Multidimensional Mellin Transforms: Fundamental Correspondence

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Outline

- Fundamental correspondence for Mellin transforms in dimension one.
- Inversion theorems for мultidimensional Mellin transforms.
- Quasi-elliptic polynomials.
- Nilsson-Passare representation for the Mellin transform of the rational function.
- Alternative representations for the Mellin transform of the rational function.
- Fundamental correspondence in several variables.

Mellin transform $(n = 1)$

$$M[f](z) = \int_0^{+\infty} f(x) x^{z-1} dx \tag{1}$$

$$\begin{aligned} f(x) &= O(x^{\alpha}),\ x \to 0 \\ f(x) &= O(x^{\beta}),\ x \to +\infty,\ \alpha > \beta \end{aligned}$$

The integral (1) converges in **the fundamental strip**

$$(-\alpha, -\beta) + i\mathbb{R}. \tag{2}$$

Example 1

Let $f(x)$ be a function $1/g(x)$ where $g(x)$ is a polynomial with the multiplicity μ at $x = 0$ and the degree d. Then the fundamental strip for $M[f](z)$ is as follows

$$(\mu, d) + i\mathbb{R}.$$

Remark: $[\mu, d]$ is the Newton polytope for the polynomial $g(x)$.

Toric view

The Mellin transform may be represented as follows

$$M[f](z) = \int\limits_{l} f(x) x^z \frac{dx}{x},$$

where l is a ray connecting points 0 and ∞ on the Riemann sphere, therefore l is a relative 1–cycle for the pair $(\overline{\mathbb{C}}, \{0, \infty\})$.

$\overline{\mathbb{C}}$ is a toric variety and $\{0, \infty\}$ is a set of $\mathbb{T}$–invariant divisors on it.

Fundamental correspondence

Flajolet et al, 1995

There is a precise correspondence between individual terms in the asymptotic expansion of an original function $f(x)$ and singularities of the transformed function $M[f](z)$:

$$\begin{array}{lcl} x^k \ln^l x \text{ at } 0 & \longleftrightarrow & \frac{(-1)^l l!}{(z+k)^{l+1}} \\ x^k \ln^l x \text{ at } \infty & \longleftrightarrow & -\frac{(-1)^l l!}{(z+k)^{l+1}} \end{array}$$

Multidimensional Mellin transforms

- The Mellin transform of $f(x)$ is defined by

$$M[f](z) = \int_{\mathbb{R}^n_+} f(x) x^{z-I} dx, \quad x^{z-I} = x_1^{z_1 - 1} \cdots x_n^{z_n - 1}. \tag{3}$$

- The inverse Mellin transform of $F(z)$ is the integral

$$M^{-1}[F](x) = \frac{1}{(2\pi i)^n} \int_{a + i\mathbb{R}^n} F(z) x^{-z} dz, \tag{4}$$

where $a \in \mathbb{R}^n$ is a fixed vector.

Classes M_Θ^U and W_U^Θ

convex domains $\Theta \subset \mathbb{R}^n$, $U \subset \mathbb{R}^n$ $\longrightarrow$ classes M_Θ^U, W_U^Θ

- M_Θ^U is the vector space of functions $\Phi(x)$ holomorphic in

$$S_{k\Theta} = \{x \in \mathcal{S} : \arg x \in k\Theta\}, \;\; k > 1,$$

$$|\Phi(x)| \leqslant C(a)|x^{-a}| \text{ for all } x \in S_{k\Theta}, \;\; a \in U,$$

- W_U^Θ is the vector space of functions $F(z) = F(u + iv)$ holomorphic in $U + i\mathbb{R}^n$ and

$$|F(u + iv)| \leqslant K(u)e^{-kH_\Theta(v)}, \;\; k > 1 \text{ with supp. f. } H_\Theta(v).$$

Theorem 1 (A., 2007)

If $f(x) \in M_\Theta^U$, then $M[f](z) \in W_U^\Theta$, and $M^{-1}M[f] = I[f]$.
If $F(z) \in W_U^\Theta$, then $M^{-1}[F](x) \in M_\Theta^U$, and $MM^{-1}[F] = I[F]$.

Tube domain $U + i\mathbb{R}^n$

The tube domain $U + i\mathbb{R}^n$ is an analog of the fundamental strip.

