Evolution of geometric ideas, and the role of Relativity

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Euclid and Newton

In **Newtonian** mechanics, the spacetime is $\mathbb{R} \times \mathbb{R}^3$ where $\mathbb{R}^3$ has the **Euclidean** geometry. In a Cartesian coordinate system $x = (x^1, x^2, x^3)$, the distance is intrinsic:

$$d(x, \overline{x})^2 = (x^1 - \overline{x^1})^2 + (x^2 - \overline{x^2})^2 + (x^3 - \overline{x^3})^2$$

By making this infinitesimal

$$(\triangle d)^2 = (\triangle x^1)^2 + (\triangle x^2)^2 + (\triangle x^3)^2$$

which induces the Euclidean metric

$$ds^2 = (dx^1)^2 + (dx^2)^2 + (dx^3)^2$$

$$= (dx^1, dx^2, dx^3) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} dx^1 \\ dx^2 \\ dx^3 \end{pmatrix}$$

The geometry appears only on the absolute space $\mathbb{R}^3$ in the form of the Euclidean metric $ds^2 = \sum_{i,j} \delta_{ij} dx^i \otimes dx^j$, as the absolute time $\mathbb{R}$ is independent of the absolute space $\mathbb{R}^3$.
The Lie group $SO(3)$ is defined as the norm-preserving linear map of $\mathbb{R}^3$

$$\langle Av, Aw \rangle =^t (Av)\delta_{ij} Aw =^t v({}^t A\delta_{ij} A)w =^t v({}^t AA)w = \langle v, w \rangle.$$

namely,

$$SO(3) = \{A \in M_3(\mathbb{R}) \mid {}^t AA = \mathrm{I}_3\}$$

In other words, all the Newtonan mechanics are invariant under the change of coordinates induces by $SO(3)$. (cf. Galilean Transforms: $+$ translation and uniform motion)

Maxwell and Minkowski

In Special Relativity, the spacetime is the Minkowski space modeled on $\mathbb{R}^4$. The intrinsic distance on it, given a Cartesian coordinate system$x = (x^0, x^1, x^2, x^3)$, is given by

$$d(x, \overline{x})^2 = -(x^0 - \overline{x^0})^2 + (x^1 - \overline{x^1})^2 + (x^2 - \overline{x^2})^2 + (x^3 - \overline{x^3})^2$$

By making this infinitesimal,

$$(\triangle d)^2 = -(\triangle x^0)^2 + (\triangle x^1)^2 + (\triangle x^2)^2 + (\triangle x^3)^2$$

which induces the Minkowski metric

$$\begin{aligned} ds^2 &= -(dx^0)^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2 \\ &= (dx^0, dx^1, dx^2, dx^3) \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} dx^0 \\ dx^1 \\ dx^2 \\ dx^3 \end{pmatrix} \end{aligned}$$

The geometry appears only on the spacetime $\mathbb{R}^4$ in the form of the Minkowski metric $ds^2 = \sum_{i,j} \eta_{ij} dx^i \otimes dx^j$.
The Lie group $SO(3,1)$ is defined as the norm-preserving linear map of $\mathbb{R}^4$

$$\langle Av, Aw \rangle =^t (Av)\eta_{ij}Aw =^t v({}^tA\eta_{ij}A)w =^t v\eta_{ij}w = \langle v, w \rangle.$$

namely,

$$SO(3,1) = \{A \in M_3(\mathbb{R}) \mid {}^tA\eta A = \eta\}$$

In other words, all the physics are invariant under the change of coordinates induces by $SO(3,1)$.
Note that the Euclidean geometry is properly contained in the Minkowskian geometry:

$$\begin{pmatrix} -1 & 0 \\ 0 & SO(3) \end{pmatrix} \subset SO(3,1)$$

"Properly" in the sense that the intrinsic distance is now mixing up the space and time, as seen in the *Lorentz boost*

$$\begin{pmatrix} \cosh t & \sinh t & 0 & 0 \\ \sinh t & \cosh t & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

which leaves *the unit sphere of radius i* invariant, and as $|t| \to \infty$, every vector asymptotically converges to the light cone

$$\{x^0 = |(x^1, x^2, x^3)|\}.$$

The physics on the Minkowski space is **Maxwell equations**:

$$dF = 0$$

$$\delta_\eta F = j$$

where $F = \frac{1}{2}\sum_{a,b=0}^{3} F_{ab}dx^a dx^b$ on $\mathbb{R}^4$ is a 2 form, d is the exterior differential on $\mathbb{R}^4$ and δ is the L^2-dual of d with respect to the Minkowski metric:

$$\int_{\mathbb{R}^4} \langle d\omega, \eta\rangle_\eta d\mu = \int_{\mathbb{R}^4} \langle \omega, \delta\eta\rangle_\eta d\mu$$

or equivalently

$$(\delta F)_a = -\sum_{b,c=0}^{3} \eta^{bc} F_{ab,c}$$

In the more classical notations, the two form $F = \frac{1}{2}\sum_{a,b=0}^{3} F_{ab} dx^a dx^b$ is written as

$$F = \begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & -B_3 & B_2 \\ -E_2 & B_3 & 0 & -B_1 \\ -E_3 & -B_2 & B_1 & 0 \end{pmatrix}$$

with E and B the electric and magnetic fields on $\mathbb{R}^3$, and $j = (\rho, J)$ with ρ charge density, J current.

