A regularity method for quantitative lower bounds on Lyapunov exponents of stochastic differential equations

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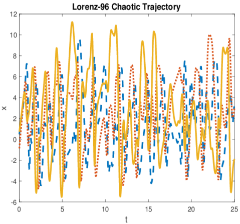
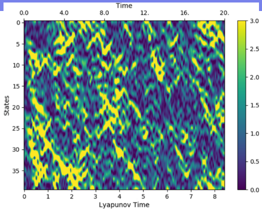
Chaos as positive Lyapunov exponent

- The Lorenz 96 system for a periodic chain of J unknowns, given by

$$\partial_t u_m = (u_{m+1} - u_{m-2}) u_{m-1} - \epsilon u_m + F_m$$

  is a common benchmark for testing numerical and analytical methods in applied mathematics.
- Observed numerically to be highly chaotic for ϵ small (equivalently F large).





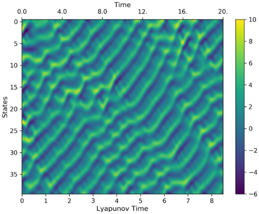


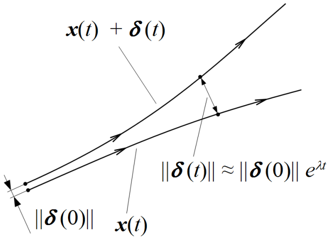
Figure: Left picture by Kang & Xu '18, right pictures Ryne Beeson & N. Sri Namachchivaya '20

Chaos as positive Lyapunov exponents

- The top Lyapunov exponent is the simplest measure of sensitivity to initial condition. If $\Phi^t : \mathcal{M} \to \mathcal{M}$ is the flow then for a given initial condition x,

$$\lambda_1 = \liminf_{t\to\infty} \frac{1}{t} \log \left| D_x \Phi^t(x) \right| .$$

- If $\lambda_1 > 0$ for a.e. x, then we're going to call the system *chaotic* (at least for this talk).
- It is extremely hard in general to verify this condition for deterministic systems, and is only known for a few examples (Lorenz 63, Axiom A systems, contact Anosov flows...)



- This is quite frustrating because chaos is so ubiquitous in nature!

SDEs as a random dynamical system (RDS)

- Moving to the random dynamical systems framework opens up significant mathematical possibilities.
- Let M be an n-dimensional, smooth, connected, orientable Riemannian manifold (possibly unbounded); consider the Stratonovich SDE for a stochastic process $x_t \in M$, $t \geq 0$

$$\mathrm{d}x_t = X_0(x_t)\,\mathrm{d}t + \sum_{k=1}^{r} X_k(x_t) \circ \mathrm{d}W_t^k$$

where $\{X_k\}_{k=0}^r$ are a family of smooth vector fields (potentially unbounded) on M and $\{W^k\}_{k=1}^r$ are independent standard Wiener processes with respect to a canonical stochastic basis $(\Omega, \mathscr{F}, (\mathscr{F}_t), \mathbf{P})$.

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Assumption

$\forall x \in M$, $\exists!$ *global solution almost surely and the solution maps* $x \mapsto x_t := \Phi_\omega^t(x)$ *comprise a (stochastic) flow of diffeomorphisms.*

Markov processes and stationary measures

- Define the Markov kernel $P_t(x, A) = \mathbf{P}(x_t \in A | x_0 = x)$ and the Markov semigroups on observables and Borel probability measures

$$\mathcal{P}_t \varphi(x) = \mathbf{E}(\varphi(x_t) | x_0 = x) = \int_M \varphi(y) P_t(x, \mathrm{d}y)$$

$$\mathcal{P}_t^* \mu(A) = \int_M P_t(y, A) \mu(\mathrm{d}y).$$

- The Markov semigroups solve PDEs $\partial_t \mathcal{P}_t = \mathcal{L} \mathcal{P}_t$ and $\partial_t \mathcal{P}_t^* = \mathcal{L}^* \mathcal{P}_t^*$ where

$$\mathcal{L}^* = (X_0)^* + \frac{1}{2} \sum_{k=1}^{r} (X_k^*)^2$$

$$\mathcal{L} = X_0 + \frac{1}{2} \sum_{k=1}^{r} X_k^2,$$

  where I am using X_0 and X_k as vector fields/differential operators.

