Amoebas of the second kind

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The moduli space

$$\mathcal{S} := \{\Gamma, d\zeta_1, d\zeta_2\} \tag{1}$$

of smooth algebraic curves with a pair of *real (imaginary)* normalized differentials are central for many aspects of the algebraic-geometrical integration theory. They provide a unifying framework for

- the Hamiltonian theory of soliton equations,
- the Whitham equations,
- WDVV equations,
- Siberg-Witten solution of $N = 2$ SUSY gauge models.
- Laplacian growth problem

Imaginary normalized (IN) differentials

By definition an imaginary normalized meromorphic differential is a differential whose periods over any cycle on the curve are imaginary.

The universality of this notion is that:

Lemma

For any fixed singular parts of poles with pure real residues, there exists a unique meromorphic differential $d\zeta$, having prescribed singular part at p_α and such that all its periods on Γ are real, i.e.

$$\mathrm{Re}\left(\oint_c d\zeta\right) = 0, \ \ \forall\ c \in H^1(\Gamma, \mathbb{Z}).$$

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IN differentials vs harmonic functions

The notion of IN differentials is almost "equivalent" to that of harmonic function. Indeed by definition the real part of the abelian integral

$$x(p) = \mathrm{Re} \int^{p} d\zeta$$

is a single valued harmonic function on $\Gamma_0 := \Gamma \setminus \{p_\alpha\}$.

Conversely: let $x(p)$ be a harmonic function, then locally there exists a unique up to an additive constant conjugate harmonic function $y(p)$. Hence, $x(p)$ uniquely defines the differential $d\zeta = dx + idy$, which by construction is *imaginary normalized* holomorphic differential on Γ_0. One can specify asymptotic behavior of $x(p)$ near the marked point by the requirement that $d\zeta$ is meromorphic on Γ and has a fixed singular part at the marked points.

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Motivation

The imaginary (real) normalized differentials of the third kind per se are not new. They were probably known to Maxwell (the real part of such differential is a single valued harmonic function on Γ equal to the potential of electromagnetic filed on Γ created by charged particles at the marked points); they were used in the, so-called, light-cone string theory, and played a crucial role in joint works of S. Novikov and the author on Laurent-Fourier theory on Riemann surfaces and on operator quantization of bosonic strings.

IN differentials from the spectral theory

Imaginary normalization as defining property of the quasi-momentum differentials in the spectral theory of linear operators with quasi-periodic coefficients was introduced by the author at the end of 1980s (they were called then absolute normalized).

Recall: If L is a difference (differential) operator with periodic coefficients then its spectral curve parameterize Bloch solutions

$$L(\lambda)\psi = 0, \quad \psi(x+\ell,\lambda) = w(\lambda)\psi$$

.

By definition $\ell^{-1} d \log w$ is a meromorphic differential on a spectral curve with periods that are

$$\oint \ell^{-1} d \log w \in \frac{2\pi i}{\ell}\mathbb{Z}$$

The imaginary normalization is an interpolation of the latter to the case of operators with quasi-periodic coefficients.

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Vanishing properties of $\mathcal{M}_{g,n}$

Constructions with real normalized differentials have found applications to the study of geometry of moduli spaces of curves with punctures. Among results are:

- a new proof the Diaz' theorem on dimension of complete subvarieties of $\mathcal{M}_g$ (Grushevsky-Kr, 2011).
- the proof of Arbarello's conjecture: the statement that any complete complex subvariety of $\mathcal{M}_g$ of dimension $g - n$ intersects the locus W_n of curves that have a Weierstrass point of weight at most n (Kr 2011).

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The key elements of the approach for a study of geometry of the moduli space of smooth curves with marked points are:

- the moduli space $\mathcal{M}_{g,k}^{(n)}$, $n = (n_1, \ldots, n_k)$ of smooth genus g Riemann surfaces with the fixed n_α-jets of local coordinates in the neighborhoods of labeled points is the total space of a *real-analytic* foliation, whose leaves $\mathcal{L}$ are locally smooth *complex subvarieties* of real codimension $2g$;
- on $\mathcal{M}_{g,k}^{(n)}$ there is an ordered set of $(\dim_{\mathbb{R}} \mathcal{L})$ continuous functions, which restricted onto the leaves of the foliation are piecewise harmonic. Moreover, the first of these function restricted onto $\mathcal{L}$ is a subharmonic function.

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The foliation structure arises through identification of $\mathcal{M}_{g,k}^{(n)}$ with the moduli space of curves with fixed *real-normalized* meromorphic differential.

