Теория чисел II

Весна 2021, лекция 10

Лекция 10.

Утв.: $$\int_{-\infty}^{+\infty} |F(t)|^2\, dt \leqslant \int_{1}^{+\infty} G(y)\, dy + \eta^2(0)\,\eta^2(1),$$

где $$G(y) = \left| \sum_{n=1}^{+\infty} \sum_{\vartheta_1, \vartheta_2 \leq X} \frac{\beta(\vartheta_1)\beta(\vartheta_2)}{\vartheta_2} \exp\left(-\frac{\pi n^2 \vartheta_1^2}{\vartheta_2^2} y^2 (\sin 2\delta + i \cos 2\delta)\right)\right|^2.$$

$\delta = \frac{1}{T}$.

Д-во:

Для $f \in L^2(-\infty; +\infty)$ верно рав-во Парсеваля

$$\int_{-\infty}^{+\infty} |f(t)|^2\, dt = \int_{-\infty}^{+\infty} |\hat{f}(t)|^2\, dt.$$

$$\int_{-\infty}^{+\infty} |F(t)|^2\, dt = \int_{-\infty}^{+\infty} |\hat{F}(t)|^2\, dt \underset{\substack{\text{чётность} \\ |\hat{F}(y)| \\ (\text{из вещ.-ти } F(y)).}}{=} 2 \int_{0}^{+\infty} |\hat{F}(y)|^2\, dy \leq$$

$$\hat{F}(y) = 2e^{\frac{1}{2}(y+i(\frac{\pi}{4}-\delta))}\sum_{n,\nu_1,\nu_2}\frac{\beta(\nu_1)\beta(\nu_2)}{\nu_2}\exp\Big(-\frac{\pi h^2\nu_1^2}{\nu_2^2}e^{2y}(\sin 2\delta + i\cos 2\delta)\Big) -$$

$$- e^{-\frac{1}{2}(y+i(\frac{\pi}{4}-\delta))}\eta(1)\eta(0).$$

$$\leq 2\Big(2\eta^2(1)\eta^2(0)\int_0^{+\infty}e^{-y}\,dy +$$

$$8\int_1^{+\infty}\Big|\underbrace{\sum_{h,\nu_1,\nu_2}\frac{\beta(\nu_1)\beta(\nu_2)}{\nu_2}\exp\Big(-\frac{\pi h^2\nu_1^2}{\nu_2^2}y^2(\sin 2\delta + i\cos 2\delta)\Big)}_{=G(y)}\Big|^2 dy\Big)$$

Упр.:

$$\int_{-\infty}^{+\infty}\Big|\int_0^h F(t+u)\,du\Big|^2 dt \lesssim h^2\int_0^{e^{1/h}}G(y)\,dy + \int_{e^{1/h}}^{+\infty}\frac{G(y)}{\ln^2 y}\,dy +$$

$$+ h^2\eta^2(0)\eta^2(1).$$

Д-во:

$$f(t) := \int_0^h F(t+u)\,du \Rightarrow \hat f(y) = \int_{-\infty}^{+\infty} \int_0^h F(t+u)\,du\; e^{-ity}\,dt =$$

$$= \int_0^h \int_{-\infty}^{+\infty} F(t)\, e^{-i(t-u)y}\,dt\,du = \hat F(y) \int_0^h e^{iuy}\,du$$

$$\int_0^h e^{iyu}\,du \le \min\left(h; \frac{2}{|y|}\right)$$

$$\int_{-\infty}^{+\infty} |f(t)|^2\,dt = \int_{-\infty}^{+\infty} |\hat f(t)|^2\,dt \le 2h^2 \int_0^{1/h} |\hat F(y)|^2\,dy + 8 \int_{1/h}^{+\infty} \frac{|\hat F(y)|^2}{y^2}\,dy \;=$$

Замена $e^y = t$:

$$= 2h^2 \left(8 \int_1^{e^{1/h}} G(y)\,dy + 4\eta^2(0)\eta^2(1) \right) + 8 \left(8 \int_{e^{1/h}}^{+\infty} \frac{G(y)}{\ln^2 y}\,dy + 2\eta^2(0)\eta^2(1) \int_{1/h}^{+\infty} \frac{e^{-y}}{y^2}\,dy \right) \le$$

$$\le 16h^2 \int_1^{e^{1/h}} G(y)\,dy + 64 \int_{e^{1/h}}^{+\infty} \frac{G(y)}{\ln^2 y}\,dy + \eta^2(0)\eta^2(1)\left(4h^2 + 16h^2 e^{-1/h}\right).$$

Утв.: $$L(x,\theta)=\int_x^{+\infty} G(u)\,u^{-\theta}\,du \quad \Big(\underset{\wedge\, e^{1/h}}{x\geq 1},\ 0\leq\theta\leq\tfrac12\Big).$$

Тогда $L(x,\theta)\ll \dfrac{\delta^{-1/2}}{\theta x^{\theta}\ln X}$ равн.-но по θ.

