The Hartogs extension phenomenon in toric varieties

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Introduction

- **The classical Hartogs extension theorem:** Let $D \subset \mathbb{C}^n (n > 1)$ be a domain and $K \subset D$ be a compact set such that $D \setminus K$ is connected. Then the restriction homomorphism $H^0(D, \mathcal{O}) \to H^0(D \setminus K, \mathcal{O})$ is an isomorphism.
- A natural question arises if this is true for complex analytic varieties.
- Let $(X, \mathcal{O})$ be a connected complex analytic variety (It means that, locally, X is an analytic set A in a domain $D \subset \mathbb{C}^n$ and $\mathcal{O}|_A = \mathcal{O}_D / I_A$, where I_A is an ideal sheaf of A).

Definition

We say that a connected complex space X admits the Hartogs phenomenon if for any domain $D \subset X$ and a compact set $K \subset D$ such that $D \setminus K$ is connected, the restriction homomorphism $H^0(D, \mathcal{O}_X) \to H^0(D \setminus K, \mathcal{O}_X)$ is an isomorphism.

- In this or a similar formulation this phenomenon has been extensively studied in many situations, including Stein manifolds and spaces, $(n-1)$-complete normal complex spaces and so on.
- Our goal is to study the Hartogs phenomenon in toric varieties.

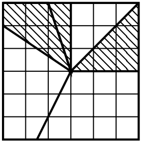
Toric varieties

Let T_N be a complex algebraic torus associated with a lattice N (note: $T_N = N \otimes_{\mathbb{Z}} \mathbb{C}^* \cong (\mathbb{C}^*)^n$, here $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$)

Definition

A normal algebraic variety equipped with an algebraic action of the torus T_N is called toric variety if it contains an open T_N-orbit.

- Ex: $(\mathbb{C}^*)^n, \mathbb{C}^n$, $\mathbb{CP}^n$.
- Toric varieties classified by fans.
- **A strictly convex cone** is a subset $\sigma = \mathbb{R}_{\geq}\langle n_i \mid i = 1, \cdots, s\rangle \subset N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$ for some $n_i \in N, i = 1, \cdots, s$
- **A fan** is a pair (Σ, N) where Σ is a finite set of strictly convex cone $\sigma \subset N_{\mathbb{R}}$ with the properties: 1) $\forall \sigma \in \Sigma \wedge \forall \tau < \sigma \Longrightarrow \tau \in \Sigma$; 2) $\forall \sigma, \sigma' \in \Sigma \Longrightarrow (\sigma \cap \sigma' < \sigma) \wedge (\sigma \cap \sigma' < \sigma')$
- The support of a fan Σ is a set $|\Sigma| := \bigcup_{\sigma \in \Sigma} \sigma$.



Examples of toric varieties

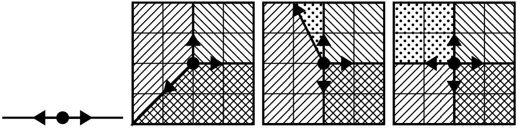


Рис.: $\mathbb{CP}^1$, $\mathbb{CP}^2$, $\mathcal{H}_2$, $\mathbb{CP}^1 \times \mathbb{CP}^1$

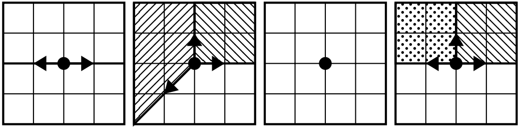
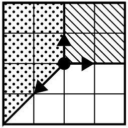


Рис.: $\mathbb{CP}^1 \times \mathbb{C}^*$,$\mathbb{CP}^2 \setminus \{pt\}$, $(\mathbb{C}^*)^2$, $\mathbb{CP}^1 \times \mathbb{C}^1$

Two examples

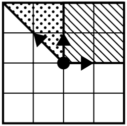
For any cone $\sigma \in \Sigma$ corresponds an affine toric variety $U_\sigma = Spec(\mathbb{C}[\sigma^\vee \cap N^*])$ which is gluing together to a toric variety $X_\Sigma = \varinjlim_{\sigma \in \Sigma} U_\sigma$ (here inductive limit is taken with respect to a partial order "to be a face")



