Germs of finite covers branched in curve germs with ADE singularities and Belyi pairs

Vik.S. Kulikov

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Denote by $\tilde{Y}$ a complex manifold, $\widetilde{X}$ a normal variety,

$F : \widetilde{X} \to \tilde{Y}$ a finite holomorphic map, $\deg F = d$.

$B := B_F = \{y \in \tilde{Y} \mid \sharp F^{-1}(y) < d\}$ the **branch locus** of F.

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- The unramified d-sheeted cover

$$F : X = \widetilde{X} \setminus F^{-1}(B) \to Y = \tilde{Y} \setminus B$$

defines a **monodromy homomorphism**

$$F_* : \pi_1(\tilde{Y} \setminus B, p) \to \mathbb{S}_d,$$

where $\mathbb{S}_d$ is the symmetric group acting on the fibre $F^{-1}(p)$.

$G_F := \operatorname{im} F_* \subset \mathbb{S}_d$ is called the **monodromy group** of F.

Theorem (Riemann - Stein). B **− effective reduced divisor in smooth complex manifold** $\tilde{Y}$**,** $Y = \tilde{Y} \setminus B$**, and** $F : X \to Y$ **− finite unramified cover** $\Rightarrow$ $\exists!$ **extension** $X \subset \widetilde{X}$ **(**$\widetilde{X}$ **− normal complex variety) and finite holomorphic map** $\tilde{F} : \widetilde{X} \to \tilde{Y}$ **s.t.** $\tilde{F}_{|X} = F$**.**

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Remark. Unramified covers $F : X \to Y$, $\deg F = d$, are in 1-to-1 correspondence with monodromies $F_* : \pi_1(Y, p) \to \mathbb{S}_d$ considered up to inner automorphisms of $\mathbb{S}_d$.

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- $\widetilde{X}$ **is connected** $\Leftrightarrow$ G_F **is a transitive subgroup of** $\mathbb{S}_d$**.**

Belyi pairs

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- C – a non-singular irreducible projective curve of genus $g = g(C)$,

  $f : C \to \mathbb{P}^1$ – a rational function on C, $f \in \mathbb{C}(C)$.

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- f_1, $f_2 \in \mathbb{C}(C)$ are **equivalent** if $\exists\ \varphi \in \mathrm{Aut}(\mathbb{P}^1) = \mathrm{PGL}(2, \mathbb{C})$

  and $\psi \in \mathrm{Aut}(C)$ s.t. $f_2 = \varphi \circ f_1 \circ \psi$.

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Theorem (Belyi) **C is defined over $\overline{\mathbb{Q}}$ if and only if**

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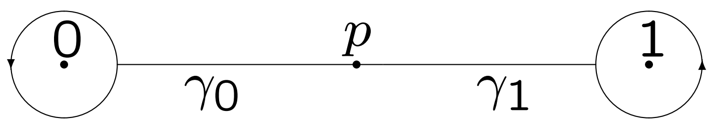
Theorem (Belyi) **C is defined over $\overline{\mathbb{Q}}$ if and only if $\exists f \in \mathbb{C}(C)$ s.t. (C, f) is a Belyi pair.**

- (C, f) – a Belyi pair. We will assume that $B_f \subset \{0, 1, \infty\}$.

- $f_* : \pi_1(\mathbb{P}^1 \setminus \{0,1,\infty\}, p) \to \mathbb{S}_{\deg f}$ – **monodromy homomorphism.**

  $\pi_1(\mathbb{P}^1 \setminus \{0,1,\infty\}, p) = \langle \gamma_0, \gamma_1 \rangle = \mathbb{F}_2$



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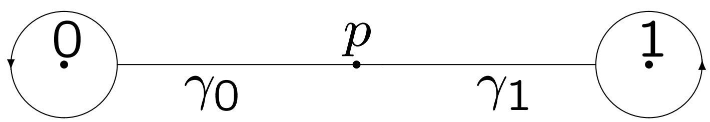

  $G_f := \mathrm{im} f_* = \langle g_0, g_1 \rangle \subset \mathbb{S}_{\deg f}$ – **monodromy group** of (C, f)

  where $g_0 = f_*(\gamma_0)$, $g_1 = f_*(\gamma_1) \in \mathbb{S}_{\deg f}$.

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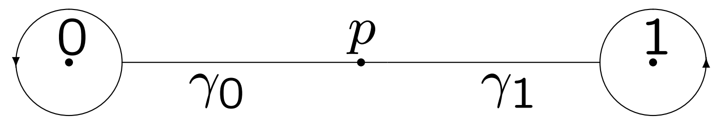

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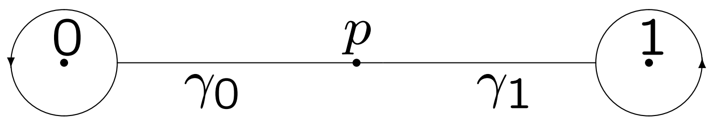

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- **$G_f = \langle g_0, g_1 \rangle$ is a transitive subgroup of $\mathbb{S}_{\deg f}$.**

- Conversely, by Riemann – Stein Theorem, **a homomorphism $f_* : \pi_1(\mathbb{P}^1 \setminus \{0,1,\infty\}, p) \to \mathbb{S}_n$, s.t. $G_f = \mathrm{im} f_*$ is a transitive subgroup of $\mathbb{S}_n \Rightarrow f_*$, defines a Belyi pair $f : C \to \mathbb{P}^1$, $\deg f = n$.**

- $c(g_i) = (m_{1,i}, \ldots, m_{n,i})$ – **cyclic type** of permutations

  $g_i = f_*(\gamma_i) \in G_f \subset \mathbb{S}_n$, $i = 0, 1, \infty$,

$$n + 2 = \sum_{j=1}^{n} (m_{j,0} + m_{j,1} + m_{j,\infty}) + g(C).$$

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Theorem. Two collections $((m_{1,0}, \ldots, m_{n,0}), (m_{1,1}, \ldots, m_{n,1}))$**,**

$\sum_{j=1}^{n} j m_{j,i} = n$ **for** $i = 0, 1$**,**

is a brief passport of some Belyi pair (C, f) **of genus** $g = g(C)$ **and** $\deg f = n \Leftrightarrow$

$$n - \sum_{j=1}^{n}(m_{j,0} + m_{j,1}) \geq 2g - 1.$$

Notations:

$$\mathcal{B}el = \bigcup_{g \geq 0} \mathcal{B}el^g$$

$$\mathcal{B}el^g = \{\textbf{Belyi pairs } f : C \to \mathbb{P}^1,\ g(C) = g\},$$

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- $(\mathbb{P}^1, f) \in \mathcal{B}el^0_2$ **iff** f **is equivalent to a cover given by**

$$(x_0, x_1) \mapsto (x_0^n, x_1^n)$$

for some $n \in \mathbb{N}$.

Germs of finite covers

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- $F : (U, o') \to (\mathbb{B}_\varepsilon, o)$ – **germ of finite cover,**

  i.e., **finite** holomorphic map s.t. $F^{-1}(o) = o'$,

  $\mathbb{B}_\varepsilon = \{(u, v) \in \mathbb{C}^2 |\ |u|^2 + |v|^2 < \varepsilon^2\}$, $\quad 0 < \varepsilon \ll 1$,

  (U, o') – **connected** germ of **normal** complex surface,

  $Sing\, U = \{o'\}$ if U is singular.