Example 2

$$f(x) = \frac{1}{1+x}, \quad M[f](z) = \Gamma(z)\Gamma(1-z),\ 0 < \mathrm{Re} z < 1,\ U = (0;1).$$

Example 3

$$f(x_1, x_2) = \frac{1}{1 + x_1 + x_2}, \quad M[f](z_1, z_2) = \Gamma(z_1)\Gamma(z_2)\Gamma(1 - z_1 - z_2)$$

is holomorphic in the tube domain $U + i\mathbb{R}^2$, where
$U = \{\mathrm{Re} z_1 > 0, \mathrm{Re} z_2 > 0, \mathrm{Re}(z_1 + z_2) < 1\}$.

Mellin transform of a rational function

$$M[1/Q](z) = \int\limits_{\mathbb{R}^n_+} \frac{x^{z-I}}{Q(x)} dx, \qquad (5)$$

$$Q(x) = \sum_{\alpha \in \mathbb{N}^n} c_\alpha x^\alpha, \ \text{supp}Q = \{\alpha \in \mathbb{N}^n : c_\alpha \neq 0\}$$

- Convergence of multiple integrals for rational functions with quasi-elliptic denominators were studied by Ermolaeva and Tsikh [SbMath, 1996].
- The Newton polytope Δ of the polynomial Q is defined to be the convex hull of supp$Q$ in $\mathbb{R}^n$.
- For each covector $a \in \mathbb{R}^{n*}$ we introduce the **truncated polynomial**

$$Q_a = \sum_{\alpha \in \Delta^a} c_\alpha x^\alpha, \ \text{where } \Delta^a = \{k \in \Delta : \langle a, k\rangle = \min_{l \in \Delta} \langle a, l\rangle\}$$

is the corresponding face of Δ.

Quasi-elliptic polynomials

Definition 1 (Ermolaeva - Tsikh, 1996)

A polynomial Q is said to be **quasi-elliptic** if for any nonzero covector $a \in \mathbb{R}^{n*}$ the truncated polynomial Q_a does not vanish in $(\mathbb{R} \setminus \{0\})^n$.

Remark

The $(\mathbb{R} \setminus \{0\})^n$ is a group with the coordinate–wise mutiplication. It has 2^n connected components. The component $\mathbb{R}^n_+$ is a subgroup. So we can define the notion of quasi-ellipticity on the $\mathbb{R}^n_+$, assuming that Q_a does not vanish in $\mathbb{R}^n_+$.

Quasi-elliptic polynomials: examples

- A polynomial Q is quasi-elliptic if all its monomials $c_\alpha x_1^{\alpha_1} \dots x_n^{\alpha_n}$ have positive coefficients c_α and even degrees α_i in each variable x_i.
  For instance, the polynomial $Q(x) = 1 + x_1^4 + x_1^8 x_2^2$ is quasi-elliptic.
- Any elliptic polynomial that does not vanish on $\mathbb{R}^n$ is quasi-elliptic; moreover, the Newton polytope $\Delta(Q)$ is the n–dimensional simplex spanned by the vertices $\alpha^0 = (0, \dots, 0), \alpha^i = (0, \dots, q, \dots, 0)$, $i = 1, \dots, n$, where $q = \deg Q$.

Quasi-ellipticity and hypoellipticity

Definition 2 (Hörmander)

A polynomial $Q(x)$ is said to be **hypoelliptic** if for any multi-index $\alpha \neq 0$

$$\frac{Q^{(\alpha)}(x)}{Q(x)} \to 0$$

when $\| x \| \to \infty$.

Theorem 2 (Zubchenkova)

If a polynomial $Q(x)$ is quasi-elliptic and its Newton polytope Δ is full, then $Q(x)$ is hypoelliptic.

Remark

The fullness of a polytope means that its projections onto all coordinate hyperplanes belong to it.

Example 4

The polynomial $Q(x) = (x_1^2 - 1)^2 + x_2^4$ is hypoelliptic, but it is not quasi-elliptic, since $Q_{(0,1)} = (x_1^2 - 1)^2$ vanishes in $(\mathbb{R} \setminus 0)^2$.

Meromorphic continuation of $M[1/Q](z)$

Theorem 3 (Nilsson-Passare, 2013)

If the polynomial Q is quasi-elliptic on $\mathbb{R}^n_+$ and its Newton polytope Δ is of full dimension, then the Mellin transform $M[1/Q](z)$ admits a meromorphic continuation of the form

$$M[1/Q](z) = \Phi(z) \prod_{k=1}^{N} \Gamma(\nu^k - \left\langle \mu^k, z \right\rangle),$$

where Φ is an entire function, $\mu^k \in \mathbb{Z}^n$ are outward normal vectors of the facets of Δ, and $\nu^k \in \mathbb{Z}$.