Следствие:

а) $$\boxed{\int_1^{e^{1/h}} G(y)\,dy} = -\int_1^{e^{1/h}} x^{\theta}\,\frac{d}{dx}L(x,\theta)\,dx = -x^{\theta}L(x,\theta)\Big|_1^{e^{1/h}} +$$
$$+\theta\int_1^{e^{1/h}} x^{\theta-1}L(x,\theta)\,dx \ll \frac{\delta^{-1/2}}{\theta\ln X}+\frac{\delta^{-1/2}}{\ln X}\cdot\frac{1}{h} \ll \boxed{\frac{\delta^{-1/2}}{h\ln X}} \text{ при } \theta\geq h.$$

б) $$\int_0^{1/2}\theta L(e^{1/h},\theta)\,d\theta = \int_{e^{1/h}}^{+\infty} G(y)\left(\int_0^{1/2}\theta y^{-\theta}\,d\theta\right)dy$$
$$=\int_{e^{1/h}}^{+\infty} G(y)\Bigg(\underbrace{\frac{\theta y^{-\theta}}{-\ln y}\Big|_0^{1/2}+\frac{1}{-\ln^2 y}\,y^{-\theta}\Big|_0^{1/2}}_{=\ -\frac{1}{2\ln y}y^{-1/2}-\frac{y^{-1/2}}{\ln^2 y}+\frac{1}{\ln^2 y}}\Bigg)dy =$$

$$\Rightarrow \int_{e^{1/h}}^{+\infty} \frac{G(y)}{\ln^2 y} - \frac{3}{2}\int_{e^{1/h}}^{+\infty} \frac{G(y)}{y^{1/2}}\,dy$$

$$\boxed{\int_{e^{1/h}}^{+\infty} \frac{G(y)}{\ln^2 y}} \leq \int_0^{1/2} \theta\, L(e^{1/h}, \theta)\, d\theta + \frac{3}{2} L\left(e^{1/h}, \frac{1}{2}\right) \ll$$

$$\ll \int_0^{1/2} \frac{d\theta}{(e^{1/h})^{\theta}} \frac{\delta^{-1/2}}{\ln X} + \frac{\delta^{-1/2}}{\ln X \cdot e^{1/(2h)}} \boxed{\ll \frac{\delta^{-1/2}}{\ln X} \cdot h}$$

б) $\boxed{\int_1^{\delta^{-2}} G(y)\,dy \ll} \int_1^{+\infty} G(y)\, y^{-\frac{1}{\ln \delta^{-1}}}\, dy \ll \boxed{\frac{\delta^{-1/2} \ln \delta^{-1}}{\ln X}}$

Кроме того, имеет место оценка

г) $$\int_{\delta^{-2}}^{+\infty} G(y)\,dy = \sum_{\substack{n_1, \vartheta_1, \vartheta_2, \\ n_2, \vartheta_3, \vartheta_4}} \frac{|\beta(\vartheta_1)\beta(\vartheta_2)\beta(\vartheta_3)\beta(\vartheta_4)|}{\vartheta_2 \vartheta_4} \int_{\delta^{-2}}^{+\infty} \exp\left(-\pi y^2 \sin(2\delta)\left(\frac{n_1^2\vartheta_1^2}{\vartheta_2^2} + \frac{n_2^2\vartheta_3^2}{\vartheta_4^2}\right)\right) dy$$

$$\int_{\delta^{-2}}^{+\infty} \exp(-Ay^2)\,dy = \left\{ \begin{array}{l} Ay^2 = t \\ y = \sqrt{\frac{t}{A}} \end{array} \right\} = \frac{1}{2}\int_{A\delta^{-4}}^{+\infty} e^{-t} \frac{t^{-1/2}}{A^{1/2}}\, dt \ll A^{-1} \delta^{2} e^{-A\delta^{-4}} \Rightarrow$$

$\beta(\vartheta) \ll 1$

$$\Rightarrow \int_{\delta^{-2}}^{+\infty} G(y)\,dy \ll \delta \sum_{\substack{n_1, n_2 \\ \partial_1,\dots,\partial_4 \le X}} \frac{1}{\partial_2 \partial_4 \left(\frac{n_1^2 \partial_1^2}{\partial_2^2} + \frac{n_2^2 \partial_3^2}{\partial_4^2}\right)} \exp\left(-\delta^{-3}\left(\frac{n_1^2 \partial_1^2}{\partial_2^2} + \frac{n_2^2 \partial_3^2}{\partial_4^2}\right)\right) \ll$$

$$\ll \delta \sum_{\substack{n_1, n_2 \\ \partial_1,\dots,\partial_4 \le X}} \frac{\partial_2 \partial_4}{(n_1^2 \partial_1^2 \partial_4^2 + n_2^2 \partial_3^2 \partial_2^2)} \exp(\dots) \ll$$

$$\ll \delta \sum_{\substack{n_1, n_2 \\ \partial_1 \dots \partial_4}} \frac{\cancel{\partial_2}\cancel{\partial_4}}{n_1 \partial_1 \cancel{\partial_4} n_2 \partial_3 \cancel{\partial_2}} \exp\left(-\delta^{-3}(\dots)\right) \ll$$

