Finite jet determination for CR mappings

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A CR mapping is a diffeomorphism between two real manifolds in complex space that satisfies tangential Cauchy-Riemann equations. We are concerned with the problem whether a CR mapping is uniquely determined by its finite jet at a point. This problem has been popular since 1970-s and the number of publications on the matter is enormous. Nevertheless, natural fundamental questions have remained open. I will present a solution to a version of the problem and discuss old and new results.

- CR manifolds and CR mappings
- Conditions on the Levi form
- Infinitesimal automorphisms of quadrics
- Finite jet determination
- 2-jet determination
- Examples

CR manifolds

Let M be a smooth real submanifold in $\mathbb{C}^n$. Recall the complex tangent space at $p \in M$

$$T_p^c(M) = T_p(M) \cap JT_p(M), \; p \in M.$$

Here $J : \mathbb{C}^n \to \mathbb{C}^n$ is the operator of multiplication by $i = \sqrt{-1}$. The manifold M is called a CR manifold if $\dim T_p^c(M)$ does not depend on $p \in M$. Then the dimension $\dim_{\mathbb{C}} T_p^c(M)$ is called the CR dimension of M and is denoted by $\dim_{CR} M$.

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The manifold M is called generic if $T_p(M)$ spans $T_p(\mathbb{C}^n) \simeq \mathbb{C}^n$ over $\mathbb{C}$ for all $p \in M$, that is,

$$T_p(M) + JT_p(M) = \mathbb{C}^n.$$

For instance, all real hypersurfaces are generic. If M is generic, then M is a CR manifold and

$$\dim_{\mathbb{C}} T_p^c(M) + \mathrm{cod}\, M = n,$$

where $\mathrm{cod}\, M$ is the codimension of M in $\mathbb{C}^n$.

CR mappings

Let M_1 and M_2 be CR manifolds. A C^1 mapping $f : M_1 \to M_2$ is called a CR mapping or a CR map if $df|_{T^c(M_1)}$ is a $\mathbb{C}$-linear mapping $T^c(M_1) \to T^c(M_2)$.

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If a CR mapping is a diffeomorphism, then it is called a CR diffeomorphism. Clearly, if $f : M_1 \to M_2$ is a CR diffeomorphism of generic manifolds in $\mathbb{C}^n$, then M_1 and M_2 should have the same dimension and CR dimension. We will consider only CR diffeomorphisms and will call them just CR mappings.

Equations of a generic manifold

We introduce coordinates $(z, w) \in \mathbb{C}^n$, $z \in \mathbb{C}^m$, $w = u + iv \in \mathbb{C}^k$, so that M has a local equation

$$v = h(z, u),$$

where $h = (h_1, \ldots, h_k)$ is a smooth real vector function with $h(0) = 0$, $dh(0) = 0$.

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Furthermore, we can choose the coordinates so that the equations of M take the form

$$v_j = h_j(z, u) = \langle A_j z, \overline{z} \rangle + O(|z|^3 + |u|^3), \qquad 1 \leq j \leq k,$$

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The matrices A_j can be regarded as the components of the vector valued Levi form of M at 0.

Conditions on the Levi form

- We say M is Levi generating at 0 if the matrices A_j are linearly independent. If this condition is not fulfilled, then the quadratic manifold $\{(z, w) \in \mathbb{C}^n : v_j = \langle A_j z, \overline{z} \rangle, 1 \leq j \leq k\}$ is foliated by CR manifolds of the same CR dimension as M. We restrict to Levi generating manifolds here.

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- We say M is strongly Levi nondegenerate at 0 if there is $c \in \mathbb{R}^k$ such that $\det \left(\sum c_j A_j \right) \neq 0$. This condition means that M lies on a Levi nondegenerate hypersurface.