- Operators are elliptic iff $\{X_k(x)\}_{k=1}^r$ span $T_x M \ \forall x \in M$.

Stationary measures

- A measure μ is called *stationary* if $\mathcal{P}_t^* \mu = \mu$ for all $t \geq 0$ or, equivalently,

$$\int_M \mathcal{P}_t \varphi(x) \mu(\mathrm{d}x) = \int_M \varphi(x) \mu(\mathrm{d}x).$$

[1] for μ-a.e. initial condition at least

[2] $P_t(x, A) > 0$ for all x, A open suffices but can get away with less.

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Assumption

There is a unique stationary measure for (x_t) process, which we denote μ.

- The pointwise ergodic theorem implies $\mathbf{P} \times \mu$-a.e. (ω, u),

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T \phi(x_t) dt = \int_M \phi(z) d\mu(z).$$

- μ determines long time behavior of statistics [1].
- The Doob-Khasminskii's theorem implies $\mathcal{P}_t : L^\infty \to C^0$ regularization + irreducibility[2] implies uniqueness of μ.
- Even if not elliptic, $\mathcal{P}_t, \mathcal{P}_t^*$ could still be smoothing – called *hypoellipticity*.

[1] for μ-a.e. initial condition at least

[2] $P_t(x, A) > 0$ for all x, A open suffices but can get away with less.

The multiplicative ergodic theorem (MET)

- Under a very mild additional tail assumption on μ, the Multiplicative Ergodic Theorem of Oscledet '68 implies there exists a (deterministic) λ_1 such that for $\mathbf{P} \times \mu$-a.e. (ω, x), there holds

$$\lambda_1 = \lim_{t \to \infty} \frac{1}{t} \log \left| D\Phi^t_\omega(x) \right| ,$$

and a (deterministic) λ_Σ such that for $\mathbf{P} \times \mu$-a.e. (ω, x), there holds

$$\lambda_\Sigma = \lim_{t \to \infty} \frac{1}{t} \log \left| \det D\Phi^t_\omega(x) \right| .$$

- These give asymptotic separation of trajectories and asymptotic compression/expansion of Lebesgue measure respectively.
- ...but how does one estimate these quantities??

The projective process

- Define

$$v_t := \frac{D\Phi^t(x)v}{|D\Phi^t(x)v|},$$

which solves the SDE

$$\mathrm{d}v_t = V_{\nabla X_0(x_t)}(v_t)\mathrm{d}t + \sum_{k=1}^{r} V_{\nabla X_k(x_t)}(v_t) \circ \mathrm{d}W_t^k,$$

where ∇ denotes the covariant derivative and, for $x \in M$ and $A : T_xM \to T_xM$ linear, the vector field V_A on $\mathbb{S}_xM$ is defined by

$$V_A(v) := Av - \langle v, Av\rangle\, v =: \Pi_v Av\,.$$

Projective process II

- The *projective process* $w = (x, v)$ evolves on $\mathbb{S}M$ (or equivalently PM the projective bundle by identifying antipodal points),

$$\mathrm{d}w_t = \tilde{X}_0(w_t)\mathrm{d}t + \sum_{k=1}^{r} \tilde{X}_k(w_t) \circ \mathrm{d}W_t^k \,, \tag{2.1}$$

with $\{\tilde{X}_k\}_{k=0}^r$ defined by[3]

$$\tilde{X}_k(x, v) := \begin{pmatrix} X_k(x) \\ V_{\nabla X_k(x)}(v) \end{pmatrix} .$$

- Has its associated Kolmogorov equations $\partial_t \tilde{\mathcal{P}}_t = \tilde{\mathcal{L}} \tilde{\mathcal{P}}_t$ etc.

Assumption

There is a unique stationary measure for the (w_t) process, which we denote ν.