Alternative representations for $M[1/Q](z)$

Theorem 4 (A-Efimov-Shchuplev-Tsikh)

Assume that Q is a quasi-elliptic polynomial on $\mathbb{R}^n_+$. For each normal vector μ^k of the Newton polytope Δ the Mellin transform $M[1/Q](z)$ admits the following representation

$$M_k(z) = \Gamma\left(-\left\langle \mu^k, z\right\rangle\right)\Gamma\left(1+\left\langle \mu^k, z\right\rangle\right)e^{-i\pi\langle\mu^k,z\rangle}\Phi_k(z)$$

with

$$\Phi_k(z) = v.p.\int\limits_{V_k} \mathrm{Res}\ \omega,$$

where $\mathrm{Res}\ \omega$ is the Lerey residue form of the integrand $\omega = \frac{x^{z-I}}{Q(x)}dx$ in the Mellin transform (5) and $v.p.$ is the principal value respectively singular points of V_k.

Sketch of proof: torus automorphisms

$$M[1/Q](z) = \int\limits_{\mathbb{R}^n_+} \frac{x^{z-I}}{Q(x)} dx$$

- The orthant $\mathbb{R}^n_+$ is a group with respect to the coordinate-wise multiplication.
- The $\mathbb{R}^n_+$ is a connected component of the real torus $(\mathbb{R} \setminus 0)^n$.
- Any automorphism of the torus is defined by a monomial transform $x = y^A$, given by an integer unimodular matrix A.
- For each outward normal vector $\mu^k = (\mu^k_1, \dots, \mu^k_n)$ of the Newton polytope $\Delta(Q)$, we define a one-parameter subgroup

$$\gamma^k = (y_1^{\mu_1^k}, \dots, y_1^{\mu_n^k}),\ y_1 \in \mathbb{R}_+.$$

- Thus, one fibers the $\mathbb{R}^n_+$ into shifts

$$c \odot \gamma^k = (c_1 y_1^{\mu_1^k}, \dots, c_n y_1^{\mu_n^k}).$$

Sketch of proof: sets V_k

Remark

The set of all shifts can be represented as $c = (y^{'})^{\eta^{'}}$, where $y^{'} = (y_2, \dots, y_n)$ and $\eta^{'}$ is a $(n \times (n-1))$-matrix s.t. the matrix $\eta = (\mu^k, \eta^{'})$ is unimodular.

- Consider the complex hypersurface

$$V = \{x \in (\mathbb{C} \setminus 0)^n : Q(x) = 0\}.$$

- Consider sections of V by the family of shifts of one-parameter subgroups γ^k

$$V_k = \bigcup_{y^{'} \in \mathbb{R}_+^{n-1}} \left(V \cap \left\{ x = y_1^{\mu^k} \odot \left(y^{'}\right)^{\eta^{'T}} : y_1 \in \mathbb{C} \setminus 0 \right\} \right).$$

Example 5: sets V_k

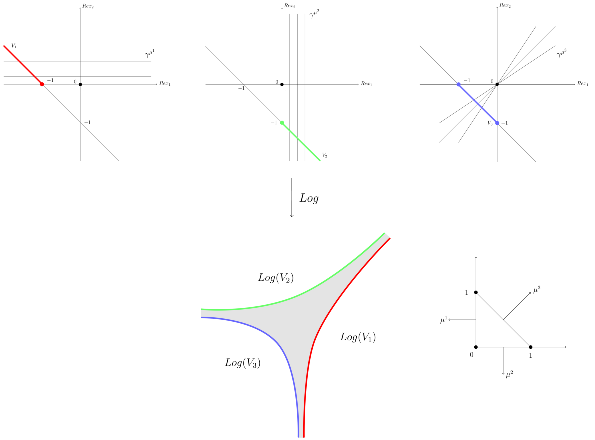

Figure: $V = \left\{ x \in (\mathbb{C} \setminus 0)^2 : 1 + x_1 + x_2 = 0 \right\}.$

Toric view in several variables

$n = 1:$

$$M[f](z) = \int_l f(x) x^z \frac{dx}{x},$$

where l is a ray connecting points 0 and ∞ on the Riemann sphere, therefore l is a relative 1–cycle for the pair $(\overline{\mathbb{C}}, \{0, \infty\})$.

$n > 1$:
The toric view extends on Mellin transforms of rational functions in several variables. In this case toric compactifications for $(\mathbb{C} \setminus 0)^n$ play the role of the Riemann sphere $\overline{\mathbb{C}}$ and the set $\{0, \infty\}$.