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- We say M is Levi nondegenerate at 0 if $\langle A_j z, \overline{\zeta} \rangle = 0$ for all j and z implies $\zeta = 0$.
- We say M is strongly Levi nondegenerate at 0 if there is $c \in \mathbb{R}^k$ such that $\det \left(\sum c_j A_j \right) \neq 0$. This condition means that M lies on a Levi nondegenerate hypersurface.
- We say M is strongly pseudoconvex at 0 if there is $c \in \mathbb{R}^k$ such that $\sum c_j A_j > 0$. This condition means that M lies on a strogngly pseudoconvex hypersurface.

Finite jet determination

The problem of finite jet determination has been a subject of work by many authors (Baouendi, Beloshapka, Bertrand, Ebenfelt, Ezhov, Kim, Lamel, Merker, Meylan, Rothschild, Schmalz, Sukhov, Zaitsev, etc).

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In spite of an enormous volume of publications on the matter, there have been fundamental open questions, in particular, when CR mappings are uniquely defined by their 2-jets. We restrict to Levi generating Levi nondegenerate CR manifolds, which we for brevity call just nondegenerate.

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Beloshapka (1988) proved that a real analytic CR automorphism of a real analytic nondegenerate CR manifold is determined by its finite jet at a point.

2-jet determination

Bertrand, Blanc-Centi and Meylan (2019-2020), prove 2-jet determination for C^3-smooth CR automorphisms of C^4-smooth generic nondegenerate manifold M with additional condition that the authors call D-nondegenerate. In particular, it implies that there is $z \in \mathbb{C}^m$ such that the vectors $\{A_j z : 1 \leq j \leq k\}$ are $\mathbb{R}$-linearly independent. This condition is quite restrictive, in particular, it implies that $\operatorname{cod} M \leq 2 \dim_{CR} M$, whereas the dimension of the space of all Hermitian forms on $\mathbb{C}^m$ is equal to m^2.

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Tumanov (2019) proves 2-jet determination for C^3-smooth CR automorphisms of C^4-smooth strongly pseudoconvex manifolds.

Both results were obtained by using the invariantness of stationary discs.

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For arbitrary large integer p, Gregerovič and Meylan (2020) constructed counterexamples for which p-jet determination fails.

We present a sufficient condition for 2-jet determination that implies both results on 2-jet determination mentioned above, that is, for strictly pseudoconvex and for D-nondegenerate manifolds. Our approach is based on infinitesimal automorphisms.

Infinitesimal automorphisms of quadrics

An infinitesimal CR-automorphism of a CR-manifold M is a vector field on M that generates a local 1-parameter group of CR-mappings (CR-automorphisms) $M \to M$.

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We first restrict to the case in which M is a nondegenerate quadric defined as before by the equations

$$v = F(z, z), \quad z \in \mathbb{C}^m, w = u + iv \in \mathbb{C}^k,$$

here $F = (F_1, \ldots, F_k)$, $F_j(z, z) = \langle A_j z, \overline{z} \rangle$.

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Let G be the group of all CR-mappings (CR-automorphisms) $M \to M$. Then G is a finite dimensional Lie group and its Lie algebra $\mathfrak{g}$ is the set of all infinitesimal automorphisms of M. The dimension of G has an estimate depending on m and k. (Beloshapka 1988, Tumanov 1988, Isaev and Kaup 2012, ...)

It turns out that all elements of G and $\mathfrak{g}$ are respectively rational and polynomial. In particular, every vector field $X \in \mathfrak{g}$ has the form

$$X = \sum f_j \frac{\partial}{\partial z_j} + \sum g_\ell \frac{\partial}{\partial w_\ell} = f \frac{\partial}{\partial z} + g \frac{\partial}{\partial w} = (f, g),$$

where f and g are polynomial vector functions in z and w that satisfy the equation

$$\operatorname{Im}(g - 2iF(f, z)) = 0, \quad (z, w) \in M.$$

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where f and g are polynomial vector functions in z and w that satisfy the equation

