Эквивалентность аффинных прямых на симплициальных торических многообразиях

По работе Ш.Калимана "Affine lines on simplicial toric varieties", arXiv:2102.07170v2

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May 21, 2021

Problem

When are two affine lines $C_1, C_2 \subset Y_{\mathrm{reg}}$ equivalent up to $\mathrm{Aut}(Y)$?

(Abhyankar–Moh 1975, Suzuki 1974)

Any two closed embeddings of $\mathbb{A}^1$ into $\mathbb{A}^2$ differ by an automorphism of $\mathbb{A}^2$. Furthermore, if $\mathbb{A}^1 \hookrightarrow \mathbb{A}^2$, $t \mapsto (p(t), q(t))$, where $p, q \in \mathbb{C}[t]$, then $\deg p \mid \deg q$ or vice versa.

(Arzhantsev–Zaidenberg 2011)

Let $X = \mathbb{A}^2/G$ for a finite $G \subset \mathrm{Aut}(\mathbb{A}^2)$. Then

- up to automorphism $G \subset \mathrm{GL}(2, \mathbb{K})$,
- if G abelian, then $G = \left\langle \begin{pmatrix} \zeta_d^e & 0 \\ 0 & \zeta_d \end{pmatrix} \right\rangle$, and each embedding of $\mathbb{A}^1$ into X_{reg} is equivalent to the image of either $C_y = \{y = 0\}$ or $C_x = \{x = 0\}$. These two images are equivalent iff $e^2 \underset{d}{\equiv} 1$.
- if G non-abelian, then no embeddings $\mathbb{A}^1 \hookrightarrow X_{\mathrm{reg}}$ exist.

Embeddings

Denote $ED(Z) = \max(2\dim Z + 1, \dim TZ)$.
The Holme Theorem: $Z \hookrightarrow \mathbb{A}^{ED(Z)}$.

(Feller, van Santen 2021)

If X is a simple algebraic group, Z smooth, $ED(Z) < \dim X$, then $Z \hookrightarrow X$.

Theorem 3.7

If Z affine and X is smooth flexible quasi-affine, $\dim X \geq ED(Z)$, then Z admits an injective immersion into X.

The properness is the bottleneck for an embedding. In Theorem 5.2 one obtains a closed (in X) embedding into X_{reg} for a toric $X = X_\sigma$, if $\dim Z < \operatorname{codim}_X X_{\text{sing}}$.

Kaliman, 1991

For a smooth Z and $n \geq 2\dim Z + 2$, all embeddings $Z \hookrightarrow \mathbb{A}^n$ are equivalent.

What for $\mathbb{A}^1 \hookrightarrow \mathbb{A}^n/G$, where $G \subset \mathbb{T}$ is finite?

Perfect families

Let $\mathcal{N}$ be a saturated collection of $\mathbb{G}_a$-subgroups in $\mathrm{Aut}(X)$, and let $G = \langle H \mid H \in \mathcal{N} \rangle$ act with an open orbit $O \subset X$.

The Transversality Theorem 2.2

There exists a family $A = H_1 \times \ldots \times H_m$ such that for any loc. closed $Y, Z \subset O$ for any g in some open subset $U(Y, Z) \subset A$:

- gZ_{reg} meets Y_{reg} transversally, and
- $\mathrm{codim}_X\, gZ_{\mathrm{reg}} \cap Y_{\mathrm{reg}} \geq \mathrm{codim}_X\, Y + \mathrm{codim}_X\, Z$ if not empty.

A family $A = H_1 \times \ldots \times H_m$, where $H_i \in \mathcal{N}$, is called perfect if

$$A \times Y \to Y \times Y,\ (h_1 \ldots, h_m, y) \mapsto (h_1 \ldots h_m y, y)$$

is smooth (gives a flat family of nonsingular varieties) on $U \times Y$, where Y is any of $X, T^*X, \mathbb{P}\mathrm{Fr}(TX), X \times X \setminus \Delta$.
If A perfect, then $A \times H, H \times A$ are also perfect. They exist for G-flexible varieties.
Any perfect A satisfies Transversality Theorem and several others.

Simplicial toric varieties

Let

- $X = X_\sigma$ of $\dim(X) = n \geq 4$ simplicial without torus factors, i.e. $\sigma \subset N_\mathbb{R}$ has simplex faces and $r = n$ rays $\rho_1, \ldots, \rho_r$;
- $\pi\colon \mathbb{A}^r \to \mathbb{A}^r/G \cong X_\sigma$, G finite, $\mathbb{A}^r = \{(x_1, \ldots, x_r)\}$;
- O_i the orbit for ρ_i, $D_i = \overline{O}_i$ divisor (normal, $\mathbb{T}$-invariant);
- $H_i \subset \mathbb{T}$ a 1-subgroup for ρ_i, $H_i(t) \cdot \chi^m = t^{\langle m, \rho_i \rangle} \chi^m$, it acts trivially on D_i;
- $\tilde{T} \cong \mathbb{G}_m^r$ acts on $\mathbb{A}^r$, $\tilde{H}_i = \{x_i \mapsto t x_i\}$, trivial on $\tilde{D}_i$.
- $U \subset X_\sigma$ consists of points where the fiber is a G-orbit.
- $U_0 = X_{\text{reg}} \subset U$ consists of points with a trivial stabilizer on the orbit.