Definition

A leaf $\mathcal{L}$ of the foliation on $\mathcal{M}_{g,k}^{(n)}$ defined to be the locus along which the periods of the corresponding differentials remain (covariantly) constant.

The leaves $\mathcal{L}$ of the foliation can be regarded as a generalization of the Hurwitz spaces of $\mathbb{P}^1$ covers.

It is basic fact of the Whitham theory:

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Coordinates along a leaf

A set of holomorphic coordinates on $\mathcal{M}_{g,k}^{(n)}$ are "critical" values of the corresponding abelian integral $F(p) = c + \int^p \Psi$, $p \in \Gamma$:

At the generic point, where zeros q_s of Ψ are distinct, the coordinates on $\mathcal{L}$ are the evaluation of F at these critical points:

$$\varphi_s = F(q_s), \;\; \Psi(q_s) = 0, \;\; s = 0, \ldots, d = \dim \mathcal{L}, \qquad (2)$$

normalized by the condition $\sum_s \varphi_s = 0$.

A direct corollary of the imaginary normalization is the statement that:

- *the real parts $f_s = \mathrm{Re}\,\varphi_s$ of the critical values depend only on labeling of the critical points*

They can be arranged into decreasing order

$$f_0 \geq f_1 \geq \cdots \geq f_{d-1} \geq f_d.$$

After that f_j can be seen as a well-defined continuous function on $\mathcal{M}_{g,k}^{(n)}$, which restricted onto $\mathcal{L}$ is a piecewise harmonic function. Moreover, f_0 restricted onto $\mathcal{L}$ is a *subharmonic function*.

Diaz' theorem revisited

Let X be a complete cycle in $\mathcal{M}_g$ and Z be its preimage under the forgetfull map: $\mathcal{M}_{g,2} \subset \mathcal{C}_g^2 \longmapsto \mathcal{M}_g$.

$\rightarrow$ On Z the function f_0 (defined by critical values of real-normalized differential with two simple poles) must achieve its maximum at some point.

$\rightarrow$ At this point the function f_0 achieves its maximum on $Z \cap \mathcal{L}$.

$\rightarrow$ Hence, it is a constant on $Z \cap \mathcal{L}$.

$\rightarrow$ If f_0 is a constant then (inductively) all the other functions f_j are constants.

$\rightarrow$ Then, $Z \cap \mathcal{L}$ is at most zero-dimensional, i.e. Z intersects $\mathcal{L}$ transversally.

$\rightarrow$ $\dim X \leq g - 2$

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The goal of the talk is to present some constructions and notions associated with algebraic curves with a pair of real normalized differentials. They generalize concepts of amoebas, Ronkin functions associated with plane curves. The latter have played crucial role in the theory of real algebraic curves (Mikhalkin) and in the theory of limiting shapes of random surfaces (Kenyon, Okounkov, Sheffield).

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Amoebas and the Ronkin function of plane curves

- The amoeba $\mathcal{A}_f$ of a holomorphic function $f : (\mathbb{C}^*)^n \to \mathbb{C}$ (where $\mathbb{C}^* = \mathbb{C} \setminus 0$) is, by definition, the image in $\mathbb{R}^n$ of the zero locus of f under the mapping $\mathrm{Log} : (\mathrm{z}_1, \dots, \mathrm{z}_\mathrm{n}) \to (\log|\mathrm{z}_1|, \dots, \log|\mathrm{z}_\mathrm{n}|)$. The terminology was introduced by Gelfand, Kapranov and Zelevinsky and reflects the geometric shape of typical amoebas, that is a semianalytic closed subset of $\mathbb{R}^n$ with tentacle-like asymptotes going off to infinity.
- All connected components of the amoeba complement $\mathcal{A}_f^c = \mathbb{R}^n \setminus \mathcal{A}_f$, are convex. When f is a Laurent polynomial, then there is a natural injective map from the set of connected components of $\mathcal{A}_f^c$ to the set of integer points of Newton polytop Δ_f of f (Forsberg, Passare, Tsikh).

This injective map is defined by the gradient $\nabla\mathcal{R}_f$ of, the so-called Ronkin function:

$$\mathcal{R}_f(x) = \frac{1}{(2\pi i)^n}\int_{\mathrm{Log}^{-1}(\mathrm{x})} \frac{\log|f(z_1,\ldots,z_n)|\,dz_1\cdots dz_n}{z_1\cdots z_n} \quad (3)$$

The Ronkin function $\mathcal{R}_f(x)$ is convex.