$$\ll \delta \left(\sum_{n_1=1}^{+\infty} \sum_{\partial_1, \partial_2 \le X} \frac{1}{n_1 \partial_1} \exp\left(-\delta^{-3} \frac{n_1^2 \partial_1^2}{\partial_2^2}\right)\right)^2 \ll$$

$$\sum_{n=1}^{+\infty} \frac{1}{n} \exp\left(-\delta^{-3} \frac{n^2 \partial_1^2}{\partial_2^2}\right) \ll \sum_{n=1}^{+\infty} \frac{1}{n} \exp\left(-\delta^{-3} X^{-2} n^2\right) \underset{X^2 \ll \delta^{-2}}{\ll} \sum_{n=1}^{+\infty} \frac{1}{n} \exp\left(-\delta^{-2} n^2\right)$$

$$\ll \exp(-\delta^{-2})$$

$$\ll \delta \exp(-\delta^{-2}) X^2 \ln^2 X \ll 1.$$

Итак:

1. $$\int_{-\infty}^{+\infty} I^2(t,h)\,dt \ll \frac{\delta^{-1/2} h}{\ln X} + h^2 X^2 \ln^2 X \ll \frac{\delta^{-1/2} h}{\ln X},$$ если $$h X^2 \ln^3 X \ll \delta^{-1/2}. \quad (*)$$

2. $$\int_{-\infty}^{+\infty} |F(t)|^2\,dt \ll \frac{\delta^{-1/2} \ln \delta^{-1}}{\ln X} + 1 + X^2 \ln^2 X \ll \delta^{-1/2} \cdot \frac{\ln \delta^{-1}}{\ln X}$$ при $(*)$.

3. $$\int_T^{2T} |\zeta\eta^2(\tfrac{1}{2}+it) - 1|^2\,dt \ll T + \int_T^{2T} |\zeta\eta^2(\tfrac{1}{2}+it)|^2\,dt \ll T,$$

т.к.

$$\int_T^{2T} |\zeta\eta^2(\tfrac{1}{2}+it)|^2\,dt \ll \int_T^{2T} |\zeta\eta^2(\tfrac{1}{2}+it)\,\pi^{-\frac{1}{4}+\frac{it}{2}}|^2 \cdot$$

$$\cdot\, |\Gamma(\tfrac{1}{4}+\tfrac{it}{2})|^2 \cdot e^{2(\frac{\pi}{4}-\delta)t}\, t^{1/2}\,dt \ll T^{1/2}\int_T^{2T} |F(t)|^2\,dt \ll T^{1/2} \delta^{-1/2} \frac{\ln \delta^{-1}}{\ln X}.$$

$(\ll T)$.

4. $$M(t,h) = \left| \int_t^{t+h} \left(\varsigma \eta^2\left(\tfrac12 + iu\right) - 1 \right) du \right| \le$$

$$\le \left| \int_{1/2}^{3/2} \left(\varsigma \eta^2(\sigma + it) - 1 \right) d\sigma \right| + \left| \int_{1/2}^{3/2} \left(\varsigma \eta^2(\sigma + i(t+h)) - 1 \right) d\sigma \right| +$$

$$+ \left| \int_t^{t+h} \left(\varsigma \eta^2\left(\tfrac32 + iu\right) - 1 \right) du \right|.$$

Вспомним, что $\eta(s) = \sum\limits_{\nu \le X} \dfrac{\beta(\nu)}{\nu^s}$, где $\beta(\nu) = \alpha(\nu)\,\ell(\nu)$,

$$\sum_{\nu=1}^{+\infty} \frac{\alpha(\nu)}{\nu^s} = \varsigma^{-1/2}(s), \qquad \ell(\nu) = \begin{cases} 1, & \nu \le \sqrt{X} \\ \dfrac{2 \ln X/\nu}{\ln X}, & \sqrt{X} < \nu \le X \\ 0, & \text{иначе} \end{cases}$$

Т.о.

$$\varsigma \eta^2\left(\tfrac32 + iu\right) - 1 = \sum_{n > \sqrt{X}} \frac{1}{n^{\frac32 + iu}} \left(\sum_{m\nu_1\nu_2 = n} \beta(\nu_1)\beta(\nu_2) \right)$$

$|\beta(\nu)| \le |\alpha(\nu)|$, $\sum\limits_{\nu_1\nu_2 = d} |\alpha(\nu_1)\alpha(\nu_2)| \le 1$, т.к.

$$\sum_{\nu=1}^{+\infty} \frac{\alpha(\nu)}{\nu^s} = \prod_p \left(1-\frac{1}{p^s}\right)^{1/2}, \quad \operatorname{Re} s > 1$$