- $d = \deg_{o'} F$ – (local) **degree** of F.

- $F_1 : (U_1, o') \to (\mathbb{B}_{\varepsilon_1}, o)$ and $F_2 : (U_2, o') \to (\mathbb{B}_{\varepsilon_2}, o)$ are **equivalent** ($F_1 \sim F_2$) if there exist $(W_1, o) \subset (\mathbb{B}_{\varepsilon_1}, o)$, $(W_2, o) \subset (\mathbb{B}_{\varepsilon_2}, o)$ and biholomorphic maps $\varphi : (W_1, o) \to (W_2, o)$, $\psi : (\widetilde{U}_1, o') \to (\widetilde{U}_2, o')$ s.t. the following diagram

$$\begin{array}{ccc} \widetilde{U}_1 & \xrightarrow{\psi} & \widetilde{U}_2 \\ {\scriptstyle F_1}\downarrow & & \downarrow{\scriptstyle F_2} \\ W_1 & \xrightarrow[\varphi]{} & W_2 \end{array}$$

is commutative, where $\widetilde{U}_1 = F_1^{-1}(W_1)$ and $\widetilde{U}_2 = F_2^{-1}(W_2)$.

- $B := B_F \subset (\mathbb{B}_\varepsilon, o)$ – **germ of branch curve** of F.

Theorem–Definition. $\pi_1^{loc}(B, o) := \pi_1(\mathbb{B}_\varepsilon \setminus B)$, $\varepsilon \ll 1$, *is called* **local fundamental group of (B, o).**

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- **G_F is a transitive subgroup of $\mathbb{S}_d$.**

- Conversely, by Riemann – Stein Theorem, **a homomorphism $F_* : \pi_1^{loc}(B,o) \to \mathbb{S}_d$, s.t. $G_F = \mathrm{im} F_*$ is a transitive subgroup of $\mathbb{S}_d$, defines a cover $F : (U,o') \to (\mathbb{B}_\varepsilon, o)$, $\deg F = d$.**

Deformations of germs of finite covers

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Equisingular deformation of curve germs.

- $D_\delta = \{\tau \in \mathbb{C} \mid |\tau| < \delta\}$,
- $(\mathcal{V}, \mathcal{B}, \mathrm{pr}_2)$ – **family** of curve germs $(B_{\tau_0}, o_{\tau_0}) \subset (V_{\tau_0}, o_{\tau_0})$,

  where $\mathcal{B}$ is an effective reduced divisor in $\mathcal{V} = \mathbb{B}_\varepsilon \times D_\delta$ s.t.

  the restriction to $\mathcal{B}$ of $\mathrm{pr}_2 : \mathcal{V} \to D_\delta$ is flat holomorphic map,

$V_{\tau_0} = \mathbb{B}_\varepsilon \times \{\tau = \tau_0\} = \mathrm{pr}_2^{-1}(\tau_0)$, $B_{\tau_0} = \mathcal{B} \cap \mathrm{pr}_2^{-1}(\tau_0))$, $o_{\tau_0} = o \times \{\tau = \tau_0\}$.

- $(\mathcal{V}, \mathcal{B}, \mathrm{pr}_2)$ – **equisingular deformation of curve germs** if

(i) Sing $\mathcal{B} = \{o\} \times D_\delta$ and $\exists$ monoidal transformations

$\widetilde{\sigma}_i : \mathcal{V}_i \to \mathcal{V}_{i-1}$, $i = 1, \dots, n$, (here $\mathcal{V}_0 = \mathcal{V}$) with centers in

smooth curves $\mathcal{S}_{i-1} \subset$ Sing $\mathcal{B}_{i-1}$ ($\mathcal{B}_0 = \mathcal{B}$ and $\mathcal{B}_i = \widetilde{\sigma}_i^{-1}(\mathcal{B}_{i-1})$);

(ii) $Sing\, \mathcal{B}_i$ is a disjoint union of sections of projection

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- For $\tau_1, \tau_2 \in D_\delta$ the curve germs (B_{τ_1}, o) and (B_{τ_2}, o) are said to be **equisingular equivalent.** Continue equisingular equivalence to **equivalence relation.**

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Remark. **$(\mathcal{V}, \mathcal{B}, \mathbf{pr}_2)$ – equisingular deformation of curve germs $\Rightarrow$ (locally) $\exists$ C^∞-trivialization of projection $\mathbf{pr}_2 \circ \widetilde{\sigma} : (\mathcal{V}_n, \mathcal{B}_n) \to D_\delta$,**

i.e., $(\mathcal{V}_n, \mathcal{B}_n) \simeq \widetilde{\sigma}^{-1}((V_{\tau_0}, B_{\tau_0})) \times D_\delta$, where $\widetilde{\sigma} = \widetilde{\sigma}_1 \circ \cdots \circ \widetilde{\sigma}_n$,

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Theorem. (Zariski, Wahl) $T[B_1] = T[B_2] \;\Rightarrow\; \pi_1^{loc}(B_1, o) \simeq \pi_1^{loc}(B_2, o)$**.**

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• $(B, o) \subset (\mathbb{B}_\varepsilon, o)$ is a **rigid germ** if for $\forall\, (B', o) \subset (\mathbb{B}_{\varepsilon'}, o)$ s.t. $T[B'] = T[B]$ there exist $(V, o) \subset (\mathbb{B}_\varepsilon, o)$, $(V', o) \subset (\mathbb{B}_{\varepsilon'}, o)$, and biholomorphic map $G : (V', o) \to (V, o)$ s.t. $G(B' \cap V') = B \cap V$.

Deformations of germs of finite covers

- $\mathcal{F}: \mathcal{U} \to \mathcal{V} = (\mathbb{B}_\varepsilon, o) \times D_\delta$ – **family of germs of finite covers**

  $F_{\tau_0} = \mathcal{F}_{|U_{\tau_0}} : U_{\tau_0} = \mathcal{F}^{-1}(\mathbb{B}_\varepsilon \times \{\tau = \tau_0\}) \to \mathbb{B}_\varepsilon \times \{\tau = \tau_0\}, \ \tau_0 \in D_\delta,$

  if $\mathcal{F}$ is a **finite** holomorphic map, $\dim \mathcal{U} = 3$,

  $o'_{\tau_0} = \mathcal{F}^{-1}(o \times \{\tau = \tau_0\})$ is a **single** point for each $\tau_0 \in D_\delta$.

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- Denote by $\mathcal{B} \subset \mathcal{V}$ the branch surface of $\mathcal{F}$ (in case $\deg \mathcal{F} > 1$).
- $\mathcal{F}$ is a **deformation** of $F_0 : (U_0, o'_0) \to (\mathbb{B}_\varepsilon \times \{\tau = 0\}, o_0)$ if

  $(\mathcal{V}, \mathcal{B}, \mathrm{pr}_2)$ is an equisingular deformation of $B_0 = \mathcal{B} \cap (\mathbb{B}_\varepsilon \times \{\tau = 0\})$.