Special to General

Minkowski, Hilbert and M. Born: over 1908-1915, made attempts to incorporate **gravitation** on the Minkowski space, as Hilbert was inspired by Gustav Mie's variational formulation of the theory of electro-magnetic scattering, with no success.
M. Grossman and Einstein: over the same period, in order to explain gravitation, consider the variable Lorentzian metric, physically it means that the speed of light is variable:

$$ds^2 = \sum_{i,j=0}^{3} g_{ij}(x) dx^i \otimes dx^j.$$

and the differential equation g satisfies

$$\Gamma_{ab}(g) = \kappa \Theta_{ab} \qquad \text{(Entwurf Theory)}$$

as an analogue of the equation of the Newtonian potential

$$\triangle \phi = 4\pi\rho$$

Two issues regarding the Entwurf theory (1913), where Γ_{ab} is the Ricci tensor ($= \partial^2 g$) of Lorentzian metric g :

- does NOT approximate the Newtonian theory in the weak gravitational setting.
- is NOT invariant under the general change of variables.

Let us recall the general change of variables. In Riemannian geometry, where g generally does not have symmetries such as $SO(3), SO(3,1)$, there is an infinite-dimensional underlying symmetries of diffeomorphisms. Given a diffeomorphism Φ of a manifold M, and a Riemainnain/Lorentzian metric g, there is a tautological isometry

$$\Phi : (M, \Phi^* g) \to (M, g)$$

where the pull-back metric $\Phi^* g$ is defined by

$$\Phi^* g(v, w) = g(d\Phi(v), d\Phi(w)).$$

In the Rimannian geometry, $(M, \Phi^* g)$ and (M, g) are identified as isometric. In the Entwurf Theory(1913), Grossman and Einstein struggled with the fact that their defining equation $\Gamma_{ab}(g) = \kappa \Theta_{ab}$ was not invariant under the diffeomorphism action:

$$\Gamma_{ab}(\Phi^* g) \neq \kappa \Phi^* \Theta_{ab}$$

Riemann and Einstein

Spacetime is a Lorentzian 4 dimensional manifold (N^4, g) where g satisfying the Einstein equations (November 1915)

$$R_{ab} - \frac{1}{2} R\, g_{ab} = T_{ab}$$

R_g: the scalar curvature, R_{ab}: the Ricci curvature of the Lorentzian metric g, and T: the energy-momentum-stress tensor of the matter fields.
Γ_{ab} of the Entwurf theory is now the LHS of the equation, which is often called Einstein tensor.

Riemann and Einstein

Einstein's self-claimed "best idea of his life": **someone falling from the roof does not feel his own weight** can be paraphrased as the equation explaining the gravitation part of the equation (RHS) should be tensorial and that tensor is divergence-free, namely, invariant under diffeomorphisms of the spacetime.

Indeed the vacuum Einstein equation is the Euler-Lagrange equation for the Hilbert-Einstein functional;

$$\mathcal{H}(g) = \int_N R_g \, d\mu_g$$

Furthermore the first variation of the energy via an infinitesimal diffeomorphism is

$$\delta\mathcal{H}(g)[L_X g] = \frac{d}{dt}\mathcal{H}(\Phi_t^* g)\Big|_{t=0} = -\int_N \langle \mathrm{Ric} - \frac{1}{2} R g, L_X g \rangle_g \, d\mu_g$$

By integration by parts, one gets

$$0 = \int_N \langle \mathrm{div}_g(\mathrm{Ric} - \frac{1}{2} R g), X \rangle_g \, d\mu_g \quad \forall X \in \mathfrak{X}^\infty(N)$$

Namely the Einstein tensor is divergence free, and hence so is the stress-energy tensor T_{ab}.

Evolution of the idea of spaces

There are three phases of geometry, in the context of relativity.

- Euclid and Newton: the mechanics as a dymamics between the absolute space and the absolute time.
- Maxwell and Minkowski: the absoluteness of the speed of light and the Erlangen program (F. Klein) of the spacetime.
- Riemann and Einstein: Riemannian metric and its curvature inducing the gravitation.

Evolution of the idea of spaces

Riemann (Habilitationsschrift 1854): The geometry is variable/inhomogeneous. (cf: Gauss' surface theory, where submanifolds can be shaped at will.)
Einstein: The freely falling person from the roof is the geodesic.