[3] The canonical Sasaki metric on the sphere bundle provides the natural decomposition $T_w \mathbb{S}M = T_x M \oplus T_v(\mathbb{S}_x M)$

Stationary measures on projective space

- This process knows about stretching and compression in the flow map: consider the projective process for the simple ODE

$$\frac{d}{dt}y_t = \begin{pmatrix} 2 & 0 \\ 0 & 1/2 \end{pmatrix} y_t.$$

- The only stationary measures of the projective process are the Dirac deltas at $(1,0)$ and $(0,1)$,
- If you consider the Markov semigroup associated with $y_t/\,|y_t|$ for this problem $\widetilde{P}_t^*$, then $\widetilde{P}_t^*\mu \to \delta_{(1,0)}$ as long as μ is mutually singular with $\delta_{(0,1)}$.
- Stationary measures of the (w_t) process encode statistics about what directions are undergoing stretching and compression in the flow map.

Furstenberg-Khasminskii

Lyapunov exponents as 'average' volume compression in projective bundle:

Theorem (Furstenberg-Khasminskii)

Define,

$$Q(x) := \operatorname{div} X_0(x) + \frac{1}{2}\sum_{k=1}^{r} X_k \operatorname{div} X_k(x)$$

$$\tilde{Q}(w) := \operatorname{div} \tilde{X}_0(w) + \frac{1}{2}\sum_{k=1}^{r} \tilde{X}_k \operatorname{div} \tilde{X}_k(w).$$

Suppose that (w_t) has a unique stationary measure ν on $\mathbb{S}M$ that projects to μ on M, and that $Q \in L^1(\mu)$ and $\tilde{Q} \in L^1(\nu)$. Then,

$$\lambda_\Sigma = \int_M Q \, \mathrm{d}\mu, \quad n\lambda_1 - 2\lambda_\Sigma = -\int_{\mathbb{S}M} \tilde{Q} \, \mathrm{d}\nu.$$

Problem: Q and $\tilde{Q}$ are not sign definite and its hard to get precise information about ν or μ outside of some simple examples.

Relative entropy à la Furstenberg

Lyapunov exponents as relative entropy growth between the stationary measure and its random push-forwards:

Theorem (Baxendale '89)

Denote $\mathrm{d}\nu(x,v) = \mathrm{d}\nu_x(v)\mathrm{d}\mu(x)$ *the disintegration of* ν, $\forall t > 0$ *there holds under suitable conditions*[4]

$$\mathbf{E}\int_M H(D\Phi^t_\omega(x)_*\nu_x|\nu_{\Phi^t_\omega(x)})\,\mathrm{d}\mu(x) = t\,(n\lambda_1 - \lambda_\Sigma)\,,$$

where H *denotes the relative entropy of defined for two measure measures* $\eta \ll \lambda$ *by*

$$H(\eta|\lambda) := \int \log\left(\frac{\mathrm{d}\eta}{\mathrm{d}\lambda}\right)\,\mathrm{d}\eta\,.$$

Proved by an identity on how matrix action deforms volume on projective space, the ergodic theorem, and a bit of trickery.

[4] More generally, one often gets the lower bound $\leq$.

à la Furstenberg

- KEY consequence: $n\lambda_1 = \lambda_\Sigma$ if and only if we have the *almost-sure* degeneracy $(D\Phi^t_\omega)_* \nu_x = \nu_{\Phi^t_\omega(x)}$.
- Ideas using this almost-sure degeneracy are called à la Furstenberg, after his seminal 1968 paper. For example it implies a lot of rigidity, easily ruled out in some cases:

Theorem (Baxendale '89, and similar work by Furstenberg, Ledrappier, Carverhill, Virtser, Royer, and others...)

If $n\lambda_1 = \lambda_\Sigma$ then one of two possibilities hold:

(a) *$\exists$ a continuous family of inner products $\langle\cdot,\cdot\rangle_x$ such that $D_{x_0}\phi^t$ is* **almost surely** *an isometry $\langle\cdot,\cdot\rangle_{x_0} \to \langle\cdot,\cdot\rangle_{x_t}$.*

(b) *$\exists$ continuous families of proper linear subspaces $L^i(x), i \leq p$ such that* **almost surely**

$$D_{x_0}\phi^t\big(\cup_{i=1}^p L^i(x_0)\big) = \cup_{i=1}^p L^i(x_t)$$

- Powerful: We used an infinite dimensional variant to prove "Lagrangian chaos" for stochastic Navier-Stokes.
- Problems: not quantitative and, great if $\lambda_\Sigma = 0$, but otherwise, it doesn't give positivity.