  F_{τ_1} and F_{τ_2}, $\tau_1, \tau_2 \in D_\delta$, are **deformation equivalent.**

- $\mathcal{F} : \mathcal{U} \to \mathcal{V} = (\mathbb{B}_\varepsilon, o) \times D_\delta$ – **family of germs of finite covers**

  $F_{\tau_0} = \mathcal{F}_{|U_{\tau_0}} : U_{\tau_0} = \mathcal{F}^{-1}(\mathbb{B}_\varepsilon \times \{\tau = \tau_0\}) \to \mathbb{B}_\varepsilon \times \{\tau = \tau_0\}, \ \tau_0 \in D_\delta,$

  if $\mathcal{F}$ is a **finite** holomorphic map, $\dim \mathcal{U} = 3$,

  $o'_{\tau_0} = \mathcal{F}^{-1}(o \times \{\tau = \tau_0\})$ is a **single** point for each $\tau_0 \in D_\delta$.
- Denote by $\mathcal{B} \subset \mathcal{V}$ the branch surface of $\mathcal{F}$ (in case $\deg \mathcal{F} > 1$).
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  F_{τ_1} and F_{τ_2}, $\tau_1, \tau_2 \in D_\delta$, are **deformation equivalent.**

  Continue deformation equivalence to **equivalence relation**

$$(F_{\tau_1} \overset{d}{\sim} F_{\tau_2}).$$

Remark. $(\mathbb{B}_\varepsilon \times D_\delta, \mathcal{B}, pr_2)$ – *equisingular deformation of curve germs,*

$(B_0, o) = \mathcal{B} \cap pr_2^{-1}(0)$ – *branch curve of finite cover*

$F_0 : (U, o') \to (\mathbb{B}_\varepsilon, o)$, $\deg_{o'} F = d$, $\quad \Rightarrow$

$\pi_1((\mathbb{B}_\varepsilon \times D_\delta) \setminus \mathcal{B}) \simeq \pi_1^{loc}(B_0, o)$ *and* $F_{0*} : \pi_1^{loc}(B_0, o) \to \mathbb{S}_d$

define deformation $\mathcal{F} : \mathcal{U} \to \mathbb{B}_\varepsilon \times D_\delta$ *of* $\mathcal{F}_{|U_0} = F_0$.

- $F : (U, o') \to (\mathbb{B}_\varepsilon, o)$ is **rigid** if $\forall\ F_1 \overset{d}{\sim} F \ \Rightarrow\ \ F_1 \sim F$.

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Theorem. (Arnol'd) $T[B] \in \{A_n, D_n, E_6, E_7, E_8\} \Rightarrow$ (B, o) **is rigid.**

ADE-**singularity types:**

- $A_n := T[v^2 - u^{n+1} = 0]$, $n \geq 0$;
- $D_n := T[u(v^2 - u^{n-2}) = 0]$, $n \geq 4$;
- $E_6 := T[v^3 - u^4 = 0]$;
- $E_7 := T[v(v^2 - u^3) = 0]$;
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Conjecture. (B,o) **is rigid** $\Rightarrow$ $T[B]\in\{A_n,D_n,E_6,E_7,E_8\}$**.**

- **Denote** by $\mathcal{R}$ the set of rigid covers $F : (U, o') \to (\mathbb{B}_\varepsilon, o)$,

  $T[B_F] \in \{A_n, D_n, E_6, E_7, E_8\}$,

$$\mathcal{R} = \left(\bigcup_{n\geq 0} \mathcal{R}_{\mathbf{A}_n}\right) \cup \left(\bigcup_{n\geq 4} \mathcal{R}_{\mathbf{D}_n}\right) \cup \left(\bigcup_{n\in\{6,7,8\}} \mathcal{R}_{\mathbf{E}_n}\right),$$

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The aim is to define a natural map $\beta:\mathcal{R}\to\mathcal{B}el$.

For this purpose I need to remind presentations of $\pi_1^{loc}(B,o)$.

Notations.

- (X, o) – germ of normal surface and $B = \bigcup_{j=1}^{m} B_j$ – the union of irreducible curve germs $(B_j, o) \subset (X, o)$.
- $\sigma : \widetilde{X} \to (X, o)$ – *resolution of singularities* of the pair (X, B), i.e., $\widetilde{X}$ is smooth and $\widetilde{B} = \sigma^{-1}(B)$ is a divisor with normal crossings.

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Assume that

- $\sigma^{-1}(o) = \bigcup_{j=1}^{k} E_j$ is a union of **rational** curves,
- **dual weighted graph** $\Gamma(\tilde{B})$ of $\tilde{B}$ is a **tree** having $m + k$ vertices v_j:

  $v_j := b_j$ with weights $w_j = 0$ correspond to B_j, $1 \leq j \leq m$,

  $v_{m+j} := e_{m+j}$ with weights $w_{m+j} = (E_j^2)_{\widetilde{X}}$ correspond to E_j, $1 \leq j \leq k$.

Two vertices v_i and v_j of $\Gamma(\widetilde{B})$ are connected by an edge (v_i, v_j) iff the corresponding curves have a non-empty intersection.

- For each pair of vertices v_i and v_j of $\Gamma(\widetilde{B})$, define

$$\delta_{i,j} = \begin{cases} 1, & \text{if } v_i \text{ and } v_j \text{ are connected by an edge in } \Gamma(\widetilde{B}), \\ 0, & \text{if } v_i \text{ and } v_j \text{ are not connected by an edge in } \Gamma(\widetilde{B}), \\ 0, & \text{if } i = j. \end{cases}$$

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Renumber the vertices $e_{1+m}, \dots, e_{k+m}$ (and corresponding them curves E_{i+m}) so that the new numbering has the following property:

in the shortest way $(e_{1+m}, e_{i_2}), (e_{i_2}, e_{i_3}), \dots, (e_{i_{n-1}}, e_{i_n})$ *from the vertex* e_{1+m} *to each vertex* e_{i_n} *along the edges* $(e_{i_j}, e_{i_{j+1}})$, $j = 1, \dots, k-1$, *we have inequalities* $i_j < i_{j+1}$.

Theorem. $\pi_1(X \setminus B) = \pi_1(\widetilde{X} \setminus \widetilde{B})$ *is generated by* $m+k$ *elements* $b_1, \dots, b_m$ *being in one-to-one correspondence with vertices* $v_1, \dots, v_m$ *and* $e_{m+1}, \dots, e_{m+k}$ *being in one-to-one correspondence with vertices* $v_{m+1}, \dots, v_{m+k}$ *of* $\Gamma(\widetilde{B})$*, and being subject to the following defining relations:*

$$e_{m+i}^{w_{m+i}} \cdot b_1^{\delta_{1,m+i}} \cdot \ldots \cdot b_m^{\delta_{m,m+i}} \cdot e_{m+1}^{\delta_{m+i,m+1}} \cdot \ldots \cdot e_{m+k}^{\delta_{m+i,m+k}} = 1, \; i = 1, \dots, k,$$

$$[b_j, e_{m+i}] = 1 \text{ if } \delta_{j,m+i} = 1,$$

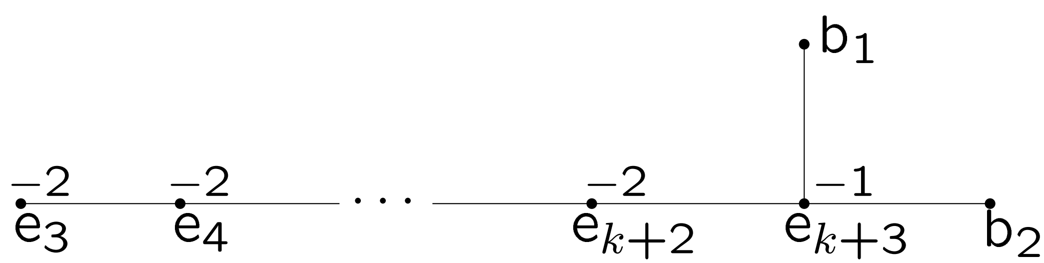
$$[e_{m+i_1}, e_{m+i_2}] = 1 \text{ if } \delta_{m+i_1,m+i_2} = 1.$$

Proof coincides almost word for word with proof of similar theorem (D. Mumford: *The topology of normal singularities of an algebraic surface and a criterian for simplisity*, Publ. Math. IHES, no. **9** (1961)).

