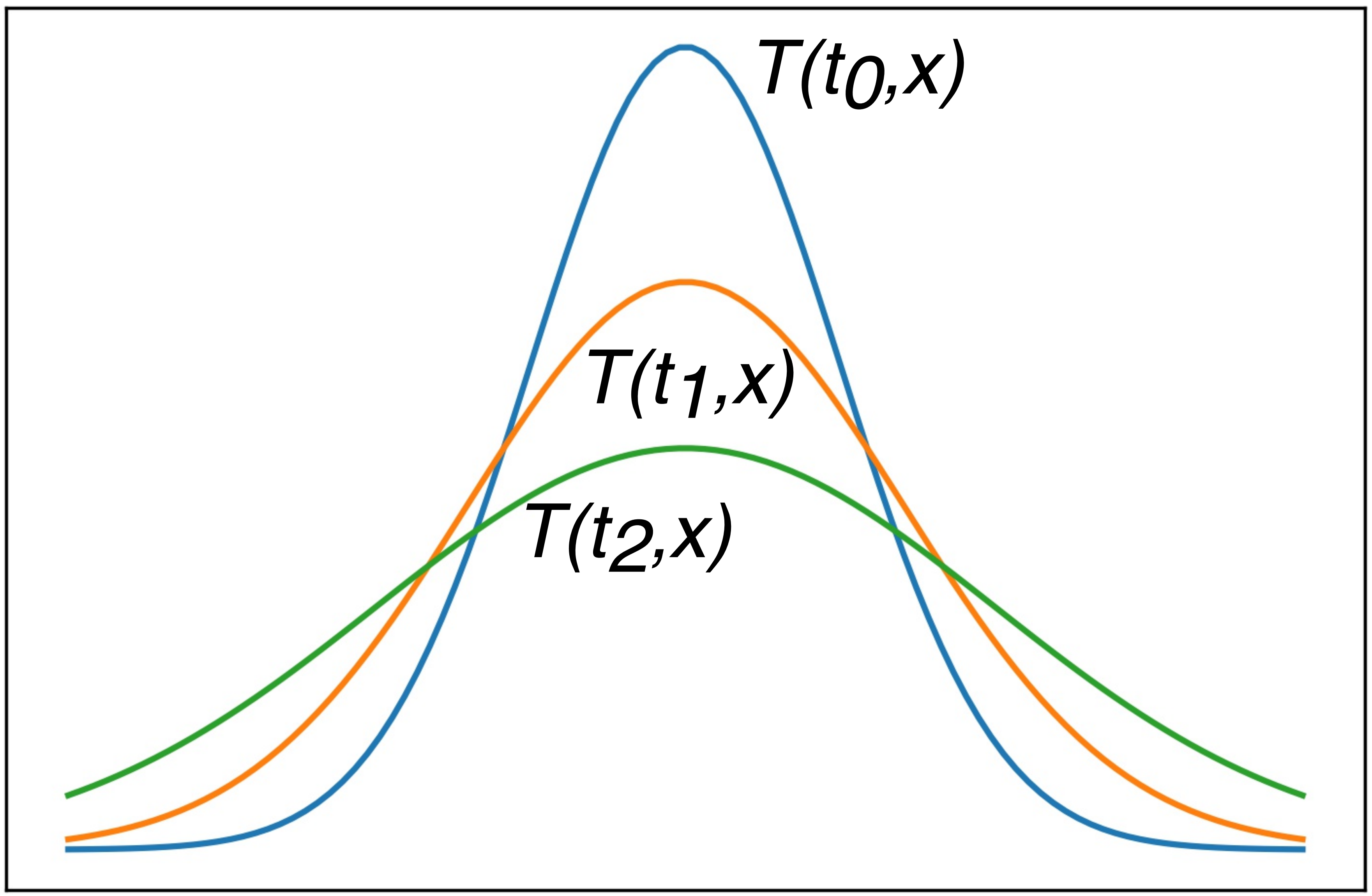
Conservation of energy

$$H(p,q) = \sum_{x=1}^{L} H_x(p,q)$$

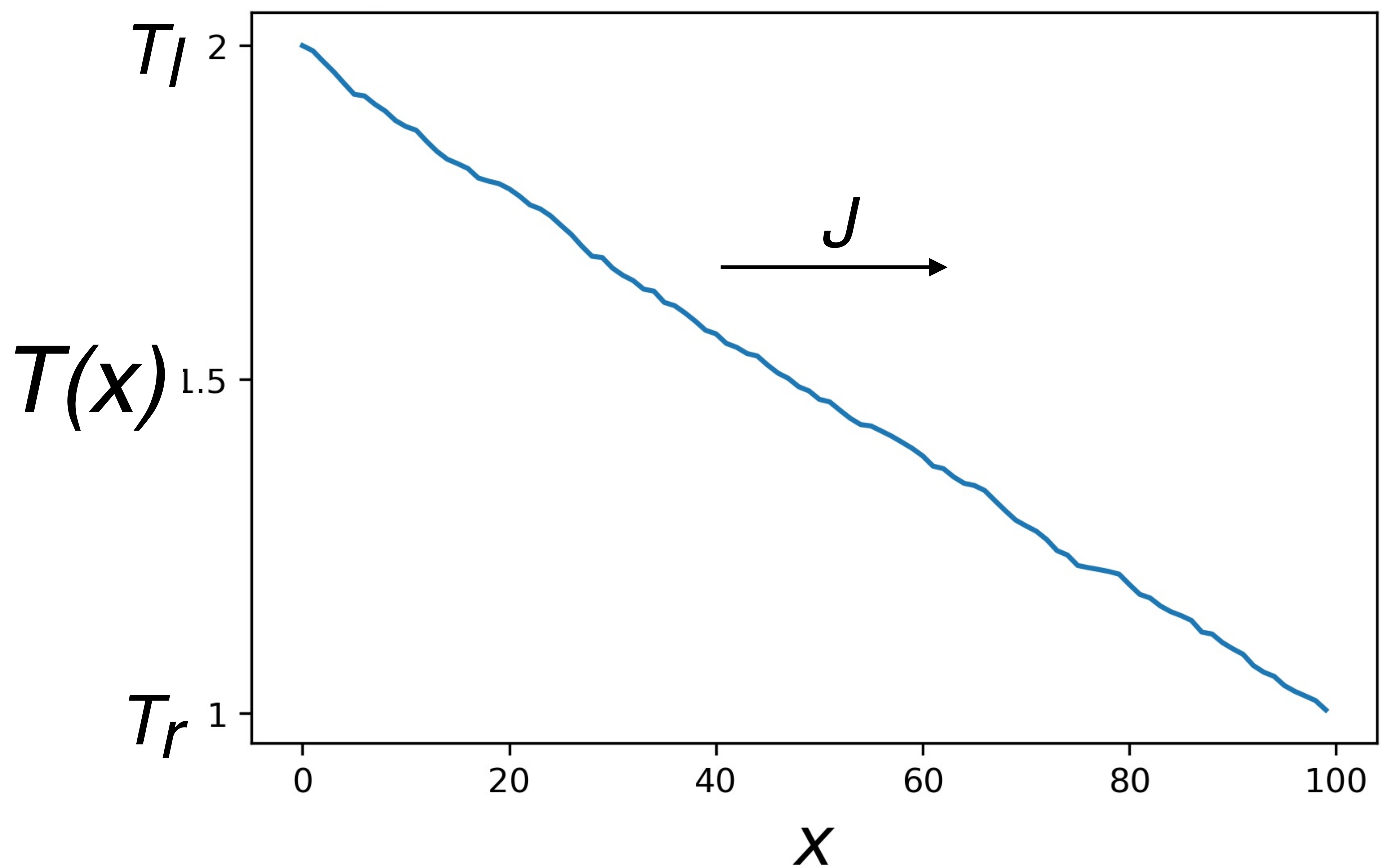
$$\frac{dH_x}{dt} = j_{x-1,x} - j_{x,x+1}$$