$\sigma_1 = \mathbb{R}_{\geq}\langle(1,0),(0,1)\rangle$
$\sigma_2 = \mathbb{R}_{\geq}\langle(0,1),(-1,-1)\rangle$
$\tau = \mathbb{R}_{\geq}\langle(0,1)\rangle$
$U_{\sigma_1} = \mathbb{C}^2_{z_1,z_2}$, $U_{\sigma_2} = \mathbb{C}^2_{w_1,w_2}$,
$U_\tau = \mathbb{C} \times \mathbb{C}^*$
$U_{\sigma_1} \hookleftarrow U_\tau \hookrightarrow U_{\sigma_2}$
Gluing map:

$$z_1 = \frac{1}{w_1}, z_2 = \frac{1}{w_1} w_2$$

$X_\Sigma = \mathcal{O}_{\mathbb{CP}^1}(1)$ or $\mathbb{CP}^2 \setminus \{pt\}$



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$X_\Sigma = \mathcal{O}_{\mathbb{CP}^1}(-1)$ or $\mathrm{Bl}_0\, \mathbb{C}^2$

Smoothness and compactness

- A toric variety X_Σ is smooth (i.e. is a complex manifold) if and only if for every cone $\sigma \in \Sigma$ there exists a $\mathbb{Z}$-basis $\{n_1, \cdots, n_r\}$ of the lattice N such that $\sigma = \mathrm{Cone}(n_1, \cdots, n_s)$ for $s \leq r$
- A toric variety X_Σ is compact if and only if $|\Sigma| := \bigcup_{\sigma \in \Sigma} \sigma = \mathbb{R}^n$.

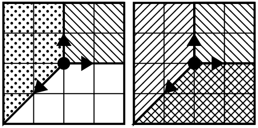


Рис.: $\mathbb{CP}^2 \setminus \{pt\}$, $\mathbb{CP}^2$

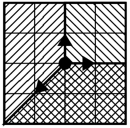
- A noncompact toric variety admits torus equivariant compactification. In terms of fans it means that an incomplete fan can be completion by cones to a complete fan.

Orbit-cone correspondence

There is 1-1 correspondence between cones in a fan Σ and a set of T_N-orbit of X_Σ. Moreover

- $\tau \leq \sigma$ if and only if $O(\sigma) \subset \overline{O(\tau)}$;
- $U_\sigma = \bigsqcup_{\tau \leq \sigma} O(\tau)$.

Example:



Let z_0, z_1, z_2 be a homogeneous coordinates in $\mathbb{CP}^2$.
$\sigma_1 = \mathbb{R}_{\geq}\langle e_1, e_2\rangle \longleftrightarrow O(\sigma_1) = [1:0:0]$
$\sigma_2 = \mathbb{R}_{\geq}\langle e_1, -e_1 - e_2\rangle \longleftrightarrow O(\sigma_2) = [0:1:0]$
$\sigma_3 = \mathbb{R}_{\geq}\langle e_2, -e_1 - e_2\rangle \longleftrightarrow O(\sigma_3) = [0:0:1]$
$\tau_1 = \mathbb{R}_{\geq}\langle e_1\rangle \longleftrightarrow O(\tau_1) = \{z_0 z_1 \neq 0, z_2 = 0\}$
$\tau_2 = \mathbb{R}_{\geq}\langle e_2\rangle \longleftrightarrow O(\tau_2) = \{z_0 z_2 \neq 0, z_1 = 0\}$
$\tau_3 = \mathbb{R}_{\geq}\langle -e_1 - e_2\rangle \longleftrightarrow O(\tau_3) = \{z_1 z_2 \neq 0, z_0 = 0\}$
$O \longleftrightarrow (\mathbb{C}^*)^2 = \{z_0 z_1 z_2 \neq 0\}$
Note that $\tau_1 < \sigma_1$ if and only if $O(\sigma_1) = [1:0:0] \subset \overline{O(\tau_1)} = \{z_2 = 0\}$

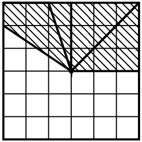
Marciniak conjecture

- In the context of toric varieties the Hartogs phenomenon was first studied by M. A. Marciniak.