Recall, that each convex function u defines the associated Monge-Ampère measure Mu. If u is a smooth convex function on $\mathbb{R}^n$, then $Mu = \det(\mathrm{Hess}(u))v$, where $\mathrm{Hess}(u)$ is the Hessian matrix and v denotes the Lebesque measure on $\mathbb{R}^n$. If u is convex but not necessary smooth ∇u can still be defined as a multifunction, and the Monge-Ampère measure of u is defined as in the smooth case.

Since $\mathcal{R}_f(x)$ is affine linear in a connected component of $\mathcal{A}_f^c$, the support of the associated Monge-Ampère measure $\mu := M\mathcal{R}_f$ is in $\mathcal{A}_f$.

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Extremal properties of Harnack curves

For $n = 2$ the area of amoeba is always bounded (Passare, H. Rullgard):

$$\mathrm{Area}(\mathcal{A}_f) \le \pi^2 \mathrm{Area}(\Delta_f) \tag{4}$$

Theorem (Mikhalkin, Rullgard)

Suppose that $\mathrm{Area}(\Delta) > 0$. Then the following conditions are equivalent.

1.

$$\mathrm{Area}(\mathcal{A}) = \pi^2 \mathrm{Area}(\Delta)$$

2. The map Log is at most 2:1 and the curve $f(z_1, z_2) = 0$ is real up to multiplication by a constant.

3. The curve $f(z_1, z_2) = 0$ is real up to multiplication by a constant and its real part is a (possibly singular) simple Harnack curve for the Newton polygon Δ.

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Amoebas of the third kind

Let us fix a pair of n-tuples of real numbers $a_j = \{a_{\alpha,j}\}$, $j = 1, 2$. Then for each smooth algebraic curve Γ with n marked points p_α we have two associated imaginary normalized differential $d\zeta_j$ with $\mathrm{Res}_{p_\alpha} d\zeta_j = a_{\alpha,j}$.

Definition

The amoeba $\mathcal{A}_\mathcal{S} \subset \mathbb{R}^2$ associated with the data $\mathcal{S} = \{\Gamma, p_\alpha, a_{\alpha,j}\}$ is the image of the map $\chi : \Gamma_0 \to \mathbb{R}^2$, $\chi(p) = (x_1(p), x_2(p))$, where $x_j(p)$ are harmonic functions on Γ_0 defined by the imaginary normalized meromorphic differentials $d\zeta_j$, i.e.

$$x_a(p) = \mathrm{Re}\left(\int^p d\zeta_a + c\right) \tag{5}$$

The following result shows that the geometric shape of generalized amoebas is the same as that of amoebas of plane curves:

Theorem

All connected components of the complement $\mathcal{A}_S^c$ are convex. There are n unbounded components separated by tentacle-like asymptotes of the amoeba.

Open problem: upper bound on the number of *connected components of $\mathcal{A}_S^c$*

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Critical points of the amoeba map

Lemma

The locus $\gamma \subset \Gamma$ of critical points of the amoeba map is a union of the locus γ_0, where the function $R(p) = \frac{d\zeta_2}{d\zeta_1}$ is real and the finite set (possibly empty) of the common zeros of the differentials $d\zeta_j$.

Let p be a regular point of the map χ. Then in the neighborhood of p we can write $\zeta_1 = x_1 + iy_1(x_1, x_2), \ \ \zeta_2 = x_2 + iy_2(x_1, x_2)$. Using the fact that the ratio R is a meromorphic function on Γ it is easy to prove the equations:

$$\partial_1 y_2 = -\frac{1}{\operatorname{Im} R}, \ \ \partial_2 y_2 = -\partial_1 y_1 = \frac{\operatorname{Re} R}{\operatorname{Im} R}, \ \ \partial_2 y_1 = \frac{|R|^2}{\operatorname{Im} R} \qquad (6)$$

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The latter imply

$$4dx_1 \wedge dx_2 = -2i(\mathrm{Im}R)\, d\zeta_1 \wedge d\bar{\zeta}_1 = -2i\mathrm{Im}R^{-1}\, d\zeta_2 \wedge d\bar{\zeta}_2\,. \quad (7)$$

Moreover: there exists a "generating" function $\tilde{\rho}(x_1, x_2)$ such that

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Generalized Ronkin function

For each $x = (x_1, x_2) \in \mathbb{R}^2$ let us define the closed subset $\Gamma_x \subset \Gamma$:

$$\Gamma_x := \{p \in \Gamma_x | \, x_1(p) \leq x_1, \; x_2(p) \leq x_2\} \,. \tag{8}$$