$$j_{x,x+1} = \frac{p_x}{m} U'(q_x - q_{x+1})$$

1) Time evolution of an energy profile



2) Steady state profile



$$J \;=\; \langle j_{x,x+1}(t)\rangle_{NESS} \quad \forall t, x$$

Scaling of *J* with *L*?

How to model the baths?

Most popular: Langevin thermostats

$$\dot{p}_1 = -\frac{\partial H}{\partial q_1} - \gamma p_1 + \sqrt{2\gamma m T_\ell}\,\xi_\ell$$

$$\dot{p}_L = -\frac{\partial H}{\partial q_L} - \gamma p_L + \sqrt{2\gamma m T_r}\,\xi_r$$

Common expectations:

1) There exists a unique NESS

2) Local equilibrium sets in:

Fix a site ***x.*** Given a local observable O_x, as $L \rightarrow \infty$

$$\langle O_x \rangle_{NESS} \quad \rightarrow \quad \langle O_x \rangle_{T(x/L)}$$

for some appropriate temperature T

Common expectations:

3) Fourier's law: As ***L —> ∞***

$$\mathsf{J} \;=\; -\kappa(\mathsf{T}(\mathsf{x}/\mathsf{L}))\,\frac{1}{\mathsf{L}}\,\nabla\mathsf{T}(\mathsf{x}/\mathsf{L}) + o(1/\mathsf{L})$$

Thermal conductivity

Normal scaling of the current

What do we know about κ?

1) From thermodynamics: *κ(T)>0*

Entropy increases: current flows from hot to cold

Cf. J-P Eckmann, C-A Pillet and L Rey-Bellet, JSP 1998

2) It is expressed by the Green-Kubo formula:

$$\kappa(T) = \frac{1}{T^2} \lim_{t\to\infty} \lim_{L\to\infty} \left\langle \left(\frac{1}{\sqrt{t}} \int_0^t \frac{1}{\sqrt{L}} \sum_{x=1}^{L-1} j_{x,x+1}(s) ds \right)^2 \right\rangle_T$$

Reminder: Gibbs state

$$\langle f \rangle_T \; = \; \frac{1}{Z(T)} \int_{\mathbb{R}^{2L}} f(p,q) \mathrm{e}^{-H(p,q)/T}$$

$$Z(T) \quad s.t. \quad \langle 1 \rangle_T = 1 \quad (probability\ measure)$$

II - Frequency mismatch

2 harmonic oscillators

$$H = \frac{p_1^2}{2m} + \frac{\omega_1^2 q_1^2}{2} + g\frac{(q_1 - q_2)^2}{2} + \frac{p_2^2}{2m} + \frac{\omega_2^2 q_2^2}{2}$$

There are two eigenmodes = eigenvectors of

$$\begin{pmatrix} \omega_1 & g \\ g & \omega_2 \end{pmatrix}$$

The eigenvectors are localized if $|g| \ll |\omega_1 - \omega_2|$

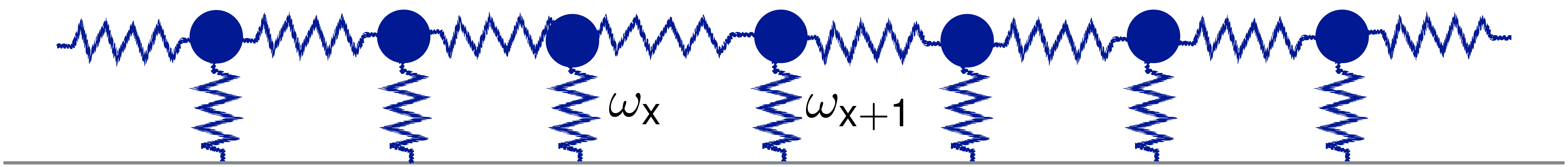
$$v_1 \sim (1, 0) \qquad v_2 \sim (0, 1)$$

Anderson Localization

Disordered harmonic chain : $(\mathsf{p},\mathsf{q}) \in \mathbb{R}^{2L}$

$$H(p,q) = \frac{1}{2} \sum_{1 \le x \le L} p_x^2 + \omega_x^2 q_x^2 + g(q_x - q_{x+1})^2$$