$\alpha(1) = 1$, α – мультипл.-ая

$$\prod_p \left(1-\frac{1}{p^s}\right)^{-1/2} = \sum_{\nu=1}^{+\infty} \frac{\alpha'(\nu)}{\nu^s}, \quad \alpha' \text{ – мультипл.-ая}$$

$$\alpha(p^k) = (-1)^k \binom{1/2}{k} = (-1)^k \frac{\frac{1}{2}\cdot\left(\frac{1}{2}-1\right)\cdot\ldots\cdot\left(\frac{1}{2}-k+1\right)}{k!}$$

$$\alpha'(p^k) = (-1)^k \frac{\left(-\frac{1}{2}\right)\cdot\left(-\frac{1}{2}-1\right)\ldots\left(-\frac{1}{2}-k+1\right)}{k!} = \frac{\frac{1}{2}\cdot\left(\frac{1}{2}+1\right)\cdot\ldots\cdot\left(\frac{1}{2}+k-1\right)}{k!}$$

$\Downarrow$

$|\alpha(n)| \leq |\alpha'(n)|$, при этом $\sum_{\nu_1\nu_2=n} \alpha'(\nu_1)\alpha'(\nu_2) = 1$

$$\Big|\underbrace{\sum_{m\partial_1\partial_2=n}\beta(\partial_1)\beta(\partial_2)}_{\tau(n)}\Big|\le\sum_{m|n}1=\tau(n)$$

$$\left|\int_t^{t+h}\left(\zeta\eta^2\left(\tfrac{3}{2}+iu\right)-1\right)du\right|=\left|\int_t^{t+h}\left(\sum_{n>\sqrt{X}}\frac{\tau(n)}{n^{\frac{3}{2}+iu}}\right)du\right|\le$$

$$\le\sum_{n>\sqrt{X}}\frac{\tau(n)}{n^{3/2}}\cdot\frac{2}{\ln n}\ll X^{-\frac{1}{4}+\varepsilon}=X^{-\frac{a}{4}}$$ (определили a).

$$\left|\int_{1/2}^{3/2}\left(\zeta\eta^2(\sigma+it)-1\right)d\sigma\right|^2\le\int_{1/2}^{3/2}X^{\frac{a}{2}\left(\frac{1}{2}-\sigma\right)}d\sigma\int_{1/2}^{3/2}X^{\frac{a}{2}\left(\sigma-\frac{1}{2}\right)}\left|\zeta\eta^2(\sigma+it)-1\right|^2d\sigma$$

Если $J_\sigma=\int_T^{2T}\left|\zeta\eta^2(\sigma+it)-1\right|^2dt$, то из 3.

$$J_{\frac{1}{2}}\ll T,\quad J_{3/2}\ll\int_T^{2T}\left|\sum_{n>\sqrt{X}}\frac{\tau(n)}{n^{3/2+it}}\right|^2dt\ll TX^{-1+\varepsilon}+\sum_{\substack{m>n\\ m,n>\sqrt{X}}}\frac{\tau(n)\tau(m)}{(mn)^{3/2}\ln\frac{m}{n}}\ll$$

$$\ll TX^{-1+\varepsilon}+X^{-\frac{1}{2}+\varepsilon}\ll TX^{-a}$$

Из th. Габриэла:

пусть $f(s)$ регулярна в области $\mathcal{D} = \{ \alpha \le \mathrm{Re}\, s \le \beta,$ $T \le \mathrm{Im}\, s \le 2T \}$, при этом $f(s) \ll e^{ct^2}$. Тогда $\forall\, q > 0$, $\gamma \in (\alpha; \beta)$ верно

$$\int_T^{2T} | f(\gamma + it) |^q \, dt \ll \left(\int_T^{2T} |f(\alpha + it)|^q \, dt + 1 \right)^{\frac{\beta-\gamma}{\beta-\alpha}} \cdot$$

$$\cdot \left(\int_T^{2T} | f(\beta + it) |^q + 1 \right)^{\frac{\gamma-\alpha}{\beta-\alpha}}$$

следует, что

$$J_\sigma \ll J_{1/2}^{\left(\frac{3}{2} - \sigma\right)} \cdot J_{3/2}^{\sigma - \frac{1}{2}} \ll \left(TX^{-a}\right)^{\sigma - \frac{1}{2}} \cdot T^{\frac{3}{2} - \sigma} \ll TX^{-a(\sigma - \frac{1}{2})},$$

$a = -1 + \varepsilon \geqslant \frac{1}{2}$.

$$\int_T^{2T} |M(t, h)|^2 \, dt \ll TX^{-\frac{a}{2}} + T \left(\int_{1/2}^{3/2} X^{-\frac{a}{2}(\sigma - \frac{1}{2})} \, d\sigma \right)^2 \ll \frac{T}{\ln^2 X} \ll \frac{Th}{\ln X},$$

если $h \geqslant \frac{A}{\ln X}$, $A \gg 1$.