Graph $\Gamma(\widetilde{B})$ of curve germ (B,o) of singularity type

$$\mathbf{A}_{2k+1} = T[u^2 - v^{2(k+1)} = 0],\ k \geq 0$$

(if $k = 0$, then weight $\omega_3 = -1$).



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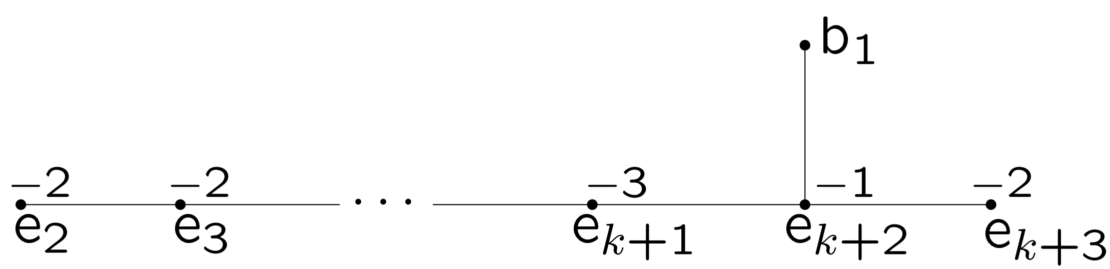
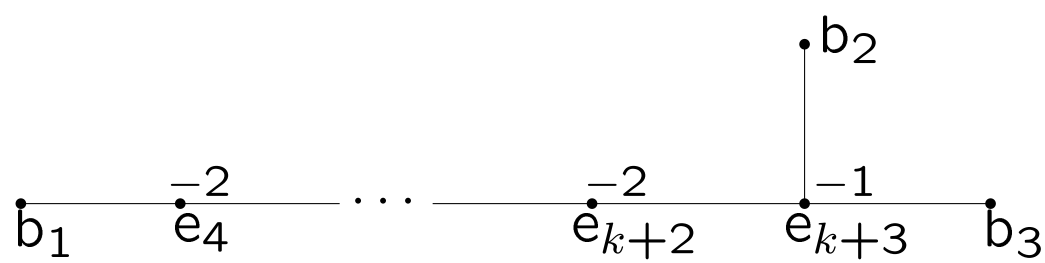


Graph $\Gamma(\widetilde{B})$ of curve germ (B, o) of singularity type

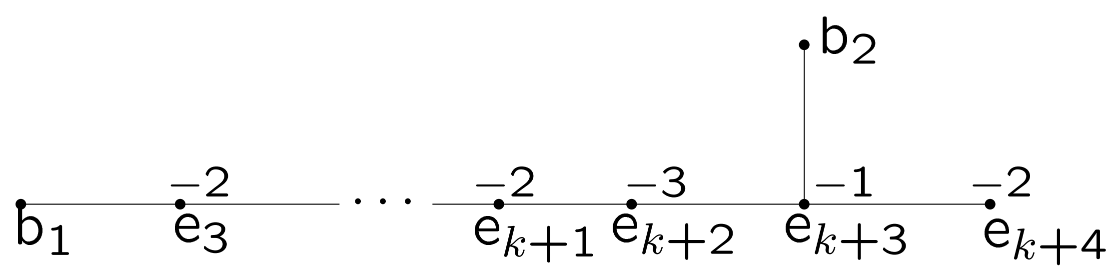
$$\mathbf{D}_{2k+2} = T[v(u^2 - v^{2k}) = 0],\ k \geq 1$$

(if $k = 1$, then weight $\omega_4 = -1$).



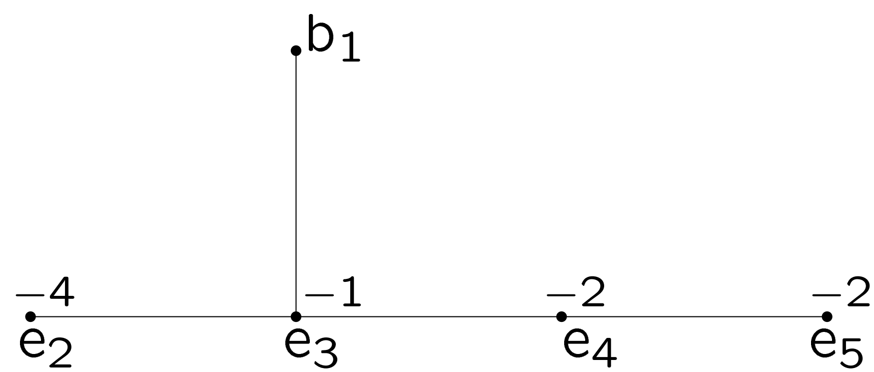
Graph $\Gamma(\widetilde{B})$ of curve germ (B, o) of singularity type

$$\mathbf{D}_{2k+3} = T[v(u^2 - v^{2k+1}) = 0],\ k \geq 1$$



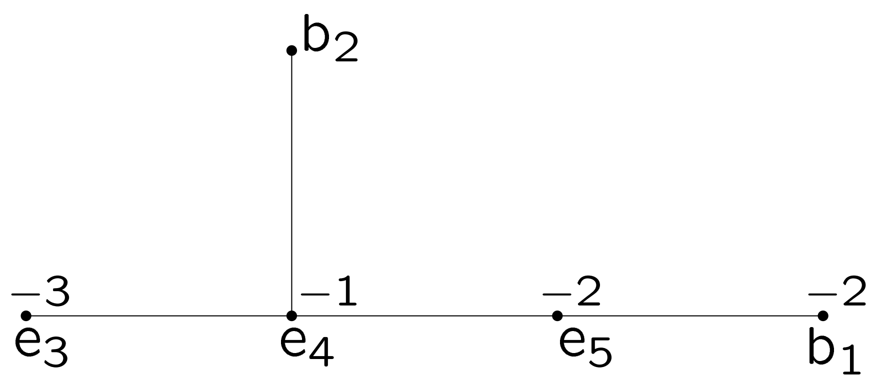
Graph $\Gamma(\widetilde{B})$ of curve germ (B, o) of singularity type

$$\mathbf{E}_6 = T[u^3 - v^4 = 0]$$



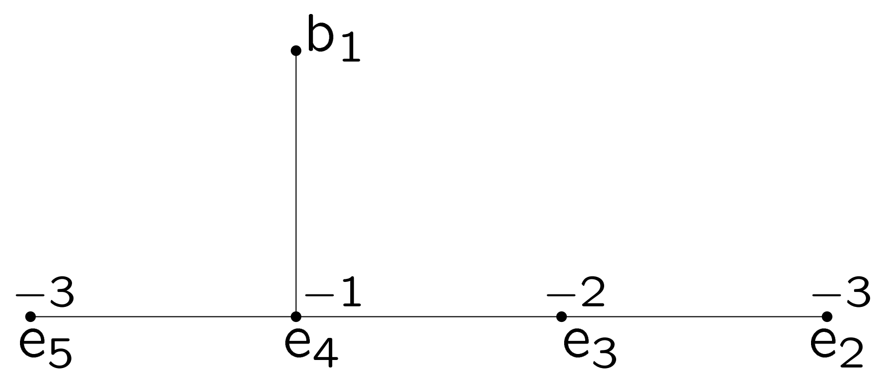
Graph $\Gamma(\widetilde{B})$ of curve germ (B,o) of singularity type