Complex Toric Variety

There are two approaches to define the toric variety X.

Definition 3.1

A connected complex analytic variety X is said to be the **toric variety** if it has the atlas with monomial transition functions.

Definition 3.2

Consider the complex algebraic torus $\mathbb{T}^n = (\mathbb{C} \setminus 0)^n$ with a group operation

$$zw = (z_1 w_1, \ldots, z_n w_n).$$

The **toric variety** is a normal variety X that contains the torus $\mathbb{T}^n$ as a dense open subset together with an action $\mathbb{T}^n \times X \to X$ of $\mathbb{T}^n$ on X that extends the natural action of $\mathbb{T}^n$ on it itself.

$\mathbb{T}$ – invariant divisors

Divisors on the toric variety X which are invariant with respect to the action of $\mathbb{T}^n$ are called the $\mathbb{T}$**–invariant divisors**.
Denote them $T_1, \dots, T_N$.

Example 6

- $\mathbb{CP}^2 = \mathbb{T}^2 \cup T_1 \cup T_2 \cup T_3$, where T_1 and T_2 complete $\mathbb{T}^2$ to $\mathbb{C}^2$, T_3 is the infinite line in $\mathbb{CP}^2$
- $\overline{\mathbb{C}} \times \overline{\mathbb{C}}$ has 4 $\mathbb{T}$–invariant divisors.

Real Toric variety

Consider the group $(\mathbb{R} \setminus 0)^n$.

- If divisors $T_1, \ldots, T_N$ are invariant with respect to the action of $\mathbb{T}^n$, then hypersurfaces $\mathcal{R}e\, T_1, \ldots, \mathcal{R}e\, T_N$ are invariant respectively the action of the group $(\mathbb{R} \setminus 0)^n$.
- Toric varieties X and $\mathcal{R}e\, X$ are compactifications of $\mathbb{T}^n$ and $(\mathbb{R} \setminus 0)^n$ correspondingly.

Theorem 5

A polynomial $Q(x)$ is quasi-elliptic if and only if the closure of the hypersurface $V = \{x \in \mathbb{T}^n : Q(x) = 0\}$ in the toric compactification of $\mathbb{T}^n$ does not intersect $\mathcal{R}e\, T_1, \ldots, \mathcal{R}e\, T_N$.

Remark

If the polynomial Q is quasi-elliptic, then the integral (5) defining the Mellin transform $M[1/Q](z)$ converges in the tube domain $\Delta^\circ + i\mathbb{R}^n$, where Δ° is the interior of the Newton polytope $\Delta(Q)$ (Ermolaeva-Tsikh, 1996).

Example 7: $M_k(z)$

- $Q(x) = 2 - x_1 + x_1^2 - x_2 + x_2^2 - x_1x_2$,
- $$M[1/Q] = \int\limits_{\mathbb{R}_+^2} \frac{x_1^{z_1} x_2^{z_2}}{2 - x_1 + x_1^2 - x_2 + x_2^2 - x_1x_2} \frac{dx}{x},$$
- $\Delta = \{\alpha_1 \geq 0\} \cap \{\alpha_2 \geq 0\} \cap \{\alpha_1 + \alpha_2 \leq 2\}$,
- $\mu^1 = (-1, 0), \mu^2 = (0, -1), \mu^3 = (1, 1)$,
- $\gamma^{\mu^1} = (y_1^{-1}, 1),\ y_1 \in \mathbb{R}_+$,
- V_1: $x_1 = \left(\frac{y_2^2 + y_2 \pm i y_2 \sqrt{7y_2^2 - 6y_2 + 3}}{2(2y_2^2 - y_2 + 1)}\right)^{-1}$, $x_2 = y_2^{-1}$,
- $M_1(z) = \Gamma(z_1)\Gamma(1 - z_1) e^{i\pi z_1} \Phi_1(z)$, where

$$\Phi_1(z) = \int\limits_0^\infty y_2^{1-z_2} \left(\frac{\left(y_2^2 + y_2 + i y_2 \sqrt{7y_2^2 - 6y_2 + 3}\right)^{1-z_1}}{2\left(2y_2^2 - y_2 + 1\right)^{1-z_1} i y_2 \sqrt{7y_2^2 - 6y_2 + 3}} - \right.$$
$$\left. - \frac{\left(y_2^2 + y_2 - i y_2 \sqrt{7y_2^2 - 6y_2 + 3}\right)^{1-z_1}}{2\left(2y_2^2 - y_2 + 1\right)^{1-z_1} i y_2 \sqrt{7y_2^2 - 6y_2 + 3}} \right) dy_2.$$