Theorem

Let X_Σ be a smooth toric surface. If the support $|\Sigma|$ of Σ is a strictly convex cone then for any compact set $K \subset X_\Sigma$ such that $X_\Sigma \setminus K$ is connected, the restriction homomorphism $H^0(X_\Sigma, \mathcal{O}) \to H^0(X_\Sigma \setminus K, \mathcal{O})$ is an isomorphism.

- Example:



- **Marciniak conjecture:** Let X_Σ be a smooth toric variety. If the complement of $|\Sigma|$ has at least one **concave** connected component then for any compact set $K \subset X_\Sigma$ such that $X_\Sigma \setminus K$ is connected, the restriction homomorphism $H^0(X_\Sigma, \mathcal{O}) \to H^0(X_\Sigma \setminus K, \mathcal{O})$ is an isomorphism.
- What is **concave** connected component in dimension at least 3?

Main results

Theorem 1

Let $X_{\Sigma'}$ be a noncompact toric variety with the fan Σ'. Assume that the complement of the fan's support $C := \mathbb{R}^p \setminus |\Sigma'|$ is connected, then $H^1_c(X_{\Sigma'}, \mathcal{O}) = 0$ if and only if $\mathrm{conv}(\overline{C}) = \mathbb{R}^p$.

Theorem 2

Let X_Σ be a noncompact toric variety with the complement $\mathbb{R}^p \setminus |\Sigma|$ being connected. The cohomology group $H^1_c(X_\Sigma, \mathcal{O})$ is trivial if and only if X_Σ admits the Hartogs phenomenon.

This allows us to specify what concavity in the conjecture formulated above means.

Definition

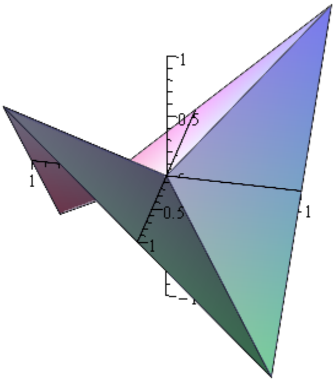
Let Σ be a fan in $\mathbb{R}^p$, and $\mathbb{R}^p \setminus |\Sigma| = \bigsqcup_j C_j$. A complement component C_j is called concave if $\mathrm{conv}(\overline{C_j}) = \mathbb{R}^p$.

Corollary

Let X_Σ be a noncompact toric variety with the fan Σ whose complement is $\mathbb{R}^p \setminus |\Sigma| = \bigsqcup_{j=1}^{n} C_j$. Then

- *if at least one of C_j's is concave then X_Σ admits the Hartogs phenomenon.*
- *if $n = 1$ then the converse is also true, i.e. if X_Σ admits the Hartogs phenomenon then $\mathbb{R}^p \setminus |\Sigma|$ is concave.*

A fan with concave components in complement



Theorem 1

Theorem 1: Let $X_{\Sigma'}$ be a noncompact toric variety with the fan Σ'. Assume that the complement of the fan's support $C := \mathbb{R}^p \setminus |\Sigma'|$ is connected, then $H^1_c(X_{\Sigma'}, \mathcal{O}) = 0$ if and only if $\mathrm{conv}(\overline{C}) = \mathbb{R}^p$.

Steps of the proof:

1. Let $X_{\Sigma''}$ be a toric compactification of $X_{\Sigma'}$. We describe $Z := X_{\Sigma''} \setminus X_{\Sigma'}$ in terms of T_N-orbits and consider an open toric variety $X_\Sigma \subset X_{\Sigma''}$ such that $Z \subset X_\Sigma$ and with the following property: any neighborhood of Z intersect with all T_N-invariant divisors of X_Σ;
2. We describe $H^1_c(X_{\Sigma'}, \mathcal{O})$ in terms of the ring $H^0(Z, \mathcal{O})$ of germs of holomorphic functions on Z;
3. We prove that any germ in $H^0(Z, \mathcal{O})$ can be represented by a Laurent series $\sum\limits_{I \in A} a_I t^I$ which converges in a neighborhood of Z.
4. We show that the support A of the Laurent series contained in $\overline{C}^{\vee} \cap N^*$.