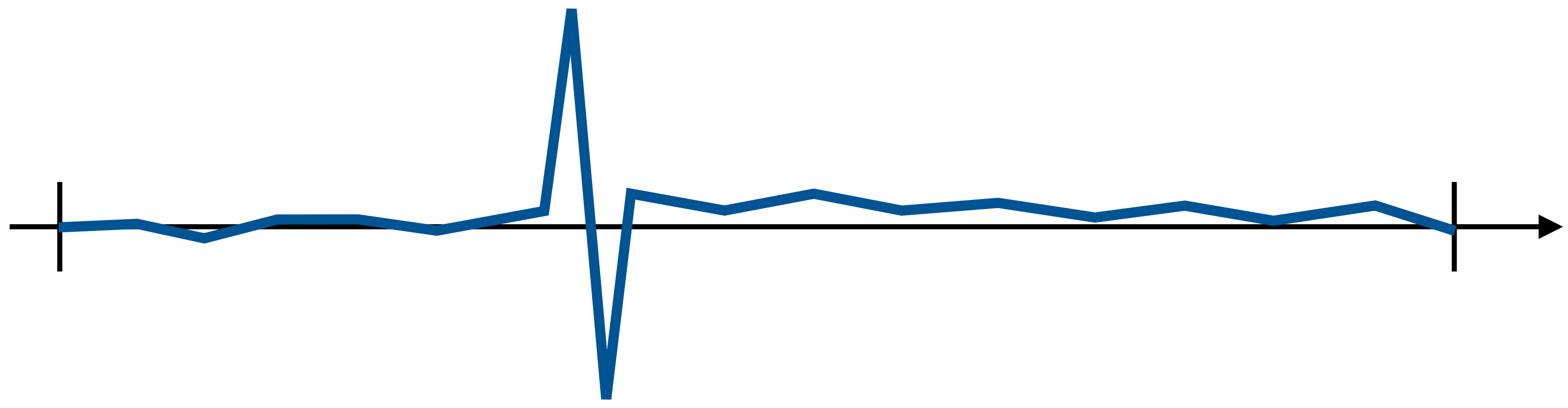
$$0 < \omega_- < \omega_x < \omega_+ \quad \text{iid}$$



Newton : $\ddot{q} = (g\Delta - V)q, \qquad V_{x,y} = \delta(x-y)\omega_x^2$

Localized eigenmodes

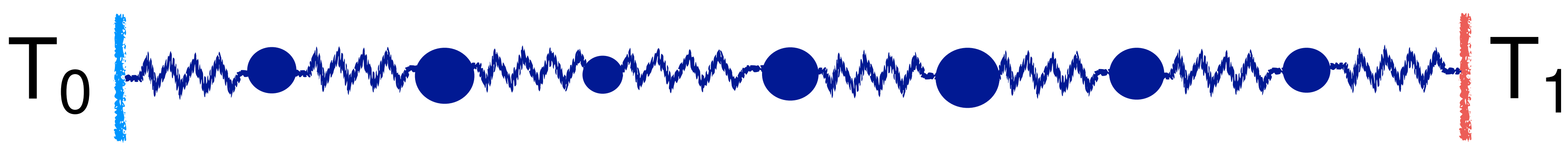
Eigenmodes of $g\Delta - V$



d = 1: localized for all values of g

d >2: localized if $|g| \ll \Delta\omega$

With heat baths:



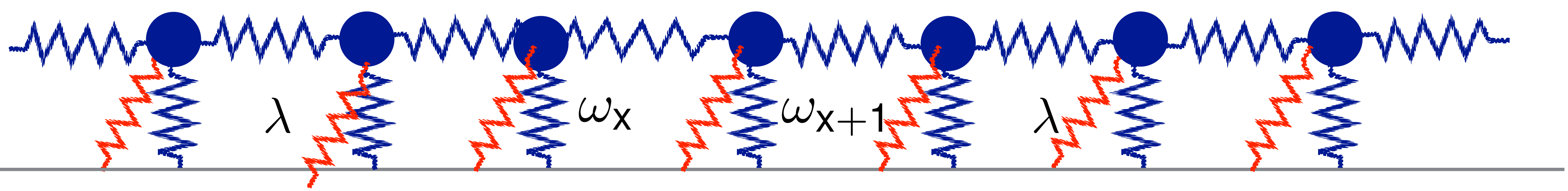
Very bad coupling, thus: $J \sim e^{-L/\xi}, \quad \xi > 0$

Introduce interactions among modes

$$H(p,q) = \frac{1}{2} \sum_{1 \le x \le L} p_x^2 + \omega_x^2 q_x^2 + g(q_{x+1} - q_x)^2 + \lambda\, q_x^4$$

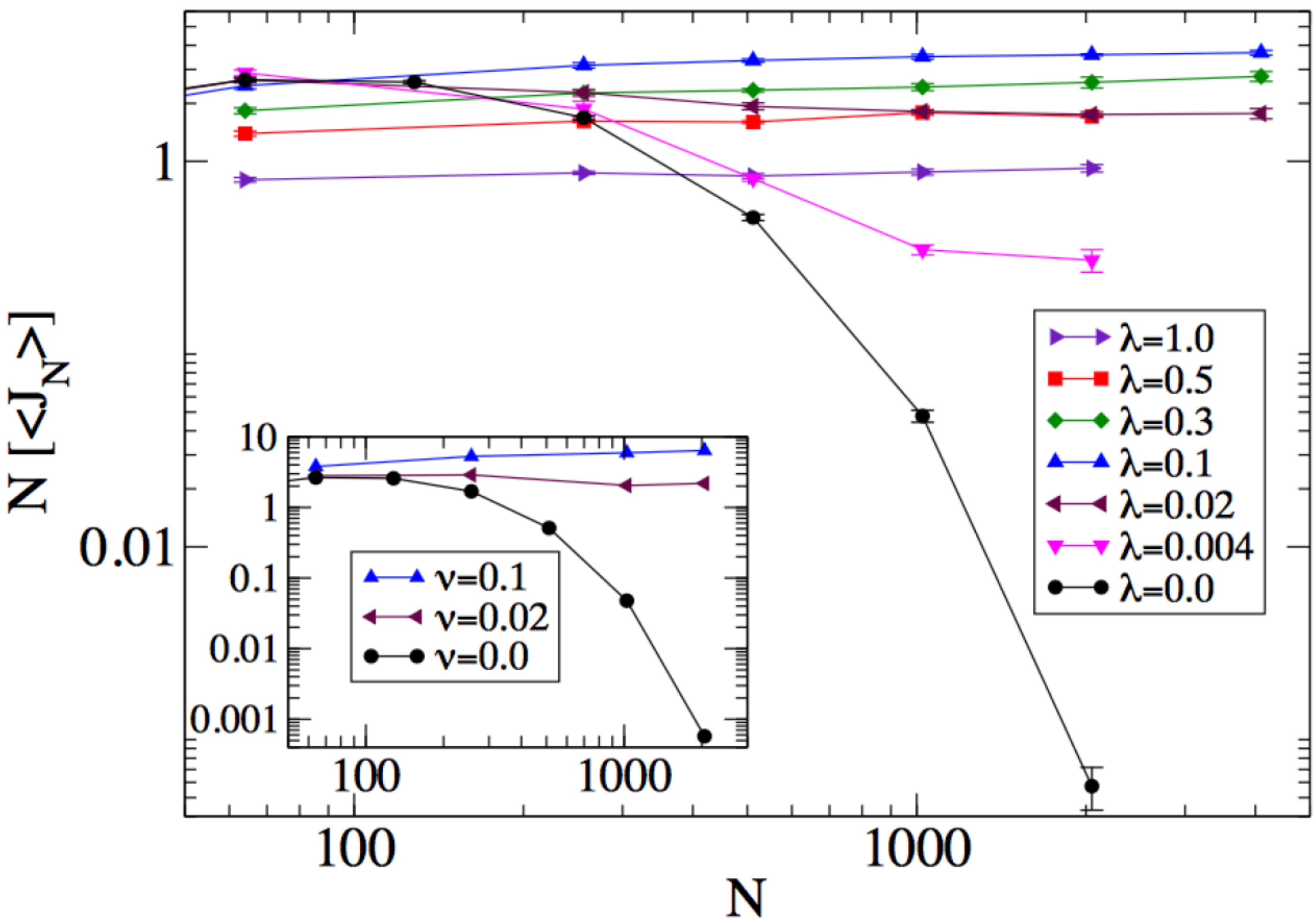
disordered harmonic chain
=
localized system