$$\mathbf{E}_7 = T[u(u^2 - v^3) = 0]$$



Graph $\Gamma(\widetilde{B})$ of curve germ (B, o) of singularity type

$$\mathbf{E}_8 = T[u^3 - v^5 = 0]$$



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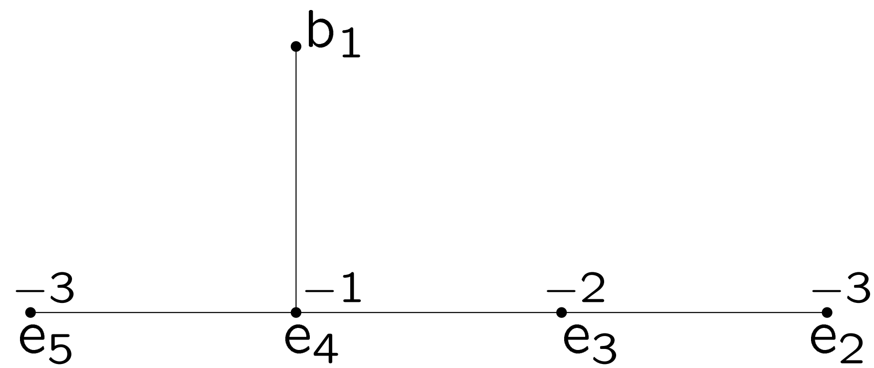


Remark. $T[(B,o)] \in ADE$, $T[(B,o)] \notin \{\mathbf{A_0}, \mathbf{A_1}\}$, $\Rightarrow$ $\Gamma(\widetilde{B})$ has the unique vertex e of valence 3 and its weight $w = -1$ (denote by E the curve corresponding to e).

Proposition. $(B, o) \subset (\mathbb{B}_\varepsilon, o)$ **is s.t.**

- **the valences of all vertices of** $\Gamma(\widetilde{B})$ **is less than** 4**,**
- e **is the unique vertex of valence** 3 **and weight of** e **is equal to** -1**.**

Then

- e **belongs to the center of** $\pi_1^{loc}(B, o)$**,**
- $\pi_1^{loc}(B, o)$ **is generated by elements** $\gamma_0, \gamma_1, \gamma_\infty$**, corresponding to the vertices of** $\Gamma(\widetilde{B})$ **connected by edges with** e**,**
- $\gamma_0\gamma_1\gamma_\infty = e$**.**

Definition of

$$\beta : \{F : U \to \mathbb{B}_\varepsilon\} \in \mathcal{R} \quad \to \quad \{f = \beta(F) : C \to \mathbb{P}^1\} \in \mathcal{B}el$$

.

$$\begin{array}{ccc} & & U \ni o' \\ & & \Big\downarrow {\scriptstyle F \in \mathcal{R}} \\ \mathbb{P}^1 \simeq E \subset \widetilde{V} & \xrightarrow[\sigma]{\hspace{6em}} & \mathbb{B}_\varepsilon \ni o \end{array}$$

$\sigma : \widetilde{V} \to \mathbb{B}_\varepsilon$ – the minimal resolution of singularity of $(B_F, o) \subset (\mathbb{B}_\varepsilon, o)$,

$\widetilde{B} = \sigma^{-1}(B_F)$.

.

$$
\begin{array}{ccc}
\widetilde{U} & \xrightarrow[\tau]{} & U \ni o' \\
\Big\downarrow \widetilde{F} & & \Big\downarrow F \\
\mathbb{P}^1 \simeq E \subset \widetilde{V} & \xrightarrow[\sigma]{} & \mathbb{B}_\varepsilon \ni o
\end{array}
$$

$\widetilde{U}$ − the normalization of $U \times_{\mathbb{B}_\varepsilon} \widetilde{V}$,

$\widetilde{B} = \sigma^{-1}(B_F)$, $\widetilde{F}$ **is branched in** $\widetilde{B}$**.**

.

$$
\begin{array}{ccccc}
\widetilde{C} \subset & \widetilde{U} & \xrightarrow{\quad\tau\quad} & U \ni o' \\
\Big\downarrow \widetilde{f} & \Big\downarrow \widetilde{F} & & \Big\downarrow F \\
\mathbb{P}^1 \simeq E \subset & \widetilde{V} & \xrightarrow{\quad\sigma\quad} & \mathbb{B}_\varepsilon \ni o
\end{array}
$$

U is normal and connected, $F^{-1}(o) = o' \;\Rightarrow\; \widetilde{C}$ is irreducible curve and ramification multiplicity of $\widetilde{F}$ in $\widetilde{C}$ is $m_e = |\langle F_*(e)\rangle|$.

.

$$
\begin{array}{ccccc}
\widetilde{C} \subset & \widetilde{U} & \xrightarrow[\tau]{\hspace{6em}} & U \ni o' \\
{\scriptstyle \widetilde{f}}\Big\downarrow & \Big\downarrow{\scriptstyle \widetilde{F}} & & \Big\downarrow{\scriptstyle F} \\
\mathbb{P}^1 \simeq E \subset & \widetilde{V} & \xrightarrow[\sigma]{\hspace{6em}} & \mathbb{B}_\varepsilon \ni o
\end{array}
$$

- $Z_e = \langle F_*(e) \rangle \subset Z(G_F) \subset G_F, \quad m_e = |Z_e|,$

.