Step 1

- Let $X_{\Sigma''}$ be a compactification of $X_{\Sigma'}$ and $Z := X_{\Sigma''} \setminus X_{\Sigma'}$. Since the set Z is T_N-invariant, it follows that $Z = \bigcup_{\tau \in \Sigma'', \mathrm{relint}(\tau) \in \mathbb{R}^n \setminus |\Sigma'|} O(\tau)$.
- Denote by Σ the fan which consists of cones σ in Σ'' such that $\sigma \subset \mathbb{R}^n \setminus int(|\Sigma'|)$. Note that $\overline{C} = |\Sigma|$.
- Orbit-cone correspondence implies that: 1) the set Z and a small neighborhood of Z is contained in X_Σ; 2) and any such neighborhood of Z intersect with all T_N-invariant divisors of X_Σ.

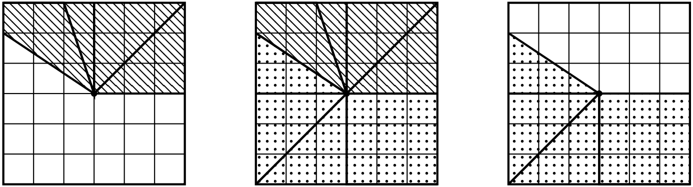


Рис.: Σ' Рис.: Σ'' Рис.: Σ

Step 1

Example:

- Let $X_{\Sigma'} = \mathbb{CP}^2 \setminus \{pt\}$ and $X_{\Sigma''} = \mathbb{CP}^2$. Then $Z = \{pt\}$ is a 0-dimensional T_N-orbit corresponding to the cone $\sigma = \mathbb{R}_{\geq 0}\langle e_1, -e_1 - e_2\rangle$ and $X_\Sigma = U_\sigma \cong \mathbb{C}^2$.
- The point Z is exactly a 0-dimensional T_N-orbit in U_σ and it corresponds to the origin in $\mathbb{C}^2$. T_N-invariant divisors in U_σ correspond to coordinate lines in $\mathbb{C}^2$.

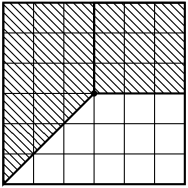
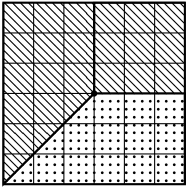


Рис.: Σ'



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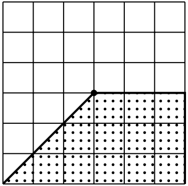

Рис.: Σ

Step 2

- We have a long exact sequence of cohomology groups

$$0 \longrightarrow H_c^0(X_{\Sigma'}, \mathcal{O}) \longrightarrow H_c^0(X_{\Sigma''}, \mathcal{O}) \longrightarrow H_c^0(Z, \mathcal{O}) \longrightarrow$$

$$\longrightarrow H_c^1(X_{\Sigma'}, \mathcal{O}) \longrightarrow H_c^1(X_{\Sigma''}, \mathcal{O}) \longrightarrow H_c^1(Z, \mathcal{O}) \longrightarrow \cdots$$

- Cohomology of toric variety: If $X_{\Sigma''}$ is a compact toric variety, then $H^i(X_{\Sigma''}, \mathcal{O}) = 0$ for all $i > 0$ and $H^0(X_{\Sigma''}, \mathcal{O}) = \mathbb{C}$.
- Since $X_{\Sigma'}$ is noncompact it follows that $H_c^0(X_{\Sigma'}, \mathcal{O}) = 0$.
- Thus $H_c^1(X_{\Sigma'}, \mathcal{O}) \cong H^0(Z, \mathcal{O})/\mathbb{C}$, here $H^0(Z, \mathcal{O}) = \varinjlim_{U \supset Z} H^0(U, \mathcal{O})$.