Fisher information formula

The following seems to be a new identity, in terms of a *Fisher information* (let q be the Lebesgue measure on SM)

Theorem (JB/Blumenthal/Punshon-Smith '20)

Let $\nu = f\mathrm{d}q$, $\mu = \rho\mathrm{d}x$ *and* $\nu_x(v) = f(x,v)/\rho(x)$. *Then,*

$$\frac{1}{2}\sum_{k=1}^{r}\int_M \frac{|X_k^*\rho|^2}{\rho}\mathrm{d}x := FI(\rho) = -\lambda_\Sigma$$

$$\frac{1}{2}\sum_{k=1}^{r}\int_M \frac{\left|\tilde{X}_k^* f\right|^2}{f}\mathrm{d}q := FI(f) = n\lambda_1 - 2\lambda_\Sigma \,.$$

and ($\mathrm{d}v$ *here denotes Lebesgue measure on* $\mathbb{S}^{d-1}$)

$$FI(f) - FI(\rho) = \frac{1}{2}\sum_{k=1}^{r}\int_M \left(\int_{\mathbb{S}_x M} \frac{|(X_k - V^*_{\nabla X_k(x)})\nu_x(v)|^2}{\nu_x(v)}\mathrm{d}v\right)\mathrm{d}\mu(x) = n\lambda_1 - \lambda_\Sigma.$$

Proof from Furstenberg-Khasminskii

- Our original argument came from formally computing $\lim_{t\to 0} \frac{1}{t}\mathbf{E}\int_M H(D\Phi^t(x)_*\nu_x|\nu_{\Phi^t(x)})\,\mathrm{d}\mu(x) = FI(f) - FI(\rho)$.

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- Easier proof from the Furstenberg-Khasminskii and the Kolmogorov equation:

$$\tilde{\mathcal{L}}^* f = \tilde{X}_0^* f + \frac{1}{2}\sum_{j=1}^{r}(\tilde{X}_j^*)^2 f = 0.$$

- Formal: pair with $\log f$ and integrate by parts: $(\tilde{X}^* g = -\tilde{X} g - (\mathrm{div}\,\tilde{X})g)$

$$-\frac{1}{2}\sum_{k=1}^{r}\int_{\mathbb{S}M}(\log f)(\tilde{X}_k^*)^2 f\,\mathrm{d}q = FI(f) + \frac{1}{2}\sum_{k=1}^{r}\int_{\mathbb{S}M}(\tilde{X}_k\,\mathrm{div}\,\tilde{X}_k)\,f\,\mathrm{d}q,$$

and

$$\int(\log f)\tilde{X}_0^* f\,\mathrm{d}q = \int \tilde{X}_0 f\,\mathrm{d}q = -\int(\mathrm{div}\,\tilde{X}_0)f\,\mathrm{d}q.$$

This yields the identity $FI(f) = n\lambda_1 - 2\lambda_\Sigma$ by Furstenberg-Khasminskii. Proof of λ_Σ is similar but on ρ instead of f. Conditional variant a consequence thereof after a tricky calculation.

The Fisher information formula: why it is a useful identity

- It is stated in terms of a non-negative quantity (unlike Furstenberg-Khasminskii)
- It is time-infinitesimal directly on the stationary measure (unlike the relative entropy formula)

$$\frac{1}{2}\sum_{k=1}^{r}\int_{\mathbb{S}M}\frac{\left|\tilde{X}_k^* f\right|^2}{f}\mathrm{d}q := FI(f) = n\lambda_1 - 2\lambda_\Sigma\,.$$

- Relates *partial regularity* of f to Lyapunov exponents. Note specifically that for $g \in C_c^\infty$,

$$\left|\left|\tilde{X}_k^* g\right|\right|_{L^1} \leq ||g||_{L^1}^{1/2}\left(\int_{\mathbb{S}M}\frac{\left|\tilde{X}_k^* g\right|^2}{g}\mathrm{d}q\right)^{1/2}$$

which is basically $W^{1,1}$ in the forcing directions.

