Rigid spheres and homogeneous Sasakian manifolds

Gerd Schmalz, University of New England
joint with Daniel Sykes, University of New England
Vladimir Ezhov, Flinders University

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1. Rigid hypersurfaces and Sasakian manifolds
2. Classification of Rigid and Sasakian structures
3. Homogeneous Sasakian manifolds

Rigid hypersurfaces and Sasakian manifolds

- Rigid hypersurface M in $\mathbb{C}^{n+1}$: $\operatorname{Im} w = f(z, \bar{z})$, no $\operatorname{Re} w$
- $Z = \frac{\partial}{\partial u} = 2 \operatorname{Re} \frac{\partial}{\partial w}$ is an infinitesimal automorphism; αZ with $\alpha \in \mathbb{R}$, as well
- Fixing Z can be considered as additional structure: Assume M is Levi-nondegenerate (or strictly pseudoconvex) then Z is the Reeb vector field for the contact form $\theta = du - \mathrm{i} \frac{\partial f}{\partial z} dz + \mathrm{i} \frac{\partial f}{\partial \bar{z}} d\bar{z}$.
- Sasakian structure: metric $g = d\theta \circ J|_D + \theta^2$, where $D = \ker \theta$
- "Homothetic equivalence" $Z \to \alpha Z$ then $g \to \frac{1}{\alpha} d\theta \circ J|_D + \frac{1}{\alpha^2} \theta^2$ corresponds to equivalence of rigid hypersurfaces, somewhat weaker than equivalence of Sasakian structures

Classification of Rigid and Sasakian structures

- Equivalence problem: Which rigid/Sasakian structures are equivalent?
- What are the rigid/ Sasakian automorphisms of a rigid hypersurface/Sasakian manifold
- What are the most symmetric objects? Homogeneous objects?

Stanton's rigid normal form

- Let $M \subset \mathbb{C}^2$, strictly pseudoconvex, real-analytic, rigid
- Rigid normal form solves the equivalence problem
  $\operatorname{Im} w = |z|^2 + \gamma_{22}|z|^4 + \gamma_{32} z^3 \bar{z}^2 + \overline{\gamma_{32}} z^2 \bar{z}^3 + \gamma_{33}|z|^6 + \cdots$
- Main differences to Chern-Moser normal form:
  1. $\gamma_{k\ell}$ are independent of $\operatorname{Re} w$
  2. "trace terms" γ_{22}, γ_{32}, γ_{33} can be different from 0
  3. determined up to $\mathbb{C}^*$ action $z \to cz$, $w \to |c|^2 w$, $\gamma_{k\ell} \to c^{k-1}\bar{c}^{\ell-1}\gamma_{k\ell}$
  4. Heisenberg sphere (as CR manifold) $v = |z|^2$ can carry many inequivalent rigid/Sasakian structures with different rigid normal forms

Classification of rigid/Sasakian structures on the Heisenberg sphere

Theorem (Ezhov, S., 2015)

Let (M, D, J) be a 3-dimensional real analytic, spherical CR-manifold and Z be an infinitesimal automorphism transversal to D. Then its rigid normal form is uniquely determined by γ_{22}, γ_{32}, γ_{33}.
Any $\gamma_{22}, \gamma_{33} \in \mathbb{R}$, $\gamma_{32} \in \mathbb{C}$ give rise to such CR manifold.
Two such pairs (M, Z) and (M', Z') are equivalent as rigid hypersurfaces/ Sasakian manifolds if and only if $\gamma'_{k\ell} = c^{k-1}\bar{c}^{\ell-1}\gamma_{k\ell}$, where $c \in \mathbb{C}^*$ / $|c| = 1$.

The infinitesimal automorphisms Z for the sphere are well known. They form the 8-dimensional Lie algebra $\mathfrak{su}(2,1)$. The transversality condition is an open condition.
Different choices of Z may give equivalent rigid/Sasakian structures.