Invariance of the defining equations:

$$SO(3) \subset SO(3,1) \text{ ``} \subset \text{''} \mathrm{Diff}^+ N^4$$

the first two Lie groups represent the special covariance, the last general covariance; **Special Covariance** is

$$g_p(v_p, w_p) = g_{\Phi(p)}(d\phi(v_p), d\Phi(w_p))$$

and the **General Covariance** is a tautological identification:

$$[\Phi^* g]_p(v_p, w_p) := g_{\Phi(p)}(d\Phi(v_p), d\Phi(w_p))$$

Einstein on Newton's shoulder

Newtonian mechanics consists of gravitational potential function $\varphi(x)$ defined on $\mathbb{R}^3$ satisfying the Poisson equation, and the equation of motion, or equivalently, Newton's third law

$$\begin{cases} \triangle_{\mathbb{R}^3}\varphi & = & 4\pi\rho(x) \\ \frac{d^2}{dt^2}x(t) & = & -\nabla\varphi(x(t)) \end{cases} \tag{1}$$

where $\rho(x)$ is the matter density at x. The underlying manifold is the four-dimensional Euclidean space $\mathbb{R} \times \mathbb{R}^3$, where the first component of the product space is the absolute time, and the second the absolute space.

Einstein on Newton's shoulder

The Einstein equation which replaces the Poisson equation and the equation of motion is replaced by the geodesic equation on the Lorentzian manifold (M, g):

$$\begin{cases} R_{ij} - \frac{1}{2}Rg_{ij} & = & 8\pi T_{ij} \\ \frac{d^2}{dt^2}x^k(t) & = & -\sum_{0\le i,j\le 3}\Gamma^k_{ij}(x(t))\frac{dx^i}{dt}\frac{dx^j}{dt}. \end{cases} \tag{2}$$

Here the indices i, j, k represent the local coordinates (x^0, x^1, x^2, x^3) and T_{ij} is the stress-energy tensor describing matters such as gas, fluids, stars and the non-gravitational fields such as the electromagnetic field. On the right hand side of the geodesic equation appear the Christoffel symbols Γ^k_{ij} which are determined by the first derivatives of the metric tensor g.

Einstein on Newton's shoulder

Newtonian mechanics versus the gravitational theory of Einstein.: Both involve potential functions; in the former case the scalar function φ, and in the latter the Lorentzian metric g. Each defining equation involves a second order partial differential equation, and in each case, the movement of free object is governed by a second order ordinary differential equation, whose right hand side contains a first derivative of the potential function/tensor.

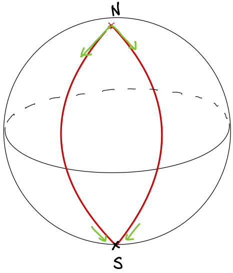
Freeeman Dyson says:

... Newton did not have the strongest muscles of intellectual concentration. Einstein's muscles were stronger. Einstein held the problem of understanding gravitation in his mind for ten years, from the discovery of special relativity in 1905 to the birth of general relativity in 1915. He held onto the problem with all his strength until it was solved. After it was solved, the new science of cosmology was born, allowing us to explore the size and shape of the universe, holding the universe in our minds as a dynamical system which we can grasp with real understanding. Einstein enlarged our vision of nature even more than Newton. Newton saw nature as dynamical. Einstein saw nature as geometrical.

"A Toast to Einstein", Freeman Dyson (2015)

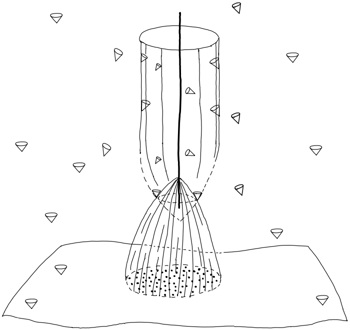
Blackhole geometry

Bonnet-Myers Theorem: Let (M^n, g) be a complete Riemannian manifold with $\mathrm{Ric} \geq (n-1)k$, then diameter is $\leq \pi/\sqrt{k}$.
(Idea of the proof: ODE $f' = Cf^2$ has blow-ups)



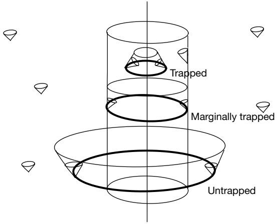
Blackhole geometry

Einstein equation $R_{ab} - \frac{1}{2}R\, g_{ab} = T_{ab}$, combined with the dominant energy condition, says that in the causal (in particular lightlike) directions, $Ric_g \geq 0$.



Blackhole geometry

Roger Penrose's idea (1965): The gravitational collapse is a conjugate point for Ricci-positive space. To locate the collapse, find a trapped surface.



(cf. Penrose's 2020 Nobel lecture
www.youtube.com/watch?v=DpPFn0qzYT0)

Concluding remarks

- Euclid and Newton: the mechanics as a dymamics between the absolute space and the absolute time.
- Maxwell and Minkowski: the absoluteness of the speed of light and the Erlangen program (F. Klein) of the spacetime.
- Riemann and Einstein: Riemannian metric and its curvature inducing the gravitation.
- Penrose and beyond: Geometry of gravitational waves

$$\text{Lorentzian metric} \Leftrightarrow \text{Wave equation}$$

The gravitational wave is distinguished from the other waves (light, radio waves, neutrino waves) for its non-interactive property. It propagates the geometric information, undisturbed by the other physics. (cf. Multi-messenger astronomy)