Quantitative estimates in small-noise limits

- This will be most useful for small noise limits:

$$\mathrm{d}x_t^\epsilon = X_0^\epsilon(x_t)\,\mathrm{d}t + \sqrt{\epsilon}\sum_{k=1}^{r} X_k^\epsilon(x_t)\circ \mathrm{d}W_t^k\,,$$

  (note we are allowing X_k^ϵ to be parameterized by ϵ as well).

- The Fisher information identity becomes

$$FI(f^\epsilon) := \frac{1}{2}\sum_{k=1}^{r}\int_{\mathbb{S}M} \frac{|(\tilde{X}_k^\epsilon)^* f^\epsilon|^2}{f^\epsilon}\,\mathrm{d}q = \frac{n\lambda_1^\epsilon - 2\lambda_\Sigma^\epsilon}{\epsilon}.$$

- Note however that the identity only contains derivatives in the directions on which the forcing acts.
- For most[5] choices of $\{X_k\}_{k=1}^r$, $\left\{\tilde{X}_k\right\}_{k=1}^r$ will not span $T_w\mathbb{S}M$.

[5] In fact, it isn't so easy to find *any* example where $\left\{\tilde{X}_k\right\}_{k=1}^r$ spans $T_w\mathbb{S}M$ but it is possible with a big enough set of nonlinear $\{X_k\}_{k=1}^r$.

Hörmander's hypoelliptic spanning condition

- Hörmander codified an essentially sharp condition for *hypoellipticity* in Kolmogorov equations – that is, regularity without $\{X_1, ..., X_r\}$ spanning the tangent space.

Definition (Hörmander spanning condition)

Given a collection of vector fields $Z_0, Z_1, \ldots, Z_r$ on a manifold $\mathcal{M}$, we define collections of vector fields $\mathscr{X}_0 \subseteq \mathscr{X}_1 \subseteq \ldots$recursively by

$$\mathscr{X}_0 = \{Z_j \,:\, j \geq 1\},$$
$$\mathscr{X}_{k+1} = \mathscr{X}_k \cup \{[Z_j, Z] \,:\, Z \in \mathscr{X}_k, \quad j \geq 0\}.$$

We say that $\{Z_i\}_{i=0}^r$ satisfies the *parabolic Hörmander condition* if exists k such that for all $w \in \mathcal{M}$,

$$\operatorname{span}\{Z(w) \,:\, Z \in \mathscr{X}_k\} = T_w\mathcal{M}. \tag{4.2}$$

Projective hypoellipticity

- By Hörmander's theorem, spanning implies $\mathcal{P}_t$ and $\mathcal{P}_t^*$ are instantly smoothing, i.e. $L^\infty \to C^\infty$ and $L^1 \to C^\infty$ for $t > 0$ (and similarly for projective spanning).
- Projective hypoellipticity would almost certainly be used in any proof of uniqueness of ν.

Uniform hypoelliptic regularity from the Fisher information

Theorem (JB/Blumenthal/Punshon-Smith '20)

Assume that $\{\tilde{X}_0^\epsilon, ..., \tilde{X}_r^\epsilon\}$ *are uniformly bounded in* C^k_{loc} $\forall k$ *and (uniform) projective spanning holds. Then,* $\exists s_* \in (0,1)$ *such that for any bounded, open set* $U \subset \mathbb{S}M$, $\exists C = C_U > 0$ *such that* $\forall \epsilon \in (0,1]$

$$||f^\epsilon||_{W^{s_*,1}(U)} \leq C\left(1 + \sqrt{FI(f^\epsilon)}\right).$$

- **Key**: No constants depend on ϵ! By the Fisher information identity we get the regularity estimate:

$$||f^\epsilon||_{W^{s_*,1}(U)} \leq C\left(1 + \left(\frac{n\lambda_1^\epsilon - 2\lambda_\Sigma^\epsilon}{\epsilon}\right)^{1/2}\right).$$

- The regularity s_* is explicitly related to how many generations of brackets (and what kinds) are required until spanning.