Definition

The generalized Ronkin function $\rho_{\mathcal{S}}$, associated with a smooth algebraic curve with two imaginary normalized differentials, i.e. associated with the data $\mathcal{S}$ is given by:

$$\rho_{\mathcal{S}}(x) = \frac{1}{8\pi i} \int\int_{\Gamma_x} \mathrm{sgn}(\mathrm{Im} R)(d\zeta_1 \wedge d\bar{\zeta}_2 - d\zeta_1 \wedge d\bar{\zeta}_2). \tag{9}$$

Theorem

The generalized Ronkin function $\rho_{\mathcal{S}}(x)$ given by (9) is a convex function on $\mathbb{R}^2$. It is affine linear in each connected component of $\mathcal{A}_{\mathcal{S}}^c$. It is smooth at the regular points of the amoeba, i.e. outside of the set F of critical values of χ, and furthermore at $x \in \mathcal{A}_{\mathcal{S}} \setminus F$

$$\mathrm{Hess}\,\rho_{\mathcal{S}}(x) = \frac{1}{2\pi} \sum_{p \in \chi^{-1}(x)} \frac{1}{|\,\mathrm{Im} R(p)|} \begin{pmatrix} 1 & \mathrm{Re} R(p) \\ \mathrm{Re} R(p) & |\,R(p)|^2 \end{pmatrix}. \tag{10}$$

Corollary

The area of the amoeba $\mathcal{A}_{\mathcal{S}}$ is not greater than

$$\mathrm{Area}(\mathcal{A}_{\mathcal{S}}) \leq \pi^2 \mathrm{Area}(\Delta_{\mathcal{S}}) \tag{11}$$

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Newton polygon

The image of the gradient map $\nabla\rho_{\mathcal{S}}$ is the polygon $\Delta_{\mathcal{S}}$ that is a convex hull of the set of points $(v_{\alpha,1}, v_{\alpha,2}) \in \mathbb{R}^2$, which are the image under $\nabla\mathcal{R}_{\mathcal{S}}$ of the unbounded components of $\mathcal{A}_{\mathcal{S}}$.
If the vectors a_α and a_β are not collinear i.e., $\varepsilon_{\alpha,\,\beta} := a_{\beta,\,1}a_{\alpha,\,2} - a_{\beta,\,2}a_{\alpha,\,1} \neq 0$, then the vertices of $\Delta_{\mathcal{S}}$ equal

$$\pm a_{\alpha,2} > 0 \Rightarrow v_{\alpha,1} = \pm \sum_{\beta \in I_\alpha^\pm} a_{\beta,2},\, I_\alpha^\pm := \{\beta \in I_\alpha^\pm \,|\, \pm a_{\beta,2} > 0, \pm\varepsilon_{\alpha,\beta} > 0\} \tag{12}$$

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The formulae (12) and (13) continuously extended to the general case when some of vectors a_α might be collinear.

Amoebas of M-curves

A smooth genus g algebraic curve Γ with antiholomorphic involution $\tau : \Gamma \to \Gamma$ is M-curve if τ has $g+1$ (maximal possible number) of fixed ovals.

Definition

The set of data $(\Gamma, p_\alpha, a_{\alpha,j})$ is called Harnack: (i) Γ is a M-curve; (ii) the marked points p_α are on one of the fixed ovals $A_0, \ldots A_g$ of the antiinvolution τ, say $p_\alpha \in A_0$; (iii) The cyclic order p_α along the cycle A_0 coincides with counterclockwise order of the vertices of the polygon Δ_S.

The antiinvolution τ on an M-curve is always of the separating type, i.e. $\Gamma = \Gamma^+ \cup \Gamma^-$, $\tau : \Gamma^+ \to \Gamma^-$, $\Gamma^+ \cap \Gamma^- = \cup_{j=0}^{g} A_j$,

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Extremal properties of Harnack data

Lemma

The amoeba map defined by imaginary normalized differentials $d\zeta_j$ corresponding to Harnack data restricted to $\Gamma^+ \subset \Gamma$ is a diffeomorphism of $\Gamma^+ \setminus \{p_\alpha\}$ with $\mathcal{A}_S$.

Theorem

The following conditions are equivalent.

1.

$$\mathrm{Area}(\mathcal{A}_S) = \pi^2 \mathrm{Area}(\Delta_S)$$

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In the framework of the spectral theory of two-dimensional difference operators

$$(L\psi)_{n,m} = \psi_{n+1,\,m} + \psi_{n,\,m+1} + u_{n,\,m}\psi_{n,\,m} \tag{14}$$

the plane curves of degree d arise as the spectral curves of d-periodic operators, $u_{n,\,m} = u_{n+d,\,m} = u_{n,\,m+d}$.