The “rigid spheres”

- Stanton's rigid spheres

$$\frac{1}{2r}\sin 2rv\left(1-\frac{2|b|^2\theta}{|c|^2}\right)=\frac{|z|^2\,\mathrm{e}^{-2\theta v}}{1+4|b|^2|z|^2+2\,\mathrm{i}(b\bar{z}-\bar{b}z)}+\frac{|b|^2}{|c|^2}(\mathrm{e}^{-2\theta v}-\cos 2rv)+ \\ +\frac{\bar{b}z}{\bar{c}(1-2\,\mathrm{i}\,\bar{b}z)}(\mathrm{e}^{-2\theta v}-\mathrm{e}^{2\,\mathrm{i}\,rv})+\frac{b\bar{z}}{c(1+2\,\mathrm{i}\,b\bar{z})}(\mathrm{e}^{-2\theta v}-\mathrm{e}^{-2\,\mathrm{i}\,rv}), \quad (1)$$

correspond to

$$Z=(b+(r+\mathrm{i}\,\theta)z)\frac{\partial}{\partial z}+(1+2\,\mathrm{i}\,\bar{b}z+2rw)\frac{\partial}{\partial w}.$$

- Relations between γ_{22}, γ_{23}, γ_{33} and θ, r, b

$$\gamma_{22}=6|b|^2-2\theta \quad (2)$$

$$\gamma_{23}=2(r-\mathrm{i}\,\theta)b+4\,\mathrm{i}\,b|b|^2 \quad (3)$$

$$\gamma_{33}=\frac{2}{3}r^2+6\theta^2+56|b|^4-\frac{112}{3}\theta|b|^2. \quad (4)$$

cannot be solved for arbitrary γ_{22}, γ_{23}, γ_{33}

Complete family of rigid spheres

$$(1-4\phi|z|^2)\frac{\sin 2rv}{2r}-\mathrm{e}^{-2\theta v}\,|z|^2-(\phi-\bar{a}z-a\bar{z}+4\phi(\phi-\theta)|z|^2)\frac{\mathrm{e}^{-2\theta v}-\cos 2rv+\frac{\theta\sin 2rv}{r}}{r^2+\theta^2}=0. \quad (5)$$

correspond to

$$Z=(\mathrm{i}\,\tau z+aw+2\,\mathrm{i}\,\bar{a}z^2+\rho zw)\frac{\partial}{\partial z}+(1+2\,\mathrm{i}\,\bar{a}zw+\rho w^2)\frac{\partial}{\partial w}$$

where

$$\tau=-\frac{\gamma_{22}}{2}=\theta-3\phi \quad (6)$$
$$a=-\frac{\gamma_{23}}{2}(=-b(r-\mathrm{i}\,\theta+2\,\mathrm{i}\,\phi))$$
$$\rho=-\frac{3}{2}\gamma_{33}+\frac{9}{4}\gamma_{22}^2=-3\phi^2-r^2+2\phi\theta.$$

and ϕ is a real solution of

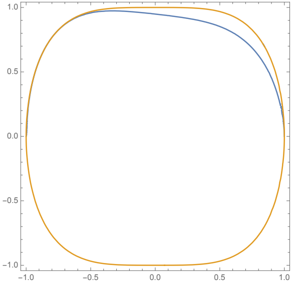
$$|a|^2=4\phi^3+4\tau\phi^2+(\tau^2-\rho)\phi. \quad (7)$$

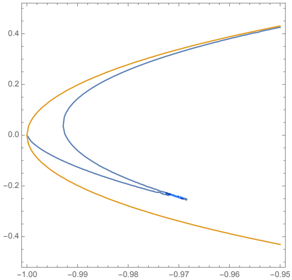
The moduli space

- Rigid spheres can be parameterised by γ_{22}, γ_{23}, γ_{33} or equivalently by τ, a, ρ modulo $\mathbb{C}^*$
- $\mathbb{C}^*$-action is $(\tau, a, \rho) \mapsto (|c|^2\tau, c\bar{c}^2 a, |c|^4\rho)$
- Moduli space of rigid spheres: $\{(0,0,0)\} \cup \mathfrak{M}$

$$\mathfrak{M} = \{(\tau, a, \rho) \in \mathbb{R}^3 \colon \tau^4 + a^{\frac{8}{3}} + \rho^2 = 1, a \geq 0\}$$

- $\mathfrak{M}$ is topologically equivalent to the closed "disc" $\{\tau^4 + \rho^2 \leq 1\} \subset \mathbb{R}^2$