Uniform hypoelliptic regularity from the Fisher information

- This theorem has nothing to do with sphere bundles etc, its a general L^1-type (uniform) hypoelliptic regularity estimate on Kolmogorov equations on an orientable manifold $\mathcal{M}$.
- It was easiest to go back to Hörmander's 1967 proof, which is very robust...

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- It was easiest to go back to Hörmander's 1967 proof, which is very robust...
- One starts with assuming you have a good bound on $||X_k^* f||_{L^1}$ for $k = 1, ..., r$. Hörmander left a hint:

$$\left|\int_{\mathcal{M}} \varphi X_0^* f \mathrm{d}q\right| \leq \frac{1}{2}\left|\int_{\mathcal{M}} \varphi (X_k^*)^2 f \mathrm{d}q\right| \leq \frac{1}{2}\sum_{k=1}^{r} ||X_k \varphi||_{L^\infty} ||X_k^* f||_{L^1}.$$

$$\mathfrak{D}(g) = \sup_{v \in C_c^\infty : ||v||_{L^\infty} + \sum_{k=1}^r ||X_k v||_{L^\infty} \leq 1} \left|\int_{\mathcal{M}} v X_0^* g \mathrm{d}q\right|.$$

- For χf with χ a smooth cutoff to a bounded set, it isn't hard to check that

$$\mathfrak{D}(\chi f) \lesssim 1 + \sqrt{FI(f)}.$$

Uniform hypoelliptic regularity from the Fisher information

- Main step: the "Hörmander inequality" $\forall g \in C_c^\infty(U)$

$$||g||_{W^{s_*,1}} \lesssim ||g||_{L^1} + \mathfrak{D}(g) + \sum_{k=1}^{r} ||X_k^* g||_{L^1} .$$

- By easy variant of a Hörmander lemma, for s depending on $\{s_k\}$.

$$||g||_{W^{s,1}} \lesssim ||g||_{L^1} + \sum_{k=0}^{r} |g|_{X_k,s_k} := ||g||_{L^1} + \sum_{k=0}^{r} \sup_{|h|\leq 1} |h|^{-s_k} \left|\left| e^{hX_k} g - g \right|\right|_{L^1} .$$

- Hörmander proved this using the CBH formula and some cleverness

$$e^{-tX} e^{-tY} e^{tX} e^{tY} = e^{\frac{t^2}{2}[X,Y]+\dots} .$$

- The *really* hard part is to recover $|g|_{X_0,1/2}$. Hörmander used a delicate regularization and duality argument to get around this issue in the L^2 framework
- In the non-self-dual L^1-L^∞ framework, we need an even more complicated regularization procedure...

A class of "Euler-like" nonlinear systems

- Interested in a class of weakly damped SDE on $\mathbb{R}^d$: A symmetric positive definite, B bilinear with

$$\mathrm{d}x_t = (B(x_t, x_t) - \epsilon A x_t)\,\mathrm{d}t + \sum_{k=1}^{r} X_k \mathrm{d}W_t^{(k)},$$

where X_k are constant ("additive noise"). We assume:

$$\nabla \cdot B(x,x) = 0, \qquad x \cdot B(x,x) = 0.$$

These conditions make the nonlinearity conserve volume in phase space and conserves the energy $E = \frac{1}{2}|x|^2$.

- Will also impose the cancellation condition[6] $B(e_k, e_k) = 0$.

[6] The $\{e_1, \ldots, e_s\}$ can be any set of vectors which form a basis for $\mathrm{span}\{X_1, \ldots, X_r\}$, not necessarily canonical coordinate vectors, though for many examples, like L96 and Galerkin-Navier-Stokes they are.

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- Will also impose the cancellation condition[6] $B(e_k, e_k) = 0$.
- There is no $\epsilon \to 0$ limit as the energy of the steady state increases like $\mathcal{O}(\epsilon^{-1})$. Need to rescale x and t:

$$\mathrm{d}x_t = (B(x_t, x_t) - \epsilon A x_t) + \sqrt{\epsilon} \sum_{k=1}^{r} X_k \mathrm{d}W_t^{(k)},$$

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A first application: projective hypoellipticity implies chaos

A first application

Theorem (JB/Blumenthal/Punshon-Smith '20)

For the above "Euler-like" systems, projective hypoellipticity implies

$$\lim_{\epsilon \to 0} \frac{\lambda_1^\epsilon}{\epsilon} = \infty.$$

In particular, $\exists \epsilon_0 > 0$ such that for $\epsilon \in (0, \epsilon_0)$, the top Lyapunov exponent is strictly positive.