Example 7: amoeba $\mathcal{A}_V$ and Log(V_k)

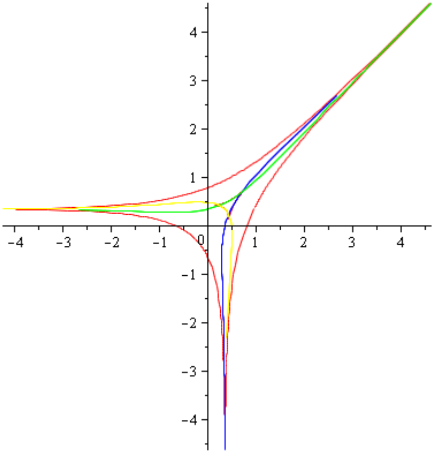

Figure: The contour of $\mathcal{A}_V$ (red), Log(V_1) (blue), Log(V_2) (green), Log(V_3) (yellow).

Example 8: $M_k(z)$

- $Q(x) = 5 + x_1 + x_2 + x_1 x_2$,
- $$M[1/Q] = \int\limits_{\mathbb{R}^2_+} \frac{x_1^{z_1} x_2^{z_2}}{5 + x_1 + x_2 + x_1 x_2} \frac{dx}{x},$$
- $\Delta = \{\alpha_1 \geq 0\} \cap \{\alpha_2 \geq 0\} \cap \{\alpha_1 \leq 1\} \cap \{\alpha_2 \leq 1\}$,
- $\mu^1 = (-1, 0), \mu^2 = (0, -1), \mu^3 = (1, 0), \mu^4 = (0, 1)$,
- $\gamma^{\mu^1} = (y_1^{-1}, 1),\ y_1 \in \mathbb{R}_+$,
- V_1: $x_1 = -\frac{5y_2+1}{y_2+1}$, $x_2 = y_2^{-1}$,
- $$M_1(z) = \Gamma(z_1)\Gamma(1 - z_1) e^{i\pi z_1} \int\limits_0^\infty (-1)^{-z_1} \frac{(1 + y_2)^{-z_1} y_2^{-z_2}}{(5y_2 + 1)^{-z_1+1}} dy_2.$$

Example 8: amoeba $\mathcal{A}_V$ and $\mathrm{Log}(V_k)$

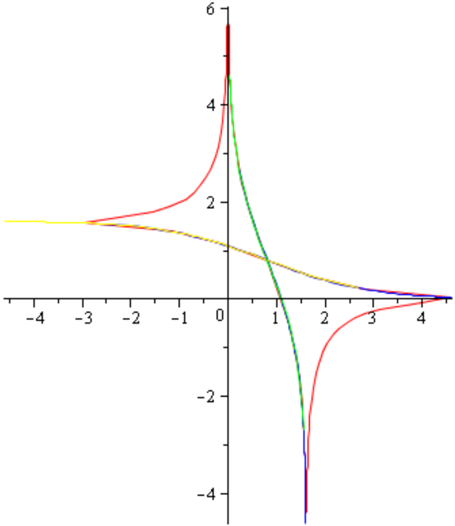

Figure: The contour of $\mathcal{A}_V$, $\mathrm{Log}(V_1)$, $\mathrm{Log}(V_2)$, $\mathrm{Log}(V_3)$, $\mathrm{Log}(V_4)$.

Elaboration: entire function $\Phi(z)$

Theorem 6

If Q is a quasi-elliptic polynomial on $\mathbb{R}^n_+$ then the function $\Phi(z)$ in the Nilsson-Passare representation

$$M[1/Q](z) = \Phi(z) \prod_{k=1}^{N} \Gamma(\nu^k - \left\langle \mu^k, z \right\rangle),$$

is of exponential type:

$$|\Phi(z)| \leq Ce^{A|z|}.$$

Singular expansions for $M[1/Q]$

Let C_v be the minimal cone with the vertex $v \in \text{Vert}\Delta$, $C_v \supset \Delta$.

Theorem 7

For each vertex $v \in \Delta$ the Mellin transform $M[1/Q]$ admits the following representation

$$\prod_{k \in I_v} \Gamma(\nu_k - \langle \mu_k, z \rangle) \Phi_v(z),$$

where $I_v = \{k : \langle \mu_k, z \rangle = \nu_k \text{ intersects} - C_v\}$, the function $\Phi_v(z)$ is holomorphic in $(-C_v)^\circ + i\mathbb{R}^n$ and of the exponential growth at most.