anharmonic pinning
=
interaction among modes



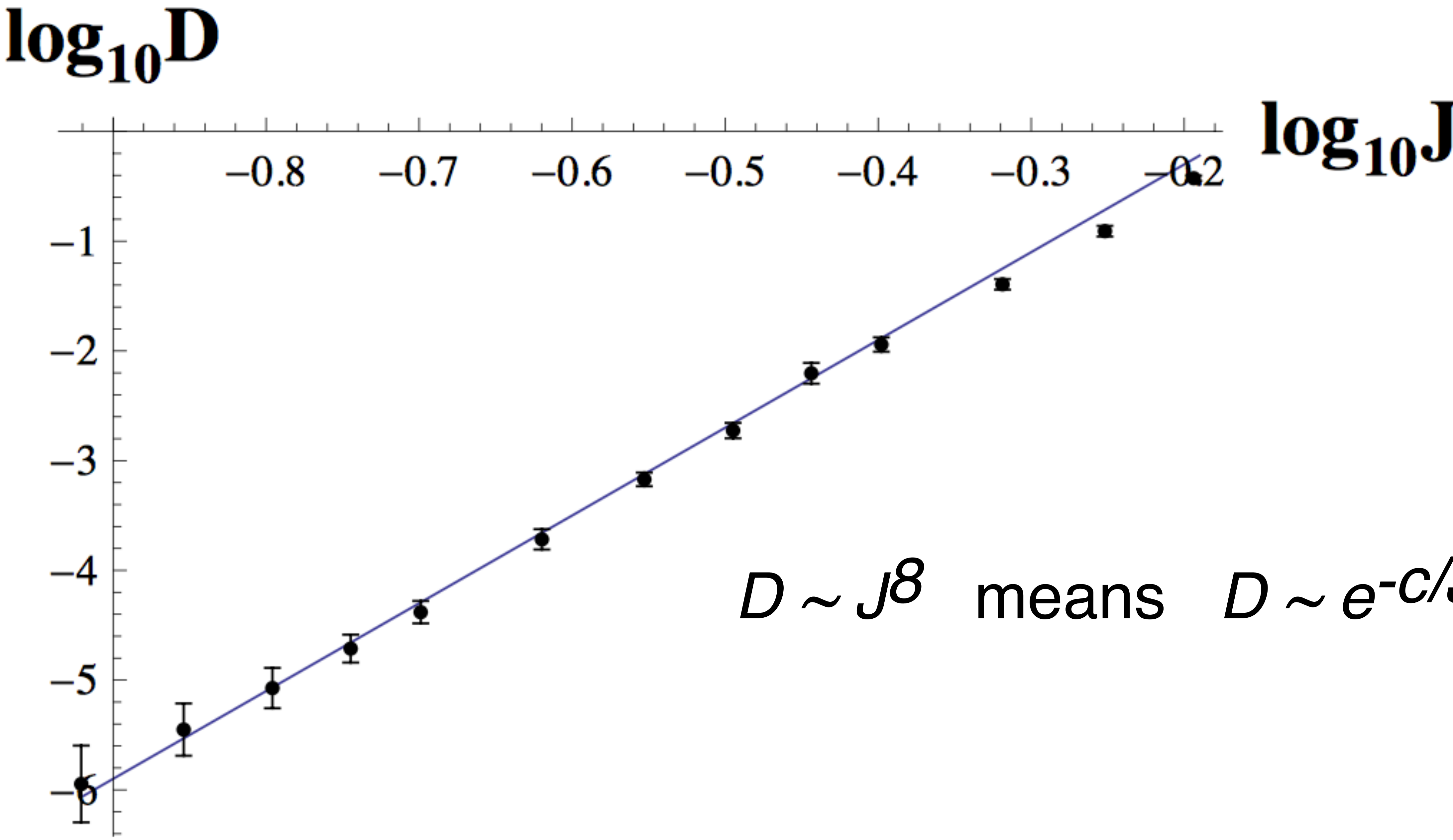
How about **κ(T)**?