- The quantitative lower bound is expected to be far from optimal of course, but its the first one of its kind nevertheless.
- Previous quantitative estimates for simple 2d nonlinear oscillators (Baxendale-Goukasian '02) and a class of simple linear SDEs (Pinsky-Whihstutz '88). These require an almost exact understanding of ν^ϵ.
- Verifying projective hypoellipticity is neither impossible nor easy...

Basic idea of the proof

- Nice qualitative properties (existence, uniqueness, regularity of μ, ν for $\epsilon > 0$ etc) follow from projective hypoellipticity, the energy identity $x \cdot B = 0$, and geometric control theory[7] to deduce projective irreducibility.
- Linear damping and $\nabla \cdot B = 0$ implies $\lambda_\Sigma = -\epsilon \mathrm{tr} A$.

[7] This is where the cancellation condition is used.

[8] Tightness is proved using tail control that comes from the energy identity again.

Basic idea of the proof

- Nice qualitative properties (existence, uniqueness, regularity of μ, ν for $\epsilon > 0$ etc) follow from projective hypoellipticity, the energy identity $x \cdot B = 0$, and geometric control theory[7] to deduce projective irreducibility.
- Linear damping and $\nabla \cdot B = 0$ implies $\lambda_\Sigma = -\epsilon \mathrm{tr} A$.
- The Fisher information formula and hypoellipticity theorem implies

$$||f^\epsilon||_{W^{s_*,1}(U)} \lesssim 1 + \left(\frac{n\lambda_1^\epsilon + 2\epsilon \mathrm{tr} A}{\epsilon}\right)^{1/2}.$$

- Assume for contradiction that $\lambda_1^\epsilon/\epsilon \lesssim 1$. Then we get uniform-in-ϵ regularity in $W^{s,1}_{loc}$.
- Compactness[8] implies $f^\epsilon \to f$ strongly in L^1 to the density of an absolutely continuous invariant measure of the *deterministic* projective process $w_t = (x_t, v_t)$ satisfying

$$\frac{d}{dt} w_t = \tilde{X}_0 w_t.$$

[7] This is where the cancellation condition is used.

[8] Tightness is proved using tail control that comes from the energy identity again.

A neat identity from the structure of Euler-like systems

- A rigidity theorem[9] shows that no such L^1 invariant measure can exist if there's any unbounded growth of $D\Phi^t$.
- The bilinearity and energy structure of the system gives a neat identity:

$$D\Phi^t(x)x = \Phi^t(x) + tB(\Phi^t(x), \Phi^t(x)),$$

- By Poincaré recurrence and $B \neq 0$ a.e., this contradicts rigidity lemma.

[9] Basically Theorem 3.23 in Arnold/Nguen/Oseledets '99, which essentially shows that an absolutely continuous measure implies Case (a) in the à la Furstenberg dichotomy, i.e. that $D\Phi^t$ is an isometry in suitable inner products.

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- By Poincaré recurrence and $B \neq 0$ a.e., this contradicts rigidity lemma.
- The same general argument for $\lambda_1^\epsilon/\epsilon \to \infty$ applies to all sorts of systems with at least unbounded growth and projective hypoellipticity.
- This is robust to lack of knowledge about the deterministic dynamics: you don't need to know about all the mechanisms for growth of $D\Phi^t$, it suffices to know *something*.
- Hopefully there is an opportunity for incremental improvements, as more dynamical 'motifs' are identified.

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Projectve spanning for Euler-like systems

For Euler-like systems with additive noise, we can reduce projective spanning to question about matrix Lie algebras.