$$\mathrm{Im}\,(g - 2iF(f, z)) = 0, \quad (z, w) \in M.$$

This equation implies

$$\deg_z f \leq 2, \quad \deg_z g \leq 1.$$

The graded algebra $\mathfrak{g}$

We give the variables and differentiations $z_j, w_j, \partial/\partial z_j, \partial/\partial w_j$ the weights 1,2,-1,-2 respectively. Let $\mathfrak{g}_p$ be the set of vector fields $X \in \mathfrak{g}$ with weighted homogeneous degree $p \in \mathbb{Z}$. Then

$$\mathfrak{g} = \sum_{p=-2}^{\infty} \mathfrak{g}_p$$

is a graded Lie algebra, that is, $[\mathfrak{g}_p, \mathfrak{g}_q] \subset \mathfrak{g}_{p+q}$. The terms $\mathfrak{g}_{-2}$ and $\mathfrak{g}_{-1}$ have the same form for all quadrics:

$$\mathfrak{g}_{-2} = \{b\frac{\partial}{\partial w} : b \in \mathbb{R}^k\}$$
$$\mathfrak{g}_{-1} = \{a\frac{\partial}{\partial z} + 2iF(z,a)\frac{\partial}{\partial w} : a \in \mathbb{C}^m\}.$$

The algebra $\mathfrak{g}_{-2} + \mathfrak{g}_{-1}$ is the Lie algebra of the group of "parallel displacements" $M \to M$

$$(z, w) \mapsto (z + a, w + b + 2iF(z, a) + iF(a, a)), \quad a \in \mathbb{C}^m, b \in \mathbb{R}^k.$$

This group acts freely and transitively on M.

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For $p \geq 0$, the structure of $\mathfrak{g}_p$ depends significantly on F.

Since F is nondegenerate, it follows that each vector $\xi \in \mathfrak{g}_p$ is uniquely determined by the map $\operatorname{ad} \xi : \mathfrak{g}_{-1} \to \mathfrak{g}_{p-1}$, here $(\operatorname{ad} \xi)(\eta) = [\xi, \eta]$.

In particular, if $\mathfrak{g}_p = 0$, then $\mathfrak{g}_q = 0$ for all $q > p$.

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Thus, the algebra $\mathfrak{g}$ is the Tanaka prolongation of $\mathfrak{g}_{-2} + \mathfrak{g}_{-1}$, that is, the maximal graded Lie algebra with the above unique determination property.

Finite jet determination

Let M be a nondegenerate CR manifold with equation

$$v = h(z, u) = F(z, z) + O(|z|^3 + |u|^3),$$

and let M_0 be the corresponding quadric with equation

$$v = F(z, z).$$

Let $\mathfrak{g}$ be the graded Lie algebra of infinitesimal automorphisms of M_0. Finite dimensionality of $\mathfrak{g}$ implies finite jet determination for CR mappings of M.

Theorem

Let M, M' be smooth non-degenerate CR manifolds defined as above. Suppose $\mathfrak{g}_p = 0$ for some $p > 0$. Then every germ at 0 of a smooth CR diffeomorphism $\Phi = (f, g) : M \to M'$ with $\Phi(0) = 0$ is uniquely determined by the jets of f and g at 0 of weights respectively p and $p + 1$.

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Corollary

Let M, M' be smooth non-degenerate CR manifolds defined as above. Suppose $\mathfrak{g}_3 = 0$. Then every germ at 0 of a smooth CR diffeomorphism $\Phi : M \to M'$ is uniquely determined by the 2-jet of Φ at 0. Conversely, if $\mathfrak{g}_3 \neq 0$, then there exists a CR diffeomorphism $\Phi : M_0 \to M_0$, $\Phi \neq \mathrm{id}$, whose 2-jet at 0 is the identity.

Beloshapka (1988) obtained the real analytic versions.