Итак,

$$\int_T^{2T} |I(t,h)|^2\,dt \ll \frac{T^{1/2}h}{\ln X},$$

$$J(t,h) \gg t^{-\frac14}\left(h - |M(t,h)|\right)$$

$$\int_T^{2T} |M(t,h)|^2\,dt \ll \frac{T}{\ln^2 X}$$

Тогда на подмн-ве интервала $(T, 2T)$ меры

$T - \dfrac{T}{h^2 \ln^2 X} \gg T$, если $h = \dfrac{A}{\ln T}$, A - дост. велико

$$J(t,h) \gg t^{-\frac14}\cdot h.$$

При этом на подмн-ве интервала $(T, 2T)$ меры $\gg T$
$I(t,h) < J(t,h)$, т.к. $I(t,h) \gg t^{-\frac14}\cdot h$ на мн-ве E
меры $\operatorname{mes} E$, где $\operatorname{mes} E\cdot T^{-\frac12} h^2 \ll \dfrac{T^{1/2}h}{\ln X} \Rightarrow \operatorname{mes} E \ll \dfrac{T}{h\ln X} \le \dfrac{T}{100}$,

если $A \gg 1$.

Осталось д-ть

Утв.: $L(x,\theta) = \int\limits_x^{+\infty} G(u) u^{-\theta}\, du \quad (x \geq 1,\ 0 \leq \theta \leq \frac{1}{2}).$

Тогда $L(x,\theta) \ll \dfrac{\delta^{-1/2}}{\theta x^{\theta} \ln X}$ равн.-но по θ, го

$$G(y) = \left| \sum_{n=1}^{+\infty} \sum_{\nu_1,\nu_2 \leq X} \frac{\beta(\nu_1)\beta(\nu_2)}{\nu_2} \exp\left(-\frac{\pi n^2 \nu_1^2}{\nu_2^2} y^2 (\sin 2\delta + i\cos 2\delta)\right)\right|^2.$$

Д-во:

$$L(x,\theta) = \sum_{\nu_1,\ldots,\nu_4 \leq X} \frac{\beta(\nu_1)\ldots\beta(\nu_4)}{\nu_2\nu_4} \sum_{n_1,n_2=1}^{+\infty} \int\limits_x^{+\infty} \exp\Bigg(-\pi y^2 \sin 2\delta \left(\frac{n_1^2\nu_1^2}{\nu_2^2} + \frac{n_2^2\nu_3^2}{\nu_4^2}\right) -$$

$$- i\pi y^2 \cos 2\delta \cdot \left(\frac{n_1^2\nu_1^2}{\nu_2^2} - \frac{n_2^2\nu_3^2}{\nu_4^2}\right)\Bigg) y^{-\theta}\, dy$$

Пусть $L_1(x,\theta)$ – это вклад от слагаемых $\dfrac{n_1^2\nu_1^2}{\nu_2^2} = \dfrac{n_2^2\nu_3^2}{\nu_4^2}$,

$L_2(x,\theta) = L - L_1.$

Пусть $q = (\partial_1\partial_4, \partial_2\partial_3)$, $\quad \partial_1\partial_4 = q a_1, \quad \partial_2\partial_3 = q a_2, \quad (a_1, a_2) = 1.$

$$\frac{n_1\partial_1}{\partial_2} = \frac{n_2\partial_3}{\partial_4} \Rightarrow n_1 a_1 = n_2 a_2 \Rightarrow n_1 = a_2\cdot n, \quad n_2 = a_1\cdot n \quad (n = 1, 2, \ldots)$$

$$L_1(x,\theta) = \sum_{\partial_1\ldots\partial_4 \le X} \frac{\beta(\partial_1)\ldots\beta(\partial_4)}{\partial_2\partial_4} \sum_{n=1}^{+\infty} \int_x^{+\infty} \exp\left(-2\pi y^2 \sin(2\delta)\cdot\frac{n^2\partial_1^2\partial_3^2}{q^2}\right) y^{-\theta} dy$$

$$\sum_{n=1}^{+\infty} \int_x^{+\infty} e^{-n^2y^2c} \frac{dy}{y^\theta} = c^{\frac{\theta-1}{2}} \sum_{n=1}^{+\infty} \int_{xn\sqrt{c}}^{+\infty} e^{-y^2} \frac{1}{n^{1-\theta}} \frac{dy}{y^\theta} =$$

$$= c^{\frac{\theta-1}{2}} \int_{x\sqrt{c}}^{+\infty} \frac{e^{-y^2}}{y^\theta} \left(\sum_{n \le \frac{y}{x\sqrt{c}}} \frac{1}{n^{1-\theta}}\right) dy$$

$$\sum_{n \le \frac{y}{x\sqrt{c}}} \frac{1}{n^{1-\theta}} = \frac{1}{\theta}\left(\frac{y}{x\sqrt{c}}\right)^\theta - \frac{1}{\theta} + K(\theta) + O\left(\left(\frac{y}{x\sqrt{c}}\right)^{\theta-1}\right), \quad |K(\theta)| \ll 1.$$

$$\frac{1}{\theta x^\theta c^{1/2}}\left(\int_0^{+\infty} e^{-y^2} dy + O(x\sqrt{c})\right) - c^{\frac{\theta-1}{2}}\left(\int_0^{+\infty} e^{-y^2} y^{-\theta} dy + O\left((x\sqrt{c})^{1-\theta}\right)\right)\left(\frac{1}{\theta} - K(\theta)\right) +$$

