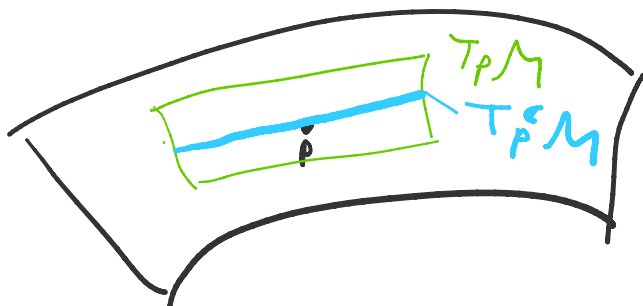


Real hypers in $\mathbb{C}^n$ m f d s
 $(M^{2n+1} \subset \mathbb{C}^{n+1})$

CR-structure:

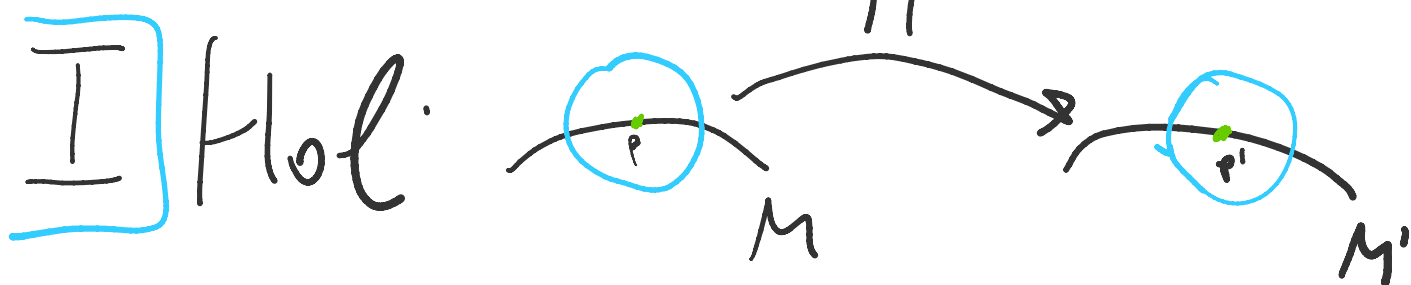
$$T^{\mathbb{C}}M := TM \cap iTM$$



Real-anal: $g(z, z) = 0$

$\mathcal{P}$ -hol, $Z = (z, w) \in \mathbb{C}^n \times \mathbb{C}^1$

CR-maps:



$H(M) \subset M'$, $H(p) = p'$, H -hol in $\mathbb{C}^{n+1}$

$H(M) \subset M$, $H(P) = P'$, H -hol in $\mathbb{C}^{n+1}$
(Poincaré)

II DG approach (Cartan)

$H \in C^\infty(M)$, $H: M \rightarrow M'$,

$H: T^c M \rightarrow T^c M'$,

$$dH \circ \gamma = \gamma' \circ dH$$

III $\hat{H}$ - formal power series Algebr approach

$$\hat{H}(z) = \sum_{n \geq 0, |n| > 0} c_n (z - p)^n$$

$$\rho'(\hat{H}(z), \overline{\hat{H}(z)}) = \hat{\alpha} \cdot P(z, \bar{z})$$

$$S(H(t), M(t)) = 2 \cdot S(z, \bar{z})$$

$$\left(\langle S'(\hat{H}, \bar{\hat{H}}) \rangle_{\text{ideal}} = \langle S(z, \bar{z}) \rangle_{\text{ideal}} \right)$$

Goal: Classify

$$\boxed{I \Rightarrow II \Rightarrow III}$$

geom
(time)

invariant /
tool

Rigidity:

$$\boxed{III \Rightarrow I}$$

under many circumstances

* „very wonder“

for now ...