Proof

Following Moser (1974) and Beloshapka (1988), we expand the equations of M and M' and the CR mapping $\Phi = (f, g)$ into Taylor series with remainders and represent them as sums of weighed homogeneous components.

$$\begin{aligned} v &= h(z,u) = F + h_3 + \dots \\ v' &= h'(z',u') = F' + h'_3 + \dots \\ z' &= f(z,w) = f_1 + f_2 + \dots \\ w' &= g(z,w) = g_2 + g_3 + \dots \end{aligned}$$

Since $z = 0$ is mapped to $z' = 0$, we have $g_1 = 0$. By linear transformations of z and w, we can put $f_1 = z$, $g_2 = w + P(z)$, where P is a quadratic polynomial, but one can see that $P = 0$. Also, one can see that $F' = F$.

By plugging z' and w' in terms of z and $w = u + ih(z, u)$ in the equation of M' we obtain an equation for the component (f_{p+1}, g_{p+2}) of Φ:

$$\mathrm{Im}\,(g_{p+2} - 2iF(f_{p+1}, z))|_{w=u+iF(z,z)} = \dots,$$

here the dots mean terms that include only f_{q+1} and g_{q+2} with $q < p$.

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Note that the corresponding homogeneous equation describes $(f_{p+1}, g_{p+2}) \in \mathfrak{g}_p$. Since $\mathfrak{g}_p = 0$, the component (f_{p+1}, g_{p+2}) is uniquely determined by the components of Φ of lower weighted degree.

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In the smooth case, we can apply the above argument to pairs of points $(z, w) \in M$, $(z', w') = \Phi(z, w) \in M'$. This results in an overdetermined PDE on Φ whose solution is uniquely determined by a jet at just one point. This completes the proof.

Results on 2-jet determination

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Tanaka (1967) gave conditions under which $\mathfrak{g}_3 = 0$. They were quite restricting, in particular, they required that the adjoint representation of $\mathfrak{g}_0$ in the spaces g_{-1} and $\mathfrak{g}_{-2}$ be irreducible. He concluded that $\mathfrak{g}_3 = 0$ if $k = 1, m^2, m^2 - 1$.

Results on 2-jet determination

Let M, F, and A_j be as above. Let $c \in \mathbb{R}^k$ be such that $\det A \neq 0$, where $A = \sum_{j=1}^{k} c_j A_j$.
Let $z \in \mathbb{C}^m$. Define matrices

$$D = (A_1 z, \ldots, A_k z), \quad B = D^* A^{-1} D.$$

We say that M and F are D-nondegenerate if there exist $c \in \mathbb{R}^k$ and $z \in \mathbb{C}^m$ such that

$$\det A \neq 0, \quad \det \operatorname{Re} B \neq 0,$$

here $\operatorname{Re} B = \frac{1}{2}(B + \overline{B})$. In this case $A_1 z, \ldots, A_k z$ are $\mathbb{R}$-linearly independent, so $k \leq 2m$.

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Theorem (Bertrand and Meylan, 2020)

Let M, M' be smooth D-nondegenerate CR manifolds. Then 2-jet determination for smooth CR diffeomorphisms $M \to M'$ takes place.

Main result

Let M, F be as above. Let $S \subset \mathbb{C}^m$ be a set. We define the orthogonal complement

$$S^F = \{z \in \mathbb{C}^m : \forall z' \in S, F(z, z') = 0\}.$$

Similarly, we can define $S^{\mathrm{Re}\,F}$. We also define

$$T(z) = \{p \in \mathbb{C}^m : \exists q \in \mathbb{C}^m : F(z, p) + F(q, z) = 0\}.$$

We say that M and F are T-nondegenerate if for all z in an open dense set we have

$$z^F \cap T(z)^F = 0.$$

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We say that M and F are T-nondegenerate if for all z in an open dense set we have

$$z^F \cap T(z)^F = 0.$$

Theorem

Let M, M' be smooth T-nondegenerate CR manifolds. Then $\mathfrak{g}_3 = 0$. Hence, 2-jet determination for smooth CR diffeomorphisms $M \to M'$ takes place.