$$+ O\left(x^{1-\theta}\ln(2+c^{-1})\right) \ll \frac{1}{\theta x^\theta c^{1/2}} + \frac{c^{\frac{\theta-1}{2}}}{\theta} K_1(\theta) + \frac{x^{1-\theta}}{\theta}\ln(2+c^{-1})$$

$$c = 2\pi \sin(2\delta)\cdot \frac{\partial_1^2 \partial_3^2}{q^2} \Rightarrow$$

$$L_1(x,\theta) \ll |S(\theta)| \frac{\delta^{-1/2}}{\theta x^\theta} + \frac{|K_1(\theta)|}{\theta} \delta^{\frac{\theta-1}{2}} |S(\theta)| +$$

$$+ O\left(\frac{x^{1-\theta} \ln(2+c^{-1})}{\theta} \sum_{\partial_1 \dots \partial_4 \le x} \frac{|\beta(\partial_1) \dots \beta(\partial_4)|}{\partial_2 \partial_4} \right), \text{ где}$$

$$S(\theta) = \sum_{\partial_1, \dots, \partial_4 \le X} \frac{\beta(\partial_1) \dots \beta(\partial_4)}{\partial_2 \partial_4} \left(\frac{q}{\partial_1 \partial_3} \right)^{1-\theta}$$

Теорема: $S(\theta) \ll \frac{X^{2\theta}}{\ln X}$ равн. по θ.

Д-во:

Определим $\gamma(d)$ по ф-ле: $q^{1-\theta} = \sum_{d|q} \gamma(d)$, т.е.

$$\gamma(n) = \sum_{d|n}{}' \mu(d) \left(\frac{n}{d}\right)^{1-\theta} = n^{1-\theta} \prod_{p|n} \left(1 - \frac{1}{p^{1-\theta}}\right). \Rightarrow 0 \le \gamma(n) \le n^{1-\theta}$$

$$S(\theta) = \sum_{\vartheta_1 \dots \vartheta_4 \le X} \frac{\beta(\vartheta_1) \dots \beta(\vartheta_4)}{\vartheta_2 \vartheta_4 (\vartheta_1 \vartheta_3)^{1-\theta}} \sum_{d \mid (\vartheta_1 \vartheta_4, \vartheta_2 \vartheta_3)} \gamma(d) =$$

$$= \sum_{d \le X^2} \gamma(d) \Bigg(\underbrace{\sum_{\vartheta_1 \vartheta_4 \equiv 0 (d)} \frac{\beta(\vartheta_1)\beta(\vartheta_4)}{\vartheta_1^{1-\theta} \vartheta_4}}_{=: T} \Bigg)^2 .$$

$\vartheta_1 = \delta_1 \vartheta_1'$, $\vartheta_4 = \delta_4 \vartheta_4'$, где $\delta_1, \delta_4 \mid d^\infty$, $(\vartheta_1' \vartheta_4', d) = 1$.

$$T = \sum_{\substack{\delta_1 \delta_4 \equiv 0 (d) \\ \delta_j \le X}} \frac{1}{\delta_1^{1-\theta} \delta_4} \sum_{\substack{\vartheta_1' \le X/\delta_1 \\ (\vartheta_1', d) = 1}} \frac{\beta(\delta_1 \vartheta_1')}{(\vartheta_1')^{1-\theta}} \sum_{\substack{\vartheta_4' \le X/\delta_4 \\ (\vartheta_4', d) = 1}} \frac{\beta(\delta_4 \vartheta_4')}{\vartheta_4'} =$$

$$= \sum_{\substack{\delta_1 \delta_4 \equiv 0 (d) \\ \delta_j \le X}} \frac{\mu(\delta_1)\mu(\delta_4)}{\delta_1^{1-\theta} \delta_4} \Bigg(\sum_{\substack{\vartheta_1' \le X/\delta_1 \\ (\vartheta_1', d) = 1}} \frac{\mu(\vartheta_1')}{(\vartheta_1')^{1-\theta}} \mathcal{L}(\delta_1 \vartheta_1') \Bigg) \Bigg(\sum_{\substack{\vartheta_4' \le X/\delta_1 \\ (\vartheta_4', d) = 1}} \frac{\mu(\vartheta_4')}{\vartheta_4} \mathcal{L}(\delta_4 \vartheta_4) \Bigg).$$

Пусть $\displaystyle S_\theta(X,a,d) = \sum_{\substack{\nu \le X/a \\ (\nu,d)=1}} \frac{\lambda(\nu)}{\nu^{1-\theta}} \frac{\ln \frac{X}{a\nu}}{\ln X}$.