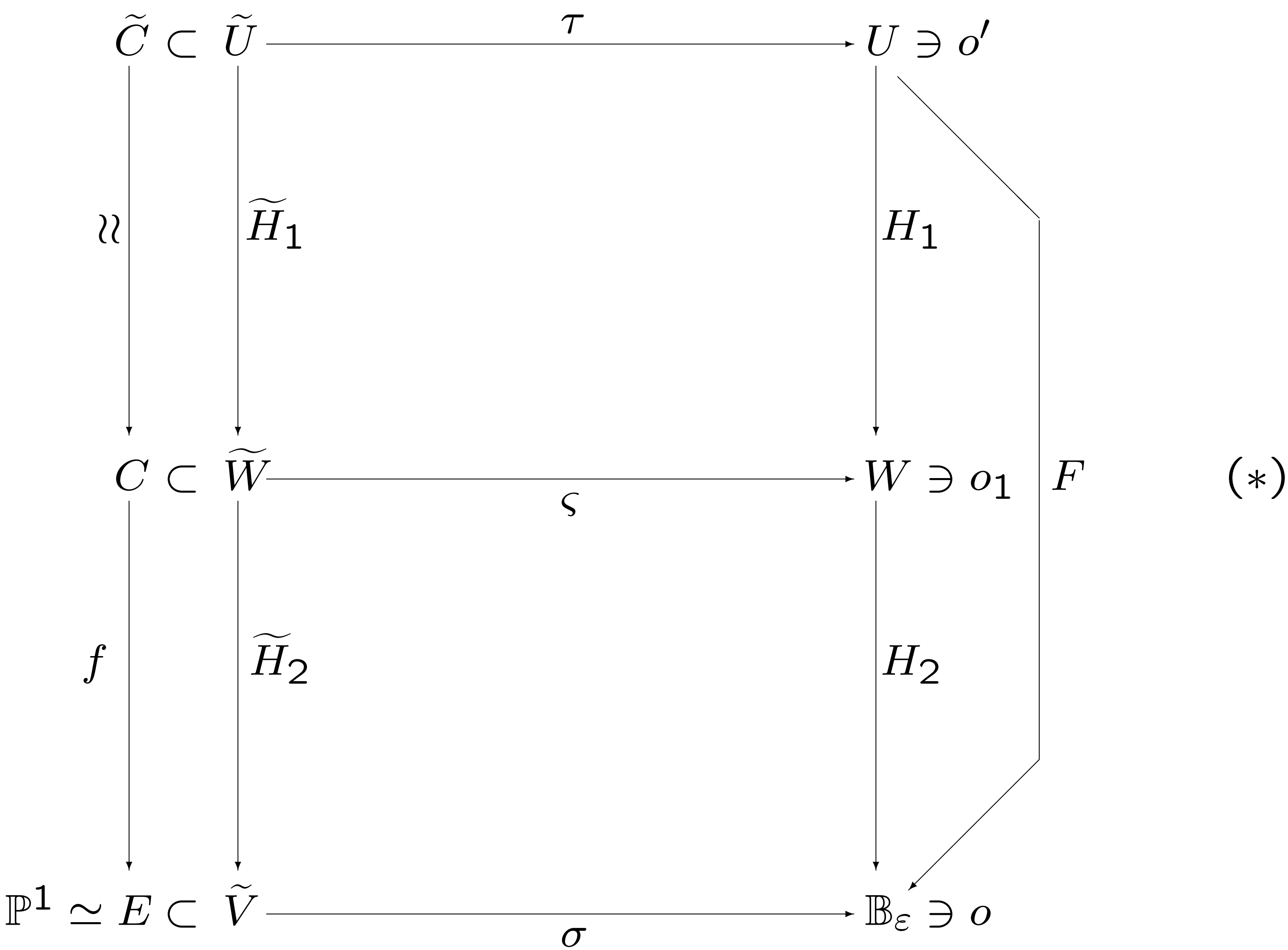
$$\begin{array}{ccc} \widetilde{C} \subset \widetilde{U} & \xrightarrow[\tau]{\qquad\qquad} & U \ni o' \\ {\scriptstyle \widetilde{f}}\downarrow \qquad \downarrow{\scriptstyle \widetilde{F}} & & \downarrow{\scriptstyle F} \\ \mathbb{P}^1 \simeq E \subset \widetilde{V} & \xrightarrow[\sigma]{\qquad\qquad} & \mathbb{B}_\varepsilon \ni o \end{array}$$

- $Z_e = \langle F_*(e) \rangle \subset Z(G_F) \subset G_F, \quad m_e = |Z_e|,$

Proposition. $F : U \to \mathbb{B}_\varepsilon$ **is a finite cover and** Z **is a subgroup of the center** $Z(G_F) \subset G_F$**. Then** Z **acts on** U **and** $\widetilde{U}$**, and** $F : U \xrightarrow{H_1} W = U/Z \xrightarrow{H_2} \mathbb{B}_\varepsilon$**,** $\widetilde{F} : \widetilde{U} \xrightarrow{\widetilde{H}_1} \widetilde{W} = \widetilde{U}/Z \xrightarrow{\widetilde{H}_2} \widetilde{V}$**.**

$$\begin{array}{ccccccc}
\widetilde{C} & \subset & \widetilde{U} & \xrightarrow{\quad\tau\quad} & U \ni o' & & \\
\wr\!\!\wr\;\big\downarrow & & \big\downarrow \widetilde{H}_1 & & \big\downarrow H_1 & & \\
C & \subset & \widetilde{W} & \xrightarrow[\quad\varsigma\quad]{} & W \ni o_1 & F & (*) \\
f\;\big\downarrow & & \big\downarrow \widetilde{H}_2 & & \big\downarrow H_2 & & \\
\mathbb{P}^1 \simeq E & \subset & \widetilde{V} & \xrightarrow[\quad\sigma\quad]{} & \mathbb{B}_\varepsilon \ni o & &
\end{array}$$

H_1 and $\widetilde{H}_1$ – quotient maps, $\deg H_1 = \deg \widetilde{H}_1 = m_e$, $E \not\subset B_{\widetilde{H}_2}$



$H_2 : (W, o_1) \to (\mathbb{B}_\varepsilon, o)$ **is a germ of finite cover,** (C, f) **is a Belyi pair.**

$$
\begin{array}{ccccc}
\widetilde{C} \subset \widetilde{U} & \xrightarrow{\quad\tau\quad} & U \ni o' & & \\
\wr\big\downarrow \qquad \big\downarrow \widetilde{H}_1 & & \big\downarrow H_1 & & \\
C \subset \widetilde{W} & \xrightarrow[\quad\varsigma\quad]{} & W \ni o_1 & F & (*) \\
f\big\downarrow \qquad \big\downarrow \widetilde{H}_2 & & \big\downarrow H_2 & & \\
\mathbb{P}^1 \simeq E \subset \widetilde{V} & \xrightarrow[\quad\sigma\quad]{} & \mathbb{B}_\varepsilon \ni o & &
\end{array}
$$

$$\beta : \{F : U \to \mathbb{B}_\varepsilon\} \in \mathcal{R} \quad \to \quad \{f = \beta(F) : C \to \mathbb{P}^1\} \in \mathcal{B}el$$

Properties of β

$$B_{\widetilde{F}} \subset \widetilde{B} = \sigma^{-1}(B_F), \quad \widetilde{H}_2 : \widetilde{U}_1 \to \widetilde{V}, \quad B_{\widetilde{H}_2} \subset \widetilde{B}' = \overline{\widetilde{B} \setminus E}.$$

$$f = \widetilde{H}_{2|C} : C \to E \simeq \mathbb{P}^1, \quad B_f \subset \widetilde{B}' = E \cap \widetilde{B}'.$$

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$$f = \widetilde{H}_{2|C} : C \to E \simeq \mathbb{P}^1, \;\; B_f \subset \widetilde{B}' = E \cap \widetilde{B}'.$$

- $F_* = \widetilde{F}_* : \pi_1(\widetilde{V} \setminus \widetilde{B}) \to G_F \subset \mathbb{S}_d, \; \langle F_*(e) \rangle \subset Z(G_F) \subset G_F \; \Rightarrow$

$$\widetilde{H}_{2*} : \pi_1(\widetilde{V} \setminus \widetilde{B}') \twoheadrightarrow G_F / \langle F_*(e) \rangle \subset \mathbb{S}_{\frac{d}{m_e}}.$$

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- $i : E \hookrightarrow \widetilde{V}$ induces

$$i_* : \pi_1(E \setminus \widetilde{B}')) \twoheadrightarrow \pi_1(\widetilde{V} \setminus \widetilde{B}') \simeq \pi_1(\widetilde{V} \setminus \widetilde{B}) / \langle e \rangle \;\; \Rightarrow \;\; f_* = \widetilde{H}_{2*} \circ i_*.$$