Comparison of various conditions

We have

$$z^F \subset z^{\mathrm{Re}\,F} \subset T(z),$$
$$z^F \cap T(z)^F \subset z^F \cap (z^F)^F.$$

Then the simpler condition

$$z^F \cap (z^F)^F = 0$$

for generic z is also sufficient for 2-jet determination.

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If M is strictly pseudoconvex, then $z^F \cap (z^F)^F = 0$ for all z, so we recover a result mentioned above.

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If M is strictly pseudoconvex, then $z^F \cap (z^F)^F = 0$ for all z, so we recover a result mentioned above.

One can see that $\det \mathrm{Re}\, B = 0$ in the definition of D-nondegeneracy implies $z^{\mathrm{Re}\, F} \cap (z^{\mathrm{Re}\, F})^{\mathrm{Re}\, F} = 0$. We have

$$z^F \cap T(z)^F \subset z^F \cap (z^{\mathrm{Re}\, F})^F \subset z^{\mathrm{Re}\, F} \cap (z^{\mathrm{Re}\, F})^{\mathrm{Re}\, F}.$$

Hence, if M is D-nondegenerate, then M is T-nondegenerate.

Proof of Main result

An element $(f, g) \in \mathfrak{g}_3$ has the following form

$$f(z, w) = A(z, z, w) + B(w, w),$$
$$g(z, w) = 2iF(z, B(\overline{w}, \overline{w})).$$

Here A and B are complex multilinear forms such that A is symmetric in the first two arguments and B is symmetric. They are characterized by the following equations:

$$F(A(z, z, F(z, a)), a) = 0,$$
$$F(A(z, z, w), a) = 4iF(z, B(\overline{w}, F(a, z))).$$

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We would like to show that $A = 0$ and $B = 0$. The difficulty is that the equations have repeated arguments. The idea is that any occurrence of $F(z, z')$ can be replaced by $F(p, p')$ for all $p, p' \in \mathbb{C}^m$ such that $F(p, p') = F(z, z')$.

We first plug $w = F(z,a)$ in the second equation and using the first equation we obtain

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Using the hypothesis that F is T-nondegenerate, we show that

$$B(F(z, a), F(z, a)) = 0.$$

Then it follows that

$$F(A(z, z, F(a, b)), b) = 0.$$

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Similarly, we finally show that $B(c, F(b, z)) = 0$, hence $B = 0$. This completes the proof.

Example 1 (Meylan)

Francine Meylan found an example of a (strictly) nondegenerate quadric for which $\mathfrak{g}_4 \neq 0$. Here $m = 4, k = 5$, $F = (F_1, \ldots, F_5)$.

$$
\begin{aligned}
F_1 &= |z_1|^2, \\
F_2 &= |z_2|^2, \\
F_3 &= \mathrm{Re}\,(z_1 \overline{z}_2), \\
F_4 &= \mathrm{Im}\,(z_1 \overline{z}_2), \\
F_5 &= \mathrm{Re}\,(z_1 \overline{z}_3 + z_2 \overline{z}_4).
\end{aligned}
$$

In this example, $z^F \cap T(z)^F \neq 0$ for all $z \neq 0$, all stationary discs are defective, and 2-jet determination fails!

Example 2

Let $m = 4, k = 3$, $F = (F_1, F_2, F_5)$ from the previous example. Then F is T-nondegenerate, so 2-jet determination takes place. This example is not D-nondegenerate and $z^F \cap (z^F)^F \neq 0$ for all $z \neq 0$.

Example 3

Let $m = 4, k = 4$, $F = (F_1, F_2, F_3, F_5)$ from Example 1. Then F is not T-nondegenerate, but one can see that $\mathfrak{g}_3 = 0$.

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Hence T-nondegeneracy is not necessary for 2-jet determination.

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Thank you!