Lemma (JB/Blumenthal/Punshon-Smith '20)

Suppose that $\{\partial_{x_k}\}_{k=1}^n$ *in span of finitely many brackets*[10] *of* $\{X_0, X_1, \ldots, X_r\}$. *Define for each* $k = 1, \ldots n$ *the following constant matrices ,*

$$H^k := \partial_{x_k} \nabla B \in \mathfrak{sl}_n(\mathbb{R})$$

and let $\mathrm{Lie}(H^1, \ldots, H^n)$ *be the matrix Lie sub-algebra of* $\mathfrak{sl}_n(\mathbb{R})$ *generated by* $H^1, \ldots H^n$. *Then projective spanning holds if*

$$\mathrm{Lie}(H^1, \ldots, H^n) = \mathfrak{sl}_n(\mathbb{R}). \tag{5.3}$$

- Note that this doesn't involve the noise anymore, it is purely a property of the nonlinearity!
- Says essentially that everywhere, it is possible to move in x infinitesimally in order to deform the linearization in any direction.

[10] not too hard to check for Lorenz 96, Galerkin-Navier-Stokes, and the shell models GOY and SABRA, for example.

Chaos for stochastically driven Lorenz 96

- Recall we have J unknowns in a periodic array (i.e. $u_k = u_{k+nJ}$)

$$\mathrm{d}u_\ell = (u_{\ell+1} - u_{\ell-2})u_{\ell-1}\mathrm{d}t - \epsilon u_\ell \mathrm{d}t + \sqrt{\epsilon} q_\ell \mathrm{d}W_t^\ell.$$

Theorem (JB/Blumenthal/Punshon-Smith '20)

If $J \geq 7$ and q_1 and q_2 are both non-zero, then

$$\lim_{\epsilon \to 0} \frac{\lambda_\epsilon^1}{\epsilon} = \infty.$$

- Unfortunately, we didn't use a robust method, we basically just played around with commutators until we found the set of elementary matrices: $E^{1,2}, E^{2,3}, \ldots, E^{n,1}$, which is a generating set for $\mathfrak{sl}_n(\mathbb{R})$.
- (here an elementary matrix $E^{i,j}_{mn} = \delta_{(i,j)=(m,n)}$)

Galerkin-Navier-Stokes

- The 2d NSE in vorticity form on[11] $\mathbb{T}^2$ for $w \in \mathbb{C}^{\mathbb{Z}^2_{\leq N}}$ satisfying

$$\partial_t w_\ell = \sum_{\ell = k+j} c_{j,k} w_j w_k - \epsilon \left|\ell\right|^2 + \sqrt{\epsilon} q_\ell dW_t^{(\ell)},$$

where $c_{j,k}$ is a rational function of j, k, $w_k = \overline{w_{-k}}$, and $\mathbb{Z}^2_{\leq N} = \left\{k \in Z^2 : \left|k\right|_{\ell^\infty} \leq N\right\}$. Writing in real variables, one can check that this is an "Euler-like" system.

Theorem (JB/Punshon-Smith '21 (work in progress!))

For all N sufficiently large[12] projective hypoellipticity holds, and hence chaos for all ϵ sufficiently small.

- Much harder due to full coupling of unknowns.
- The proof uses some basic formalisms from complex geometry, ideas from bilinear control theory for dealing with matrix Lie algebras, and computational algebraic geometry...
- May provide a blueprint for deducing similar hypoelliptic results for other PDEs...

[11] The results hold for all rectangular torii too, not just square.

[12] I think $N \geq 4$ suffices but don't quote me on the exact number yet.

Looking forward

- We haven't even used the full power of the lower bound we have... so there is a lot of work to be done and the results still leave a lot to be desired, but we are hoping its a reasonable start... There should be room for improvements in:
  - Even to use the compactness-rigidity: need more robust methods for proving growth of $D\Phi^t$ in deterministic dynamical systems;
  - Better hypoelliptic tools (better lower bounds by precision norms, more advantageous ϵ dependence, problem specific etc);
  - Using more information about the dynamics (i.e. more 'motifs') in order to get better lower bounds on $||f^\epsilon||_{W^{s,1}}$ and similar quantities;
- Infinite dimensions would be great, but there are a lot of challenges to this...

Thank you for your attention!