The points of the spectral curve parameterize Bloch solution of the equation $L\psi = 0$, i.e.

$$\psi_{n+d,\,m} = z_1\psi_{n,\,m}, \quad \psi_{n,\,m+d} = z_2\psi_{n,\,m} \tag{15}$$

$$f(z_1, z_2) := \det(L(z_1, z_2)) = 0 \tag{16}$$

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Spectral theory of $2D$ difference operators

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Quasi-periodic integrable operators

Theorem (Kr 1985)

Let p be a point of Γ. Then for any g-dimensional vector Z the function

$$\psi_{m,n} = \frac{\theta(A(p) + mU + nV + Z)}{\theta(mU + nV + Z)} \, e^{m\Omega_1(p) + n\Omega_2(p)}, \qquad (17)$$

where $U = A_(p_1) - A(p_3)$, $V = A(p_2) - A(p_3)$, satisfies the difference equation

$$\psi_{m,n+1} = \psi_{m+1,n} + u_{m,n}\psi_{m,n} \qquad (18)$$

with

$$u_{m,n} = \frac{\tau_{m+1,n+1}\tau_{m,n}}{\tau_{m+1,n}\tau_{m,n+1}} \qquad (19)$$

where

$$\tau_{m,n} = c_1^m c_2^n \, \theta(mU + nV + Z) \qquad (20)$$

- In general the coefficient of the difference equation is a complex quasiperiodic function (possibly singular) of the variables (m, n).
- If the vectors U and V are d-periodic points of the Jacobain, then $u_{m,n}$ are d-periodic. (The later condition is equivalent to the condition that on Γ there exist functions z_1 and z_2 having pole of order d at p_3 and zeros of order d at p_1 and p_2, respectively).
- If on Γ there is an antiholomorphic involution and if the marked points p_j are fixed by the involution, then u is real for real Z (but still might be singular).
- If Γ is M-curve, and the marked points are on one of the fixed ovals of the involution, then for all real Z the coefficients of the difference equation are non-singular for all (n, m).

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Under the gauge transformation $\Psi_{m,n} = (-1)^n \tau_{m,n} \psi_{m,n}$, equation (18) takes the following form

$$\tau_{m+1,n} \Psi_{m,n+1} + \tau_{n,n+1} \Psi_{m+1,n} + \tau_{m+1,n+1} \Psi_{m,n} = 0 \qquad (21)$$

Corollary

The coefficients of equation (21) are real positive numbers if and only if the corresponding set of algebraic-geometrical data is a Harnack set.

Amoebas versus mushrooms and other creatures.

In the most general form the amoeba map $\chi : \Gamma_0 \to \mathbb{R}^2$ of a smooth algebraic curve Γ with punctures p_α can be defined by any pair of IN differentials (second kind if they have no residues)

- *Example 1.* The elliptic Calogero-Moser system $d\zeta_1, d\zeta_2$ be imaginary normalized differentials on Γ having pole at a marked point p_0 of the form $(z^{-2} + O(1))dz$ and $i(z^{-2} + O(1))dz$. Notice, that a different choice of the local coordinate z corresponds to a linear transformation of the pair of differentials.

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Example 2. KP2 hierarchy

The following example of a pair of imaginary normalized differentials having poles of the form $d\zeta_1 = (-z^{-2} + O(1))$ and $d\zeta_2 = (-2z^{-3} + O(1))dz)$ is connected with the spectral theory of nonstationary Shrödinger equation.

- The corresponding map χ is of degree zero.
- There is one infinite connected component of a complement of the image of χ. It is bounded by a curve which asymptotically is the parabola $x_2 = x_1^2$.
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Common zeros of real normalized differentials

Solutions of the Whitham equations are singular at points of the moduli space where the differentials dE, dQ have common zeros. The study of the singularity locus is ongoing project with Grushevsky.

Central problem:

- *What is the maximal number of common zeros of two real normalized differentials having fixed orders of poles?*

is connected with a study of singularities of families of generalized Jacobians over singular spectral curves ...

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Plane curves of degree d are defined by the equation

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Kulikov constructed a families of curves with large number of cusps that give

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Upper bound

For a long time the best upper bound was than obtained by Hirzebruch

$$s(d) \leq \frac{5}{16}d^2 - \frac{3}{8}d \simeq 0.3125d^2 + O(d)$$

In 2004 Lander using generalization of Bogomolov-Miyaoka-Yau inequality proved

$$s(d) \leq \frac{125 + \sqrt{73}}{432}d^2 \simeq 0.309d^2$$

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