Step 3

- Assume that X_Σ is a **smooth** toric variety. It admits an open covering $X_\Sigma = \bigcup_{\sigma \in \Sigma(p)} U_\sigma$, where $U_\sigma \cong \mathbb{C}^n$ is an affine chart corresponding to a cone $\sigma \in \Sigma(n)$ of dimension n.
- A set $Z \cap U_\sigma = \bigcup_{\tau < \sigma, \mathrm{relint}(\tau) \in \mathrm{int}(|\Sigma|)} O(\tau) \subset U_\sigma$ is a union of coordinate subspaces of U_σ.
- Consider equivalence class $[f, V] \in H^0(Z, \mathcal{O})$ that is a function f is holomorphic in a neighborhood V of Z.
- A function $f|_{U_\sigma \cap V} \in H^0(U_\sigma \cap V, \mathcal{O})$ can be represented as a convergent power series in a sufficiently small neighborhood W_σ of $Z \cap U_\sigma$.
- Choose a neighborhood D of Z such that $D \subset \bigcup_{\sigma \in \Sigma(n)} W_\sigma$ and $D \cap U_\sigma \subset W_\sigma$.

  So, $f|_D = \sum_{I \in A} a_I t^I$, where $t = (t_1, \cdots, t_n)$ are coordinates in the torus T_N.
- Thus $[f, V] = [\sum_{I \in A} a_I t^I, D]$.

Step 4

- Since the order of vanishing of a Laurent monimial t^I along a T_N-invariant divisor $V(\rho), \rho \in \Sigma(1)$ is equal to $\langle u_\rho, I \rangle$, it follows that $A \subset |\Sigma|^\vee \cap N^*$ (here u_ρ is a primitive generator of 1-dimension cone ρ).
- Since $\overline{C} = |\Sigma|$, it follows that $H^0(Z, \mathcal{O}) = \{[\sum_{I \in A} a_I t^I, D] \mid \text{ series converges in } D, A \subset \overline{C}^\vee \cap N^*\}$. Moreover $H^0(Z, \mathcal{O}) = \mathbb{C}$ if and only if $\overline{C}^\vee = O$
- Note that if A is a closed (but not necessarily convex) cone, then $\mathrm{conv}(A) = A^{\vee\vee}$.
- We obtain that $\mathrm{conv}(\overline{C}) = \mathbb{R}^p$ if and only if $\overline{C}^\vee = O$.

Theorem 2

Theorem 2: Let X_Σ be a noncompact toric variety with the complement $\mathbb{R}^p \setminus |\Sigma|$ being connected. The cohomology group $H^1_c(X_\Sigma, \mathcal{O})$ is trivial if and only if X_Σ admits the Hartogs phenomenon.

The proof of Theorem 2 is based of the following lemmas
Let $(X, \mathcal{O})$ be a noncompact connected complex analytic variety.

1. If $H^1_c(X, \mathcal{O}) = 0$, then for any compact set $K \subset X$ such that $X \setminus K$ is connected, the restriction homomorphism $H^0(X, \mathcal{O}) \to H^0(X \setminus K, \mathcal{O})$ is an isomorphism.
2. Suppose X admits a compactification X' such that X' is a compact connected complex analytic variety and $H^1(X', \mathcal{O}) = 0$. If $H^1_c(X, \mathcal{O}) = 0$ then the Hartogs phenomenon holds in X.
3. Suppose X admits a compactification X' as in (2) and a compact exhaustion $\{V_n\}_{n \in \mathbb{N}}$, such that $X \setminus V_n$ is connected. If for any $n \in \mathbb{N}$ the restriction homomorphism $H^0(X, \mathcal{O}) \to H^0(X \setminus V_n, \mathcal{O})$ is an isomorphism, then the natural homomorphism $H^1_c(X, \mathcal{O}) \to H^1(X', \mathcal{O})$ is monomorphism.

Conclusion

Corrected Marciniak conjecture: Let X_Σ be a noncompact toric variety. If the complement of $|\Sigma|$ has at least one concave connected component then the Hartogs phenomenon holds in X_Σ

Theorems 1 and 2 implies the validity of the corrected Marciniak conjecture.

Corollary

Let X_Σ be a non-compact toric variety with the fan Σ whose complement is $\mathbb{R}^p \setminus |\Sigma| = \bigsqcup_{j=1}^{n} C_j$. Then

- *if at least one of C_j's is concave then X_Σ admits the Hartogs phenomenon.*
- *if $n = 1$ then the converse is also true, i.e. if X_Σ admits the Hartogs phenomenon then $\mathbb{R}^p \setminus |\Sigma|$ is concave.*