Properties of β

$$B_{\widetilde{F}} \subset \widetilde{B} = \sigma^{-1}(B_F), \quad \widetilde{H_2} : \widetilde{U}_1 \to \widetilde{V}, \quad B_{\widetilde{H}_2} \subset \widetilde{B}' = \overline{\widetilde{B} \setminus E}.$$

$$f = \widetilde{H}_{2|C} : C \to E \simeq \mathbb{P}^1, \quad B_f \subset \widetilde{B}' = E \cap \widetilde{B}'.$$

- $F_* = \widetilde{F}_* : \pi_1(\widetilde{V} \setminus \widetilde{B}) \to G_F \subset \mathbb{S}_d$, $\langle F_*(e) \rangle \subset Z(G_F) \subset G_F \ \Rightarrow$

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$$i_* : \pi_1(E \setminus \widetilde{B}')) \twoheadrightarrow \pi_1(\widetilde{V} \setminus \widetilde{B}') \simeq \pi_1(\widetilde{V} \setminus \widetilde{B}) / \langle e \rangle \quad \Rightarrow \quad f_* = \widetilde{H}_{2*} \circ i_*.$$

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Properties of β

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- $\beta(\mathcal{R}) = \mathcal{B}el$.

- Below we **assume** that $\{F : (U, o') \to \mathbb{B}_\varepsilon\} \in \mathcal{R}_T^{sm}$.

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Theorem. $\{F : (U, o') \to (\mathbb{B}_\varepsilon, o)\} \in \mathcal{R}^{sm} \ \Rightarrow \ \exists$ **coordinates** z, w **in** U **and** u, v **in** $\mathbb{B}_\varepsilon$ **s.t.** F **is given by**

$$u = \frac{f_1(z, w)}{g_1(z, w)}, \qquad v = \frac{f_2(z, w)}{g_2(z, w)},$$

where $f_i(z, w)$ **and** $g_i(z, w) \in \overline{\mathbb{Q}}[z, w]$ **for** $i = 1, 2$.

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The following Theorem states that $\beta(\mathcal{R}_{\mathbf{D_4}}^{sm}) = \mathcal{B}el^0$.

Theorem. *Let* $(\mathbb{P}^1, f) \in \mathcal{B}el^0$, $\deg f = n > 1$ *and* $B_f \subset \{0, 1, \infty\}$,

be given by coprime homogeneous forms $h(x_1, x_2)$, $h_2(x_2, x_2)$,

$$f : \ (x_1 : x_2) \mapsto (h(x_1, x_2) : h_2(x_2, x_2)),$$

and let $p_1 = (0, 1)$ *and* $p_2 = (1, 0) \in \mathbb{P}^1$ *be s.t.*

$\{f(p_1), f(p_2)\} \cup B_f = \{0, 1, \infty\}$ *and* $f(p_1) = 1$ *if* $f \in \mathcal{B}el^0_2$.

Then for $m_1, m_2 \in \mathbb{N}$ *s.t.* $G.C.D.(m_1, m_2) = 1$

a cover $F : (U, o') \to (V, o)$ *given by functions*

$$u = h_1(z^{m_1}, w^{m_2}), \ v = h_2(z^{m_1}, w^{m_2}) \qquad (*)$$

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Conversely, $\forall\ F \in \mathcal{R}^{sm}_{\mathbf{D}_4}$ *can be given by functions of the form* (*) *and* $\beta(F) = f : (x_1 : x_2) \mapsto (h(x_1, x_2) : h_2(x_2, x_2))$.

- $(\mathbb{P}^1, f) \in \mathcal{B}el^0$. For each $T \in ADE$ the set $\beta^{-1}(f) \cap \mathcal{R}_T^{sm}$ can be described in terms of brief passport $\Pi'(f)$.

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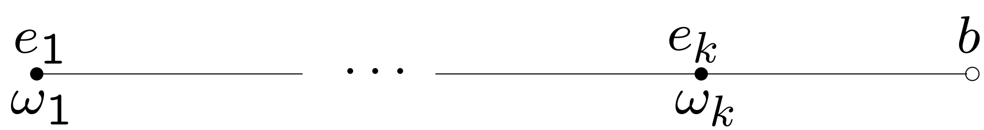
- **If** $T[B] \in ADE \setminus \{A_0, A_1\} \;\Rightarrow\; \widetilde{B}' = \bigsqcup_{j=1}^{3} \widetilde{B}'_j$**,** $\Gamma(\widetilde{B}')$ **consists of three connected components and each component is a chain.**



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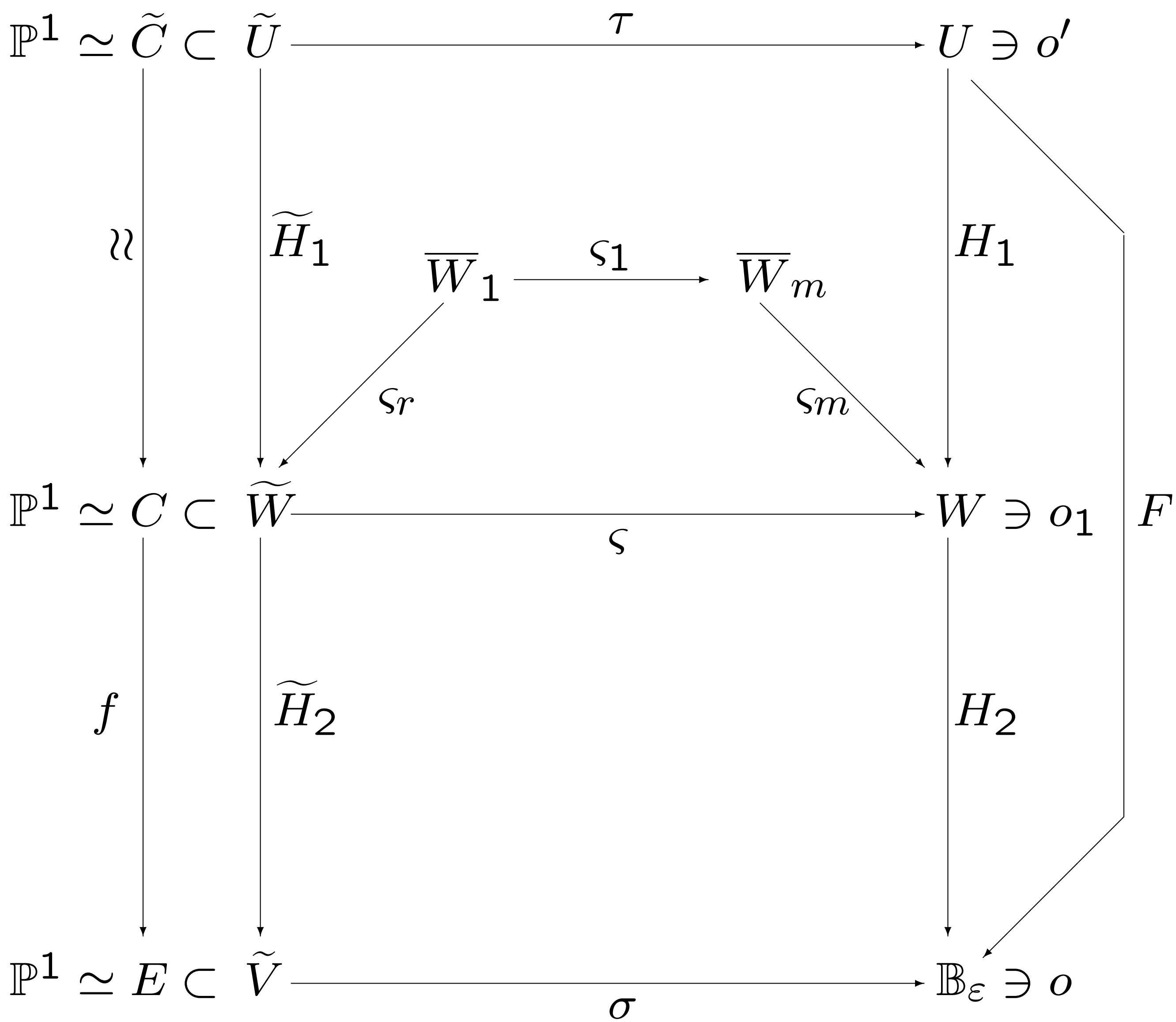
Lemma. **$\pi_1(X \setminus \widetilde{B}'_j)$ is a cyclic group generated by e_1, where X is a tubular neighborhood of $\widetilde{B}'_j$.**