Normal conductivity



From Dhar and Lebowitz, PRL 2008

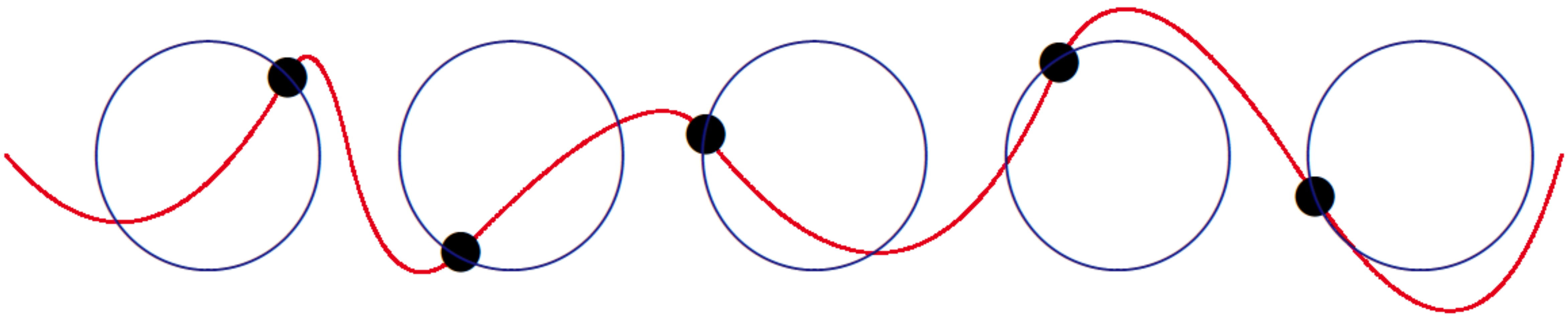
Small normal conductivity



$D \sim J^8$ means $D \sim e^{-c/J}$?

From Oganesyan, Pal and Huse, PRB 2009

The rotor chain



$$H(\omega,\theta) \;=\; \sum_{x=1}^{L} \frac{\omega_x^2}{2} + g(1-\cos(\theta_x-\theta_{x+1}))$$

For small g, ω_x are almost i.i.d. in the Gibbs state:

same phenomenology, cf. KAM, Nekhoroshev at finite L

III - Rigorous results

Difficulty and way out

Problem: We don't know whether $\kappa(T, \lambda) < +\infty$

Introduce some noise s.t.

1) Energy still conserved
2) Make G-K integral convergent
3) Not too large to keep Hamiltonian effects

E.g.: add velocity flip

Liouville operator

Noise

$$\mathcal{L} = \mathcal{A} + \gamma \mathcal{S}$$

$$\mathcal{S} f(p, q) = \sum_x f(\dots, -p_x, \dots, q) - f(p, q)$$

Results

Chains like before:

$$\mathsf{H}(\mathsf{p},\mathsf{q}) = \sum_{\mathsf{x}=1}^{\mathsf{L}} \frac{\mathsf{p}_\mathsf{x}^2}{2\mathsf{m}} + \frac{\omega_\mathsf{x}^2 \mathsf{q}_\mathsf{x}^2}{2} + \lambda(\mathsf{q}_\mathsf{x} - \mathsf{q}_{\mathsf{x}+1})^4 \qquad \text{(disordered)}$$

$$\mathsf{H}(\theta,\omega) = \sum_{\mathsf{x}=1}^{\mathsf{L}} \frac{\omega_\mathsf{x}^2}{2} + \lambda(1 - \cos(\theta_\mathsf{x} - \theta_{\mathsf{x}+1})) \qquad \text{(strongly anharmonic)}$$

Theorem

$$\forall r \in \mathbb{N}, \exists C(r,T) \quad s.t. \quad \kappa(T,\lambda,\gamma) \leq C(r,T)\left(\frac{\lambda^{2r}}{\gamma} + \gamma\right)$$

Results

Taking $\gamma = \lambda^r$ we get

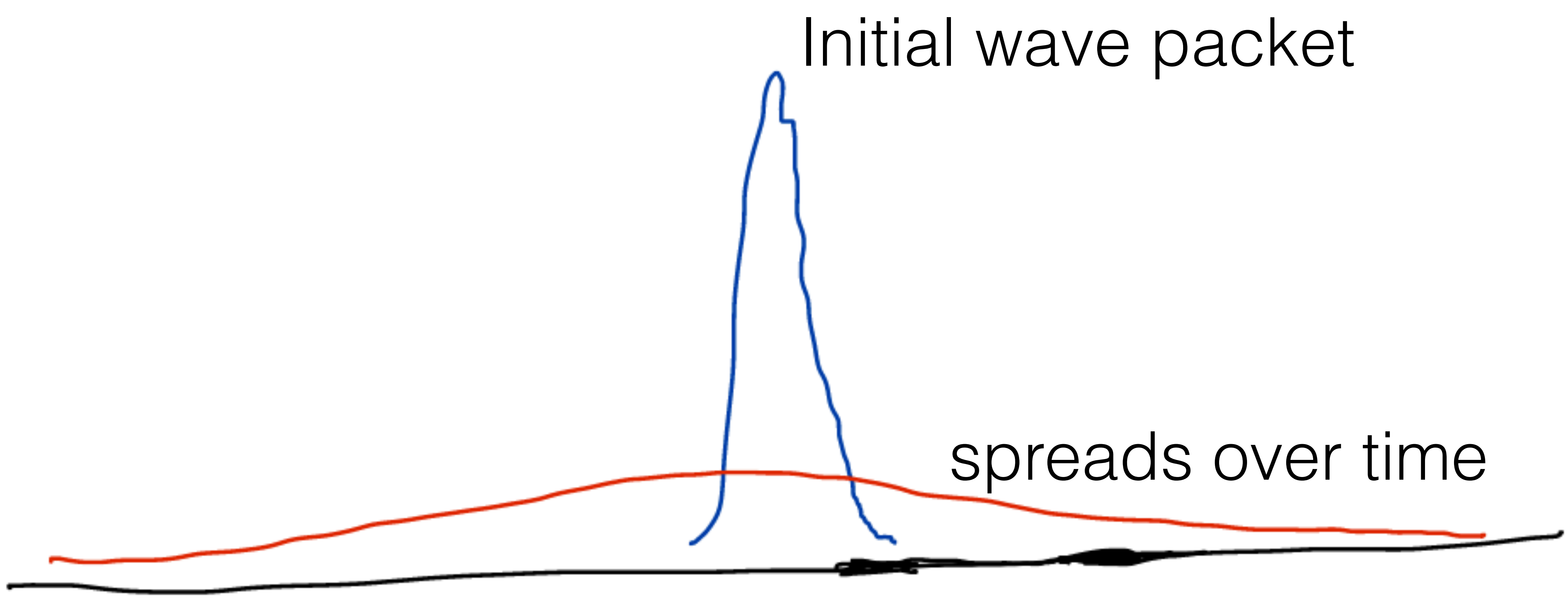
$$\kappa(T, \lambda, \lambda^r) \ \leq \ C(r, T)\lambda^r$$