Заметим, что при $a\nu \le \sqrt{X}$

$$1 = 2\frac{\ln X/a\nu}{\ln X} - 2\frac{\ln \sqrt{X}/a\nu}{\ln X}.$$ П.о.

$$\sum_{\substack{\nu \le X/\delta \\ (\nu,d)=1}} \frac{\lambda(\nu)}{\nu^{1-\theta}} \mathcal{L}(\delta\nu) = 2S_\theta(X,\delta,d) - 2S_\theta(\sqrt{X},\delta,d)$$

Мы д-ем, что $\displaystyle S_\theta(X,\delta,d) \ll \frac{(X/\delta)^\theta}{\sqrt{\ln X}} \prod_{p|d}\left(1+\frac{1}{p}\right)^{1/2}$

Тогда

$$T \ll \sum_{\substack{\delta_1\delta_4 \equiv 0 (d) \\ \delta_j \le X}} \frac{|\lambda(\delta_1)\lambda(\delta_4)|}{\delta_1\delta_4} \frac{X^\theta}{\ln X} \prod_{p|d}\left(1+\frac{1}{p}\right)$$

$$\sum_{\substack{\delta_1\delta_4 \equiv 0 (d) \\ \delta_j \le X}} \frac{|\lambda(\delta_1)\lambda(\delta_4)|}{\delta_1\delta_4} = \sum_{D\equiv 0(d)} \frac{1}{D} \sum_{\delta_1\delta_4 = D} |\lambda(\delta_1)\lambda(\delta_4)| \le \sum_{\substack{D\equiv 0(d) \\ D \mid d^\infty}}' \frac{1}{D} \le$$

$$\le \frac{1}{d} \prod_{p\mid d}\left(1-\frac{1}{p}\right)^{-1} \ll \frac{1}{d} \prod_{p\mid d}\left(1+\frac{1}{p}\right)$$

$$T \ll \frac{X^\theta}{\ln X} \frac{1}{d} \prod_{p\mid d}\left(1+\frac{1}{p}\right)^2$$

$$S(\theta) \ll \sum_{d\le X^2} \gamma(d)\, T^2 \ll \sum_{d\le X^2} \frac{d^{1-\theta}}{d^2} \frac{X^{2\theta}}{\ln^2 X} \prod_{p\mid d}\left(1+\frac{1}{p}\right)^4 \ll$$

$$\ll \frac{X^{2\theta}}{\ln^2 X} \sum_{d\le X^2} \frac{1}{d^{1+\theta}} \prod_{p\mid d}\left(1+\frac{1}{p}\right)^4$$

$$\prod_{p\mid d}\left(1+\frac{1}{p}\right)^4 \ll \prod_{p\mid d}\left(1+\frac{4}{p}\right) \ll \prod_{p\mid d}\left(1+\frac{1}{\sqrt{p}}\right) \ll \sum_{n\mid d}\frac{1}{\sqrt{n}}$$

$$\sum_{d\le X^2} \frac{1}{d^{1+\theta}} \sum_{n\mid d}\frac{1}{\sqrt{n}} \le \sum_{n\le X^2} \frac{1}{n^{3/2+\theta}} \sum_{d\le X^2/n} \frac{1}{d^{1+\theta}} \ll \sum_{d\le X^2}\frac{1}{d} \ll \ln X$$

$$S(\theta) \ll X^{2\theta} \ln^{-1} X.$$

$$L_1(x,\theta) \leq |S(\theta)| \frac{\delta^{-1/2}}{\theta x^\theta} + \frac{|K_1(\theta)|}{\theta} \delta^{\frac{\theta-1}{2}} |S(\theta)| +$$

$$+ O\Big(\frac{x^{1-\theta} \ln(2+c^{-1})}{\theta} \sum_{\nu_1 \ldots \nu_4 \leq x} \frac{|\beta(\nu_1) \ldots \beta(\nu_4)|}{\nu_2 \nu_4} \Big) \ll$$

$$\ll \frac{\delta^{-1/2}}{\theta x^\theta \ln X} + \frac{\delta^{\frac{\theta-1}{2}}}{\theta} \frac{X^{2\theta}}{\ln X} + \frac{x^{1-\theta} \ln \delta^{-1}}{\theta} X^{2+\varepsilon} \ll$$

$$\ll \frac{\delta^{-1/2}}{\theta x^\theta \ln X}, \text{ если } \left(\delta^{\frac{1}{2}} X^2 x\right)^\theta \ll 1 \quad \left(\begin{array}{l} X = \delta^{-\varepsilon} \\ x \leq e^{1/h} \leq e^{\frac{\ln T}{A}} \end{array} \right).$$

Оценим $L_2(x,\theta)$:

$$\int_x^{+\infty} e^{-Py^2+iQy^2}\frac{dy}{y^\theta} = \frac{1}{2}\int_{x^2}^{+\infty}\frac{e^{-Pv}}{v^{\frac{\theta+1}{2}}}e^{iQv}\,dv \ll \frac{e^{-P}}{Q\,x^\theta}$$

$$L_2(x,\theta) \ll x^{-\theta}\sum_{\partial_1\ldots\partial_4}\frac{1}{\partial_2\partial_4}\sum_{n_1,n_2}\frac{1}{\left|\frac{n_1^2\partial_1^2}{\partial_2^2}-\frac{n_2^2\partial_3^2}{\partial_4^2}\right|}\exp\left(-\pi\left(\frac{n_1^2\partial_1^2}{\partial_2^2}+\frac{n_2^2\partial_3^2}{\partial_4^2}\right)\right)\delta)$$