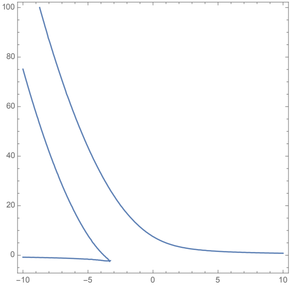


The boundary curve of Stanton's cases

- Stanton's cases: $\phi = |b|^2 \geq 0$, $r^2 \geq 0$
- If $\phi = 0$ then $a = 0$, $\tau = \theta$ and $\rho = -r^2 \leq 0$.
- If $r^2 = 0$ then parametric curve ($a \neq 0$ fixed, parameter ϕ)

$$\tau = \pm \frac{|a|}{\sqrt{\phi}} - \phi \qquad \rho = \pm 2|a|\sqrt{\phi} + \phi^2 \tag{8}$$

where $\phi \in (0, \infty)$.



(Interior of $\mathfrak{M}$, $a \neq 0$, scaled to $a = 1$.)

The discriminant of the cubic

- Cubic equation on ϕ: $4\phi^3 + 4\tau\phi^2 + (\tau^2 - \rho)\phi - |a|^2 = 0$ has at least one real solution. Is there a continuous branch on $\mathfrak{M}$?
- The discriminant (for $a = 2$):
  $\rho^3 - 2\rho^2\tau^2 + \rho\tau^4 + 72\rho\tau - 8\tau^3 - 432 = 0$
- Mystery: The discriminant set and the Stanton boundary set (with $a = 2$) coincide!
- There is no continuous branch of a real solution: The cusp is a branch point; the double solution is larger/smaller than the third solution for $\phi < 1$ / $\phi > 1$.

Homogeneous Sasakian manifolds

Theorem

Let M be a 3-dimensional Sasakian manifold with underlying spherical CR manifold. Let (without loss of generality) $Z = (\mathrm{i}\,\tau z + aw + 2\,\mathrm{i}\,\bar{a}z^2 + \rho zw)\frac{\partial}{\partial z} + (1 + 2\,\mathrm{i}\,\bar{a}zw + \rho w^2)\frac{\partial}{\partial w}$ be the Reeb vector field. Then the Lie algebra of infinitesimal Sasaki automorphisms is

1. *4-dimensional in the case when $\tau = 0, a = 0, \rho = 0$. This is the Heisenberg sphere $v = |z|^2$ with the affine automorphisms $(z, u) \mapsto (e^{\mathrm{i}\,\phi}(z + p), u + q - 2\,\mathrm{Im}\, z\bar{p})$, where $p \in \mathbb{C}$, $q \in \mathbb{R}$, $\phi \in [0, 2\pi)$.*
2. *4-dimensional in the case $a = 0$, $\rho = \tau^2$ and $\tau > 0$. This corresponds to the rigid sphere $v = \log(1 + |z|^2)$ and can be globally realised as the round sphere $|z|^2 + |w|^2 = 1$ in $\mathbb{C}^2$ with its natural Sasakian structure. The Sasakian automorphisms are induced by the unitary transformations in $\mathbb{C}^2$.*
3. *4-dimensional in the case $a = 0$, $\rho = \tau^2$ and $\tau < 0$. This corresponds to the rigid sphere $v = -\log(1 - |z|^2)$ and can be globally realised as the hyperboloid $|z|^2 - |w|^2 = 1$ in $\mathbb{C}^2$ with its natural Sasakian structure. The Sasakian automorphisms are induced by the pseudounitary transformations in $\mathbb{C}^2$.*
4. *2-dimensional in all other cases.*

Homogeneous rigid manifolds

- Look at all homogeneous CR manifolds (Cartan's list) and transversal CR automorphisms preserved up to scale
- This adds two homogeneous rigid manifolds with respect to homothetic automorphisms
  1. Type (E) $v = y^\alpha$ with $|\alpha| < 1$, $\alpha \neq 0$
  2. Type (F) $v = y \log y$

  where the Sasakian structure is produced by fixing the Reeb vector field as either $\operatorname{Re} \frac{\partial}{\partial z}$ or $\operatorname{Re} \frac{\partial}{\partial w}$.