From physics, we expect the noise to enhance transport, so we think

$$\kappa(T, \lambda, 0) \ \leq \ C(r, T)\lambda^r$$

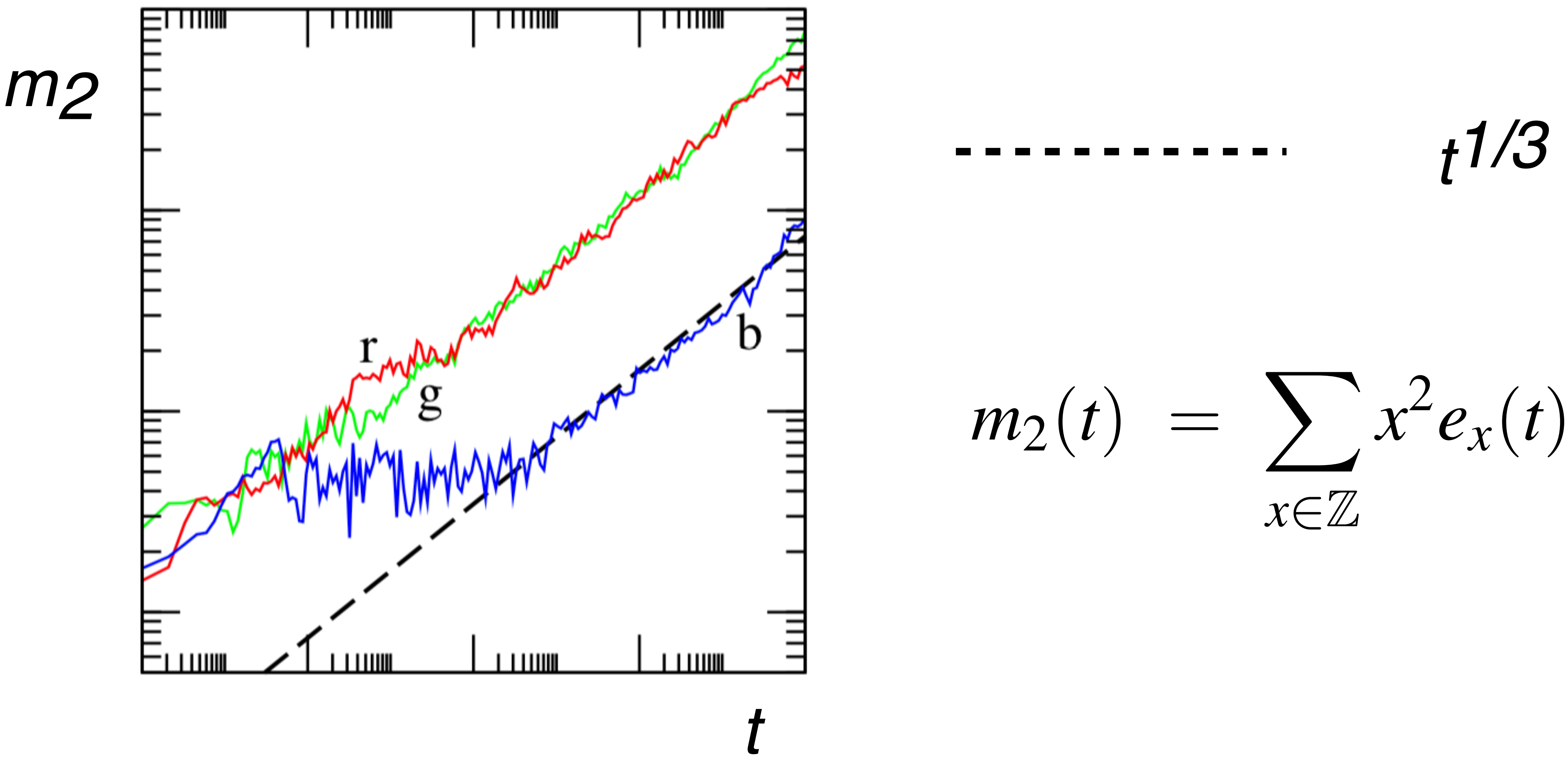
IV - Puzzles at T=0

Spreading at zero temperature



$$H(p,q) = \frac{1}{2} \sum_{1 \leq x \leq L} p_x^2 + \omega_x^2 q_x^2 + g(q_{x+1} - q_x)^2 + \lambda \, q_x^4$$

Spreading at zero temperature



$$m_2(t) \;=\; \sum_{x\in\mathbb{Z}} x^2 e_x(t)$$

From Flach, Krimer and Skokos, PRL 2009

This clashes with our result

We expect local equilibrium to set in, hence

$$\partial_t E \; = \; \partial_x \kappa(T,\lambda)\partial_x E, \quad T = T(E)$$

with $T(t,x) \to 0 \quad \forall x \quad as \quad t \to +\infty$

For this system:

$$\kappa(T,\lambda) = \overline{\kappa}(T\lambda)$$

i.e. low temperature = low effective anharmonic interactions

$m_2(t) \sim t^{1/3}$ only possible if $\overline{\kappa}(T\lambda)$ approach 0 **polynomially**

Way out?

Our theorem does not apply directly to this set-up

Wait longer? $\sqrt{m_2(t)} \sim t^{1/6}$ **very slow**!