$\frac{n_1\partial_1}{\partial_2} > \frac{n_2\partial_3}{\partial_4} \Rightarrow$ вклад в сумму по n_1,n_2

$$\ll \sum_{n_1=1}^{+\infty} e^{-\pi\frac{n_1^2\partial_1^2}{\partial_2^2}\delta}\sum_{n_2<\frac{n_1\partial_1\partial_4}{\partial_2\partial_3}}\left(\frac{n_1^2\partial_1^2}{\partial_2^2}-\frac{n_2^2\partial_3^2}{\partial_4^2}\right)^{-1}$$

$$\frac{n_1^2\partial_1^2}{\partial_2^2}-\frac{n_2^2\partial_3^2}{\partial_4^2} \geqslant \frac{n_1\partial_1}{\partial_2}\left(\frac{n_1\partial_1}{\partial_2}-\frac{n_2\partial_3}{\partial_4}\right) = \frac{n_1\partial_1(n_1\partial_1\partial_4-n_2\partial_2\partial_3)}{\partial_2\cdot\partial_2\partial_4}$$

$$\sum_{n_2}\frac{1}{n_1\partial_1\partial_4-n_2\partial_2\partial_3} \leq 1+\frac{1}{\partial_2\partial_3}+\frac{1}{2\partial_2\partial_3}+\ldots = 1+O\left(\ln\frac{n_1X}{\partial_2\partial_3}\right)$$

$$\sum_{n_1,n_2} \ll \frac{\vartheta_2^2\vartheta_4}{\vartheta_1}\sum_{n_1=1}^{+\infty}\left(\frac{1}{n_1}+\frac{\ln(n_1X)}{n_1\vartheta_2\vartheta_3}\right)e^{-\pi n_1^2\vartheta_1^2\vartheta_2^{-2}\delta} \ll$$

$$\ll \frac{\vartheta_2^2\vartheta_4}{\vartheta_1}\left(1+\frac{\ln X}{\vartheta_2\vartheta_3}\right)\ln\frac{X^2}{\delta}+\frac{\vartheta_2\vartheta_4}{\vartheta_1\vartheta_3}\ln^2\frac{X^2}{\delta} \ll \frac{\vartheta_2^2\vartheta_4}{\vartheta_1}\ln\delta^{-2}+$$

$$+\frac{\vartheta_2\vartheta_4}{\vartheta_1\vartheta_3}\ln^2\delta^{-1}$$

$$L_2(x,\sigma) \ll x^{-\sigma}\sum_{\vartheta_1,\dots,\vartheta_4}\left(\frac{\vartheta_2}{\vartheta_1}\ln\delta^{-1}+\frac{1}{\vartheta_1\vartheta_3}\ln^2\delta^{-1}\right) \ll \frac{X^4}{x^\sigma}\ln^2\delta^{-2}.$$

Д/з:

38. Повторить идею д-ва th. Сельберга о нулях:

а) определение $I(t,h)$, $J(t,h)$

б) выражение $J(t,h)$ через $M(t,h)$.

в) при каком h и на подмн-ве какой меры интервала $(T, 2T)$ хотим показать справедливость нер-ва $J > I$?

г) Из каких оценок среднего для $I(t,h)$, $M(t,h)$ хотим получить, что I, M дост. малы (и как именно малы)?

д) Если доказано, что на подмн-ве интервала $(T, 2T)$ меры $\geq \frac{9}{10}T$ справедлива оценка $|I| \leq C$, и на подмн-ве $(T, 2T)$ меры $\geq \frac{9}{10}T$ $|I| > C$. На подмн-ве какой меры (оценка снизу) $|I| < |J|$?

39. Выписать условие на h, X, δ, которое (которые) необходимы для д-ва теоремы Сельберга, где $\delta = T^{-1}$.

40.

$$\sum_{\substack{m,n \geq N \\ m>n}} \frac{1}{(m \cdot n)^a \ln \frac{m}{n}} \ll_a N^{2-2a} \ln N, \quad a>1.$$

Идея: $m = n+r, \quad r = 1, 2, \ldots$

41. Определим при $0 \leq \theta \leq \frac{1}{2}$, $d \in \mathbb{N}$

$$S_\theta(X, d) = \sum_{\substack{\nu \leq X \\ (\nu, d)=1}} \frac{\mu(\nu)}{\nu^{1-\theta}} \frac{\ln \frac{X}{\nu}}{\ln X}.$$

Д-ть, что $|S_\theta(X, d)| \ll \frac{X^\theta}{\sqrt{\ln X}} \prod_{p|d} \left(1+\frac{1}{p}\right)^{1/2}$.

Идея: ф-ла Перрона.