LNDs

6.1. π lifts $\partial_{\rho_i,e}$ into $\prod_{k\neq i} x_k^{\pi^*(e)_k} \frac{\partial}{\partial x_i}$
6.2. $\theta_{ij} = \mathrm{Cone}(\rho_i, \rho_j)$, $H_{ij} = \langle H_i, H_j \rangle$,

$$\operatorname{Spec} \mathbb{K}[\theta_{ij,M}^{\vee}] = D_{ij} = X_\sigma // H_{ij} = D_i \cap D_j$$

is an invariant divisor in both. Assume all θ_{ij} are regular.
6.3. The $\mathbb{T}$-equivariant morphism $\kappa_i \colon X_\sigma \to D_i$ (quotient by the $\mathbb{G}_a$-action of $\partial_{\rho_i,e} \colon \chi^m \mapsto \langle m, \rho_i \rangle \chi^{m+e}$, i.e., dual to $\tau_i \hookrightarrow \sigma$) is smooth over the open orbit $O(\theta_{ij}) \subset D_{ij}$ with fibers $\mathbb{A}^1$.
6.4. There is an open $V_i \subset D_i \cap X_{\mathrm{reg}}$ big in D_i, and $\forall v \in V_i$ $\exists \delta = g\partial_{\rho_i,e}$, $g \in \mathbb{K}(D_i)$ (not $\mathbb{K}[D_i]$), nonvanishing on $\kappa_i^{-1}(v)$.
6.5. Let $Z \subset V_i$ be closed in D_i, $s \colon Z \to \mathbb{A}^r$ a section of $\pi|_Z$, and $s(Z) \subset \mathbb{A}^r$ closed. Then $\exists \delta \sim \partial_{\rho_i,e}$ nonvanishing on $\kappa_i^{-1}(Z)$.
6.6. Let $Z \cong \mathbb{A}^1$, $\delta = \pi_* \tilde{\delta}$. Then $\forall h(t) \in \mathbb{K}[t] \exists g \in \ker \delta$ such that for $\delta' = \pi^*(g)\tilde{\delta}$ there holds $x_i(\exp(\delta')(t)) = h(t)$, $t \in \pi^*(Z)$ is the coordinate.

Curves on X_σ

X_σ simplicial: every face is a simplex, $n = r$, G finite.
$\kappa_i \colon X_\sigma \to D_i = X_\sigma // \exp(t\partial_{\rho_i,e})$. $V_i' = \kappa_i^{-1}(V_i)$.

Lemma 7.1
$\dim X \geq 4$, $C \subset X_{\mathrm{reg}}$ smooth curve. For θ_{jk} let $\psi_{\theta_{jk}} : X_\sigma \to D_{jk}$ induced by $\tau_j \cap \tau_k \hookrightarrow \sigma$. The for any j, k up to $\mathrm{Aut}\, X_\sigma$:

1. C is in $V = \bigcap V_i'$;
2. $\kappa_j(C)$ meets $W_j = V_j' \setminus \bigcap_k V_k'$ at finite set;
3. $\kappa_j|_C$ is a closed embedding into D_j;
4. $\psi_{\theta_{jk}}|_C$ is birational.

Proof sketch: V_k contains an open subset of D_{lj}, so V_k' contains an open subset of D_j, so it is big in X. Then V is flexible and any perfect family on V extends to X. For a general $\alpha \in A$ all conditions hold for $\alpha(C)$.
Lemma 7.2. Let C as in 7.1 and δ' as in 6.6 with $h = ct + d$ for general c, d. Then $\exp(\delta')(C)$ also satisfies conditions in 7.1.

Th.7.3. Equivalence of $C, C' \cong \mathbb{A}^1$ in X_{reg}

Proof sketch: Since $\pi^{-1}(C)$ is a locally trivial principal G-bundle, it admits a section, see [Feller–van Santen]. Then there are $\tilde{C}, \tilde{C}' \subset \mathbb{A}^n$ such that $\pi\colon \tilde{C} \overset{\cong}{\to} C, \tilde{C}' \overset{\cong}{\to} C'$. Let t and t' be coordinates on C and C', $t = \phi^* t'$, where $\phi\colon C \to C'$. Applying Lemma 7.2 for $i = 1, \ldots, r$, we may assume that $x_i(t) = c_i t + d_i$ for all i, where $(c_1, d_1, \ldots, c_r, d_r)$ is a general point in $\mathbb{A}^{2r}$. Doing the same for C', we choose general points equal.