(may explain the apparent stability of this regime)

Main idea:

describe numerically how *κ(λT)* behaves,

that this is consistent with the scaling of m_2,

that it must change by virtue of a theorem

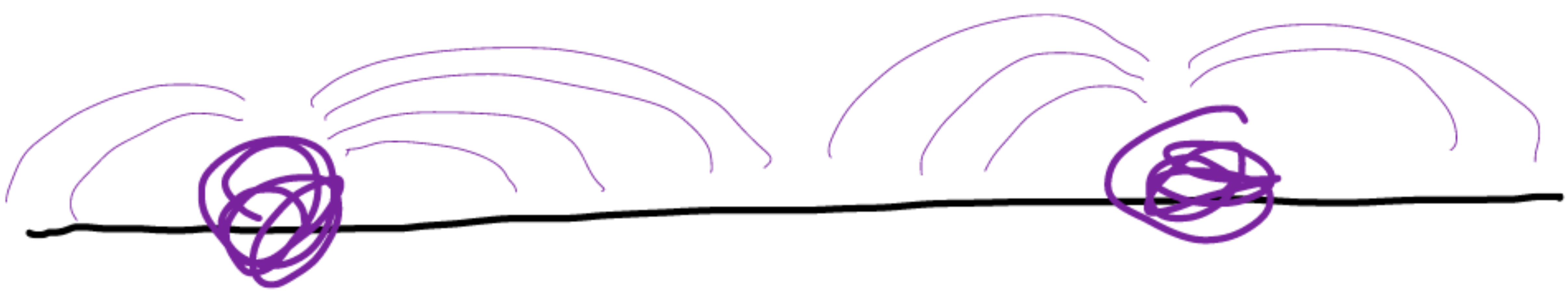
V - Many Body Localization

Classical vs Quantum

For classical systems, why do we expect

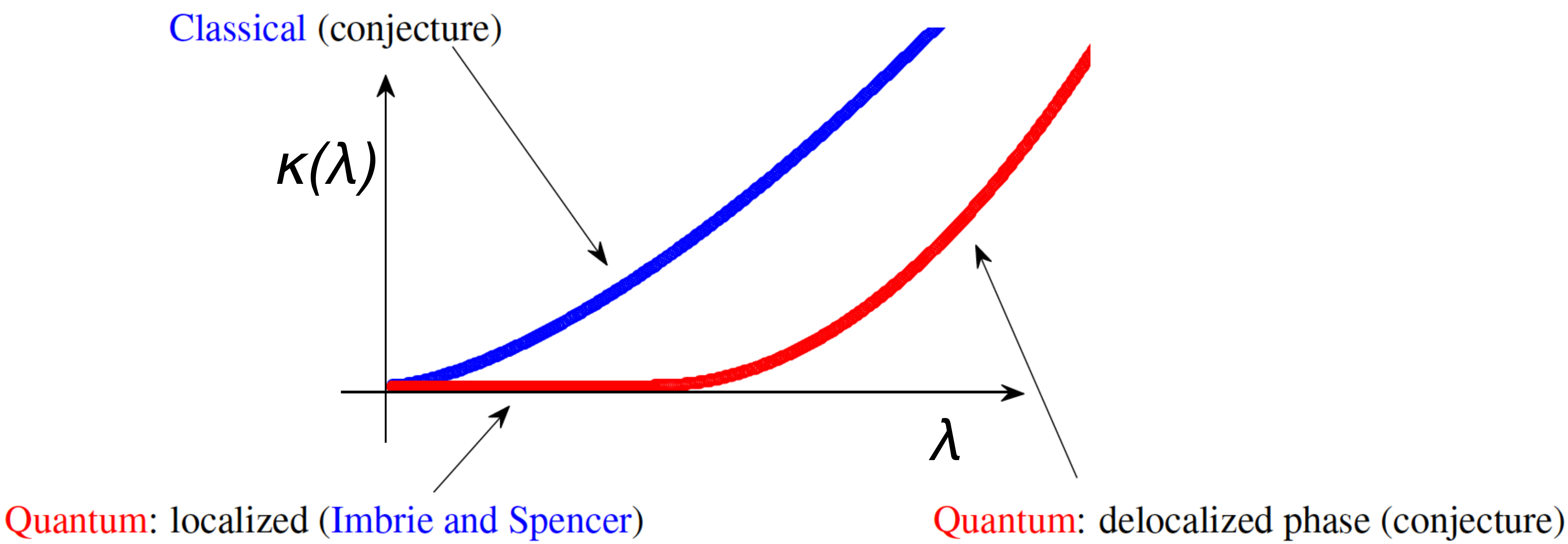
$$\kappa(\lambda, T) > 0 \qquad \text{for} \qquad \lambda > 0\ ?$$

Naive idea: a few oscillators can yield chaos:



This does not need to be true for quantum systems!

Quantum vs Classical

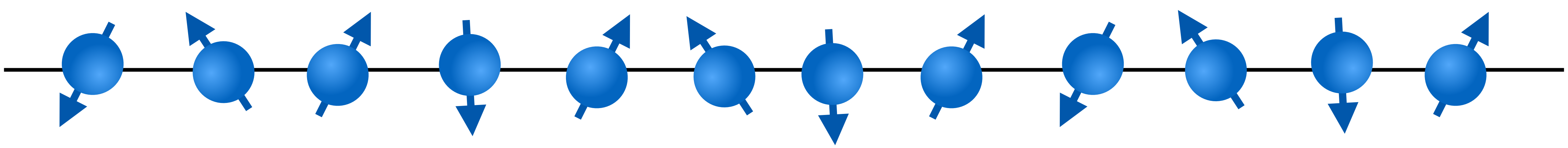


Many-body localized phase

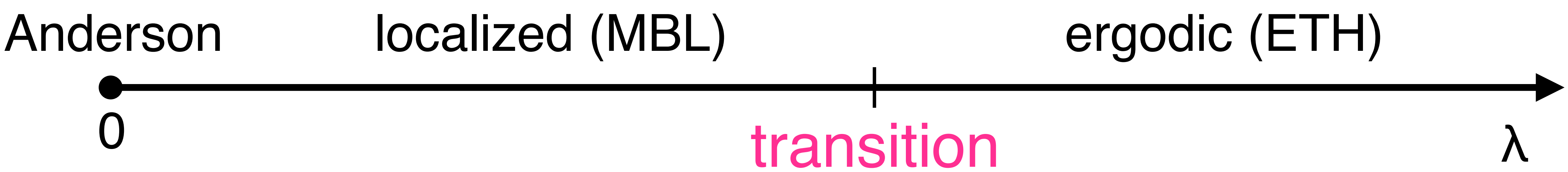
disordered quantum spin chain:

$$H = \sum_{i=1}^{L} h_i \sigma_i^{(z)} + J \sum_{i=1}^{L-1} \sigma_i^{z} \sigma_{i+1}^{(z)} + \lambda \sum_{x=1}^{L} \sigma_i^{(x)}$$

classical Ising model spin flip



Expected phase diagram



Localized means: eigenstates look still a bit like the eigenstates at λ=0:

2^L eigenstates of the classical Ising model:

$$|++-+-+++\cdots+\rangle$$

Localized vs ergodic

Thermal: $$\langle E|\sigma_0^{(z)}|E\rangle \;\sim\; \langle \sigma_0^{(z)}\rangle_{T(E)} \;\sim\; 0$$

where ~0 holds at high temperature

Localized: $$\langle E|\sigma_0^{(z)}|E\rangle \;\sim\; \pm 1 + \mathcal{O}(\lambda)$$

This is like KAM: keeps memory of initial conditions forever

Green-Kubo formula

$T+\Delta T$ ——— NESS ——— $T-\Delta T$

$$\langle J \rangle \sim \kappa(T) \cdot \frac{2\Delta T}{L}$$

Physical limit:

$$\lim_{\Delta T \to 0} \lim_{L \to \infty} \frac{L \langle J \rangle}{2 \Delta T} = \kappa(T)$$

Fourier

Other limit:

$$\lim_{L \to \infty} \lim_{\Delta T \to 0} \frac{L \langle J \rangle}{2 \Delta T} = \kappa(T)$$

(1) $$(L-1) \langle J \rangle_{NESS} = \left\langle \sum_{x=1}^{L-1} j_{x,x+1} \right\rangle_{NESS} = \langle \mathcal{J} \rangle_{NESS}$$

(2) $\Delta T = 0$: $\langle \cdot \rangle_{NESS} = \langle \cdot \rangle_T$

$$\rho : \langle f \rangle_{NESS} = \int f \cdot \rho \, dp \, dq$$

$$\rho = \rho_0 \left(1 + (\Delta T) f + (\Delta T)^2 f^2 + \dots\right)$$

?

Forget

$$\rho_0 = \frac{1}{Z(T)} e^{-H/T}$$

(3) f ? ρ is invariant.

$\mathcal{L}$: generator

$\mathcal{L}^+$: adjoint of $\mathcal{L}$ w.r.t. ρ_0

$$\mathcal{L}^+ \left(1 + \Delta T f + O(\Delta T^2)\right) = 0$$

$$\mathcal{L} = \mathcal{L}_0 + \Delta T \mathcal{L}_1$$

$$\mathcal{L}_1 = \gamma \left(\frac{\partial^2}{\partial p_1^2} - \frac{\partial^2}{\partial p_2^2} \right)$$

$$\left(\mathcal{L}_0^+ + (\Delta T) \mathcal{L}_1^+\right)\left(1 + \Delta T f + O(\Delta T^2)\right) = 0$$

$$\mathcal{L}_0^+ 1 = 0 \qquad \text{OK}$$

$$\mathcal{L}_0^+ f = -\mathcal{L}_1^+ 1$$

$$f = (-\mathcal{L}_0^+)^{-1} \mathcal{L}_1^+ 1$$

4) $$\frac{(L-1)\langle J\rangle_{NESS}}{2\Delta T} = \frac{\langle J\rangle_{NESS}}{2\Delta T}$$

$\frac{d\rho}{d\rho_0}$

$$= \frac{\langle J\cdot(1+\Delta T f + O(\Delta T^2))\rangle_T}{2\Delta T}$$

$$= \frac{\langle J\rangle_T}{2\Delta T} + \frac{\langle J\cdot f\rangle_T}{2} + O(\Delta T)$$

$O(\Delta T) \to 0$

$\frac{\langle J\cdot f\rangle_T}{2} = K(T)$

$\lim_{L\to\infty}$

$$K(T) = \frac{1}{2}\langle J\cdot(-\mathcal{L}_0^+)\mathcal{L}_1^+ 1\rangle_T$$

(5) $$\mathcal{L}_1^+ 1 = \frac{1}{T^2}(\gamma p_1^2 - \gamma p_L^2)$$

$$= \frac{-1}{T^2}(j_{0,1} + j_{L,L+1})$$

$$K(T) = \frac{-1}{2T^2}\langle J, (-\mathcal{L}_0^+)^{-1}(j_{0,1}+j_{L,L+1})\rangle$$

(6) Energy conservation:

$$\mathcal{L}_0 e_x = j_{x-1,x} - j_{x,x+1}$$

$\mathcal{L}_0 = \frac{d}{dt}$

$$\mathcal{L}_0 (e_x + e_{x+1} + \dots + e_y) = j_{x-1,x} - j_{y,y+1}$$

$$j_{0,1} = \frac{1}{L-1} \sum_{x=1}^{L-1} -j_{x,x+1} + \mathcal{L}_0^+ (e_1 + \dots + e_x)$$

(7) $$\langle J, (-\mathcal{L}_0^+)^{-1} \mathcal{L}_0^+ e_z \rangle_T$$

$$= \langle J, e_z \rangle_T = 0$$

$$\quad \text{``} \ p\,\phi(q) \qquad \text{``} \ p^2$$

$$K(T) = \lim_{L\to\infty} \frac{1}{L-1} \frac{1}{T^2} \langle J, (-\mathcal{L}_0)^{-1} J \rangle$$

$$\tilde{J} = \frac{1}{\sqrt{L-1}} \sum_{x=1}^{L-1} j_{x,x+1}$$

$$K(T) = \lim_{L\to\infty} \frac{1}{T^2} \langle \tilde{J}, (-\mathcal{L}_0)^{-1} \tilde{J} \rangle$$

8) $$(-\mathcal{L}_0)^{-1} = \int_0^\infty dt\, e^{t\mathcal{L}_0}$$

$$\mathcal{L}_0 1 = 0$$

$$\lambda \in \sigma(\mathcal{L}_0), \ Re(\lambda) \lesssim 2$$

$$\Rightarrow \kappa(T) = \lim_{L\to\infty} \frac{1}{T^2} \int_0^\infty dt \, \langle \tilde{J}, e^{t\mathcal{L}_0} \tilde{J} \rangle_T$$

$$= \lim_{L\to\infty} \frac{1}{T^2} \int_0^\infty dt \, \mathbb{E}_T \left(\tilde{J}(0) \tilde{J}(t) \right)$$

$$= \lim_{L\to\infty} \lim_{t\to\infty} \frac{1}{T^2} \mathbb{E}_T \left(\left(\frac{1}{\sqrt{t}} \int_0^t ds \, \tilde{J}(s) \right)^2 \right)$$