* „Very nondegenerate“, for ex. Levi-nondeg
(Cartan, Tanaka, Chern, Moser)

Strong non-integr of $T^c M$

Convergence of formal maps

$$\left(\frac{\partial^2 \rho}{\partial z_i \partial \bar{z}_j} \right)_{i,j} \Big|_{T_p M}$$

Levi Form

* „Very degenerate“

- Levi-flat $\sim \mathbb{C}^n \times \mathbb{R}$

($\exists \text{ II} \Rightarrow \exists \text{ I}$)

- $\mathbb{C}^n$ mfd $\sim \mathbb{C}^n$

- Totally real $\sim \mathbb{R}^k$

Formal problem: are

formally equivalent CR-
structures equiv.

holomorphically?

Formal Principle: generically,
yes (due to many authors)

Artin Approx (algebra): $F(x, y) = 0$; $\hat{y}(x)$ -sol $\Rightarrow$
 $y^j(x)$ -sol-s, $y^j(x) \xrightarrow[\text{jets}]{j \rightarrow \infty} y(x)$

So, Formal principle $\approx$ CR-Artin

Importance: *distinguishing CR-inf's
by means of their finite jets

*Producing formal normal forms

Tools: * Artin
* DG (canonical objects: Chern-Moser)
* (!) Complete systems of PDEs
" " " " " "

(1) complete systems + 1 DEs
(ODEs) for H

$$j^k H = \text{Hol}(j^{k-1} H)$$

Divergence Theorem: (Kossovski,
Shafikov
2013)

∃ real-anal hyper in $\mathbb{P}^{n+1}$, $n \geq 1$,

for which the formal principle

FAILS!

(comes from Dyn. systems)

So, ~~III $\Rightarrow$ I~~

(Naive) Question:

QUESTION

Maybe $\underline{\text{III}} \Rightarrow \underline{\text{II}}$?

(CR Borel principle)

Obvious troubles:

* Why would $\hat{H} = \gamma(H)$?

Taylor jet map

* Even if so - why would $H(M) = M'$ hold?

„Troublemakers”: Flat functions

1/10/17 ... THAT FUNCTIONS

Ex: $M = \{ |z|^2 + |w|^2 = 1 \}$, $p = (1, 1)$

$$H: (z, w) \rightarrow (z + e^{-\frac{1}{4|z|^2}}, w)$$

$$H: M_p \rightarrow M'_p \quad \text{formally}$$

But not truly

NEVERTHELESS!

Theorem: (Kossovsky -
Lamell -
Stolovitch ²⁰¹⁶)

CR-Borel principle holds

for $n=1$ ($\text{III} \Rightarrow \text{II}$ in $\mathbb{C}^2$)

Conclude ...

Corollary: formal principle

holds for real-alg hyper in $\mathbb{C}^2$

(in fact even formal $\Rightarrow$ algebraic!)

Theorem (Ebenfelt -
Kossovskiy -
Lamel, 2020)

List of holomorphic / smooth
normal forms for real-anal
hyper in $\mathbb{C}^2$

Which „miracle“ makes
it possible to overcome

is possible to overcome
the obvious troubles?

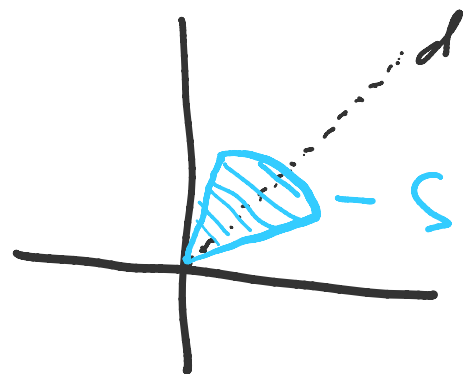
(Multi) Summability

Comes from one variable

Theory of how to canonically
realize formal power series as
asymptotic expansions of $f \in O(s)$

Comes from Dyn
systems

$$\begin{array}{ccc} X & \xrightarrow{\hat{H}} & X_0 \\ F & \xrightarrow{\hat{H}} & F \end{array}$$



Vect field
diff eq

$$F \xrightarrow{H} F_0$$

diffeo

object is not even topologically equivalent to its normal form

$$\underline{\text{Ex:}} \quad x \frac{\partial}{\partial x} - y \frac{\partial}{\partial y} + \dots \xrightarrow{\hat{H}} x \frac{\partial}{\partial x} - y \frac{\partial}{\partial y} + P$$

dynamics is totally different

The discovery was that $\hat{H}$ is still
 „trackable“ analytically is
 sectors $\rightarrow$ cocycle of hol norm maps

$$H_1, \dots, H_n \sim \hat{H}$$

$$H_i - H_{i+1} \sim 0$$

is „small“



Generically nontrivial functions
 gluing H_i with H_{i+1} are called

Stokes data; Stokes Phenomena

Poincaré, Leroy, Watson, Borel,
Ecalle, Malgrange, Ramis, Sibuya,
Balser, Martinez, Braaksma...

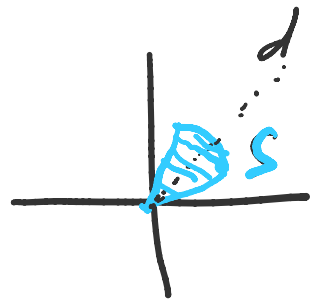
At a glance...

* Poincaré asympt expansion:

$$f \in \mathcal{O}(S)$$

$$\hat{f} = \sum_{n=0}^{\infty} c_n z^n$$

$$f \sim \hat{f}$$



$$\forall n, \quad \left| f(z) - \sum_{j=0}^n c_j z^j \right| \xrightarrow[z \in S']{z \rightarrow 0} 0$$

* Gevrey formal series:

$$\exists A, C, S > 0: \forall n, |c_n| \leq C \cdot A^n \cdot (n!)^S$$

$$\exists H, C, S > 0: \forall n, |c_n| \leq C \cdot H \cdot (n!)^S$$

$$S = \frac{1}{k} \text{ - Gevrey order}$$

* Gevrey asymptotic expansion:

$$f \in O(S), \hat{f} \sim f, \text{ and furthermore}$$

$$\exists C, A, S: \forall n, |f(z) - \sum_{i=0}^n c_i z^i| \leq C \cdot A^n \cdot |z|^n \cdot (n!)^S$$

$$\frac{1}{k} = S \text{ - Gevrey order}$$

Fact: $f \sim 0 \Rightarrow |f| \leq C \cdot e^{-\frac{\lambda}{|z|^k}}$

Gevrey is order S ($n < \frac{1}{|z|} \tilde{c} < n+1$) - and apply Gevrey

Watson Thm: $f \sim 0$ in a "large" sector

$$(\text{opening} > \frac{\pi}{k}) \Rightarrow \boxed{f \equiv 0}$$

Corollary: γ (= jet map) is injective on the

space of Gevrey functions in a "large" sector!!

Uniqueness

... (we need)

Def: such Gevrey functions in a "large" sector bisected by d are called k -summable in the dir. d .

Towards existence ...

Borel transform:



$$Bf(z) = B_1 f(z) := \frac{1}{z^{1/k}} \int_0^z f(\zeta) e^{\frac{\zeta}{z}} \frac{d\zeta}{\zeta^2}$$

exist limits to
exp; assume also
 $|f| \leq C \cdot |z|^\lambda, \lambda > 0$

$$p_k(f(z)) := f(z^{1/k})$$

$$B_k \in O(|\arg z - d| < \varepsilon)$$

$$B_k f(z) := p_{1/k} \circ B \circ p_k(f)$$

Fact: $B(z^\lambda) = \frac{z^{\lambda-1}}{\Gamma(\lambda)}, \lambda > 0 \Rightarrow$

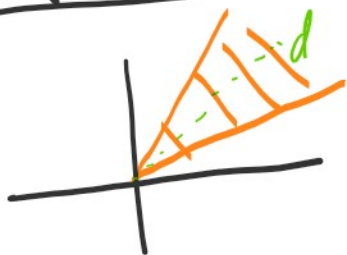
$$B \hat{f} := \sum_{n=1}^{\infty} c_n \cdot \frac{z^{n-1}}{(n-1)!}; \quad B_k \hat{f} := \sum_{n=1}^{\infty} c_n \cdot \frac{z^{n-k}}{\Gamma(\frac{n}{k})}$$

$\parallel \sum c_n z^n$

(... ..) $\hat{f}$ - Gevrey of order 1

So, we conclude that for $\hat{f}$ -Gevrey of $\text{ord} = \frac{1}{k}$,
 $B_k \hat{f} \in \text{hol}(0)$!

Laplace transform: $L_k f(z) := k \int_d f(s) e^{-\frac{s}{2k} z^{k-1}} ds$



$$|f| \leq C e^{R|z|^k}, \quad |z| \rightarrow \infty$$

$$|f| \leq C |z|^{k-1}, \quad |z| \rightarrow 0$$

$$B_k \circ L_k = L_k \circ B_k = \text{Id}$$

So, roughly, the way to sum $\hat{f}$:

$$\hat{f} \rightarrow B_k(\hat{f}) \rightarrow L_k \circ B_k(\hat{f}) := f$$

just as this step we use
 .. "truncated" Laplace

Borel-Ritt-Gevrey Thm:

For such considered $\hat{f}$, and $\forall d$,
 — — — — — "small"

F admits $\hat{f}$ as a symp expan
in a sector bisected by d with opening $\frac{\pi}{k}$.

So, we get the coverings, cocycles,
etc.

Multi-Summability: $\hat{f}$ is $(k_1, \dots, k_r)$ multi-
summable, if $\hat{f} = \hat{f}_1 + \dots + \hat{f}_r$, where
 $\hat{f}_j$ is k_j -summable in the same dir d ...

Merom ODEs

$$w^m \frac{dY}{dw} = F(Y, w)$$

Brankov's Thm: $\forall$ formal solution

$\hat{Y}(w)$ is multi-summable in "almost"
all directions (all except
(some directions))

Stores rays: $d_1, \dots, d_n$.

Proof by normal form theory for ODEs
+ the above Gevrey theory tools.

Application to CR

* Fundamental correspondence

$M \rightarrow E(M)$, $E(M)$ - a singular
or nonsingular system of diffeos
E. Cartan, Segre, Sukhov. ...

Functoriality: $H: M \rightarrow M' \Rightarrow H: E(M) \rightarrow E(M')$

* $(\mathbb{C}^2)$ $\hat{H}_{z^e} = \text{Hol}(z, w, \hat{H}_2, \dots, \hat{H}_{2e-1})$

$$w^k \frac{dY}{dw} = \text{Hol}(w, Y),$$

$$Y = (\hat{H}_1(0, w), \hat{H}_2(0, w), \dots, \hat{H}_{2e-1}(0, w)).$$

$\downarrow$ n

$\downarrow$ 1

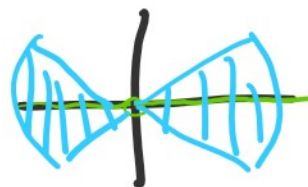
$\downarrow$ $\dots$ $\downarrow$

* Braaksma Thm: $\hat{H}(z, w) = H^{\text{hol}}(z, \hat{Y}(w)) =$
 $= \sum_{j=0}^{\infty} z^j \hat{h}_j(w), \hat{h}_j(w) \text{ multi-summable}$

„anal in z and multi-summable in w “
 multi-summ. in „almost“ all dir

Thm: if a formal series of the latter kind maps onto each other real-anal hyper, then its canonical (Gevrey) sectorial realization maps the manifolds onto each other.

$H \sim \hat{H}, H: \Delta \times S^{\pm}$
 polydisc



(in case $\mathbb{R}^{\pm}$ - Stokes, we write slightly $S^{\pm}$)

Proof: $\rho(\hat{H}(z, z + i\psi(z, \bar{z}, w)), \text{conj.}) = 0$

$M: V = \psi(z, \bar{z}, w)$

$z=0$: holds formally $\Rightarrow$ by uni-summable $\equiv 0$

not

$\frac{d^2 L}{dz d\bar{z}}$: same argum. 

Normal forms are
obtained in a similar spirit

THANKS!