$$\begin{array}{ccccc}
\widetilde{C} \subset \widetilde{U} & \xrightarrow{\tau} & U \ni o' & & \\
\wr\downarrow \qquad \downarrow \widetilde{H}_1 & & \downarrow H_1 & & \\
C \subset \widetilde{W} & \xrightarrow[\varsigma]{} & W \ni o_1 & F & (*) \\
f\downarrow \qquad \downarrow \widetilde{H}_2 & & \downarrow H_2 & & \\
\mathbb{P}^1 \simeq E \subset \widetilde{V} & \xrightarrow[\sigma]{} & \mathbb{B}_\varepsilon \ni o & &
\end{array}$$

U **is smooth** $\Rightarrow$ $\tilde{C} \simeq C \simeq \mathbb{P}^1$, **i.e.** $(C, f) \in \mathcal{B}el^0$



$(*)$

U is smooth, $T[B_F] \in ADE$, ς_r and ς_m resolutions of singular points

- The cyclic group $Z_e = \langle g = F_*(e) \rangle$ acts on (U, o') by the rule

$$g : (z, w) \mapsto (exp(2\pi i/m_e)z, exp(2\pi qi/m_e)w),$$

where $m_e = m_0 m_1 m_2 = |Z_e|$, $\mathrm{GCD}(m_1, m_2) = \mathrm{GCD}(m_0, q) = 1$.

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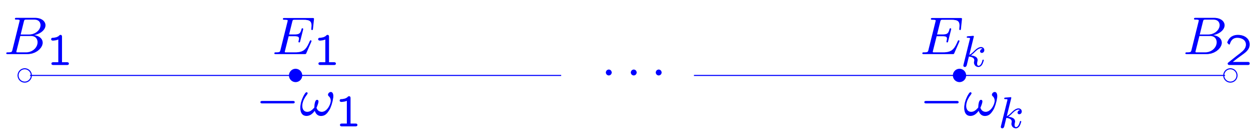
where $m_e = m_0 m_1 m_2 = |Z_e|$, $\mathrm{GCD}(m_1, m_2) = \mathrm{GCD}(m_0, q) = 1$.

- $(W, o_1) = (U, o')/Z_e$ **has** (so called) **cyclic quotient singularity,**

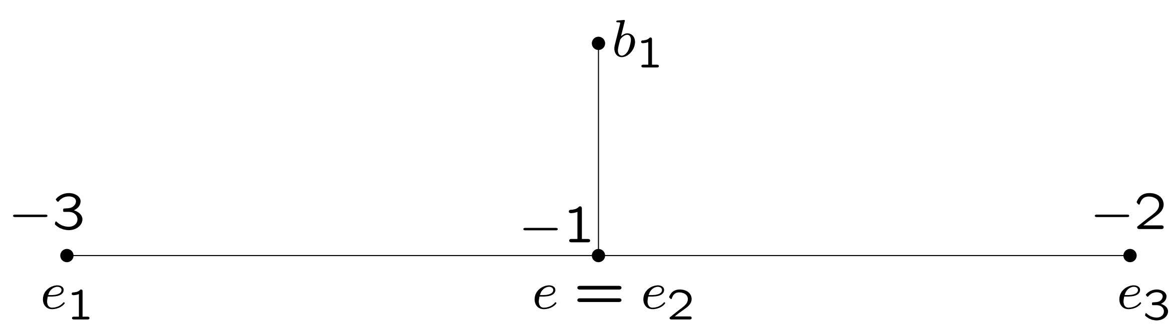
$\varsigma_m : \overline{W}_m \to W$ **– the resolution of the singular point** $o_1 \Rightarrow$

$$\varsigma_m^{-1}(o_1) = \bigcup_{i=1}^{k} E_i, \qquad E_i \simeq \mathbb{P}^1, \qquad (E_i^2)_{\overline{W}_m} = -\omega_i.$$



$$\frac{m_0}{q} = \omega_1 - \cfrac{1}{\omega_2 - \cfrac{1}{\omega_3 - \cfrac{1}{\dots - \cfrac{1}{\omega_k}}}}.$$

Example. $(\mathbb{P}^1, f) \in \mathcal{B}el^0, \;\; F \in \beta^{-1}(f) \cap \mathcal{R}^{sm}_{\mathbf{A_2}}.$



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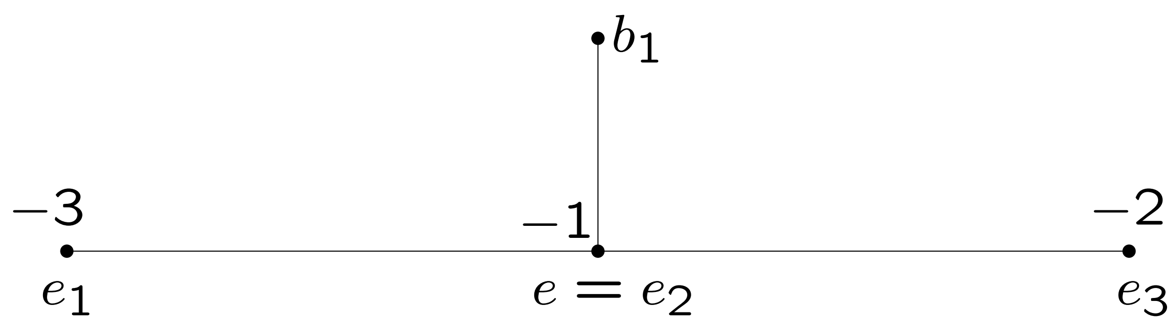

- In diagram $(*)$ $\widetilde{H}_2 : \widetilde{W} \to \widetilde{V}$ is ramified only over B_1, E_1, and E_3 and it is not ramified in $C = \widetilde{H}^{-1}(E)$ $\Rightarrow$ $\widetilde{H}_2^{-1}(E_1)$ **and** $\widetilde{H}_2^{-1}(E_3)$ **are disjoint unions of curves** $\simeq \mathbb{P}^1$**.**

Lemma. X **is a tubular neighborhood of** $E_j \simeq \mathbb{P}^1$**,** $(E_j^2)_X = -n \quad \Rightarrow$ $\pi_1(X \setminus E_j) = \mathbb{Z}_n$**.**

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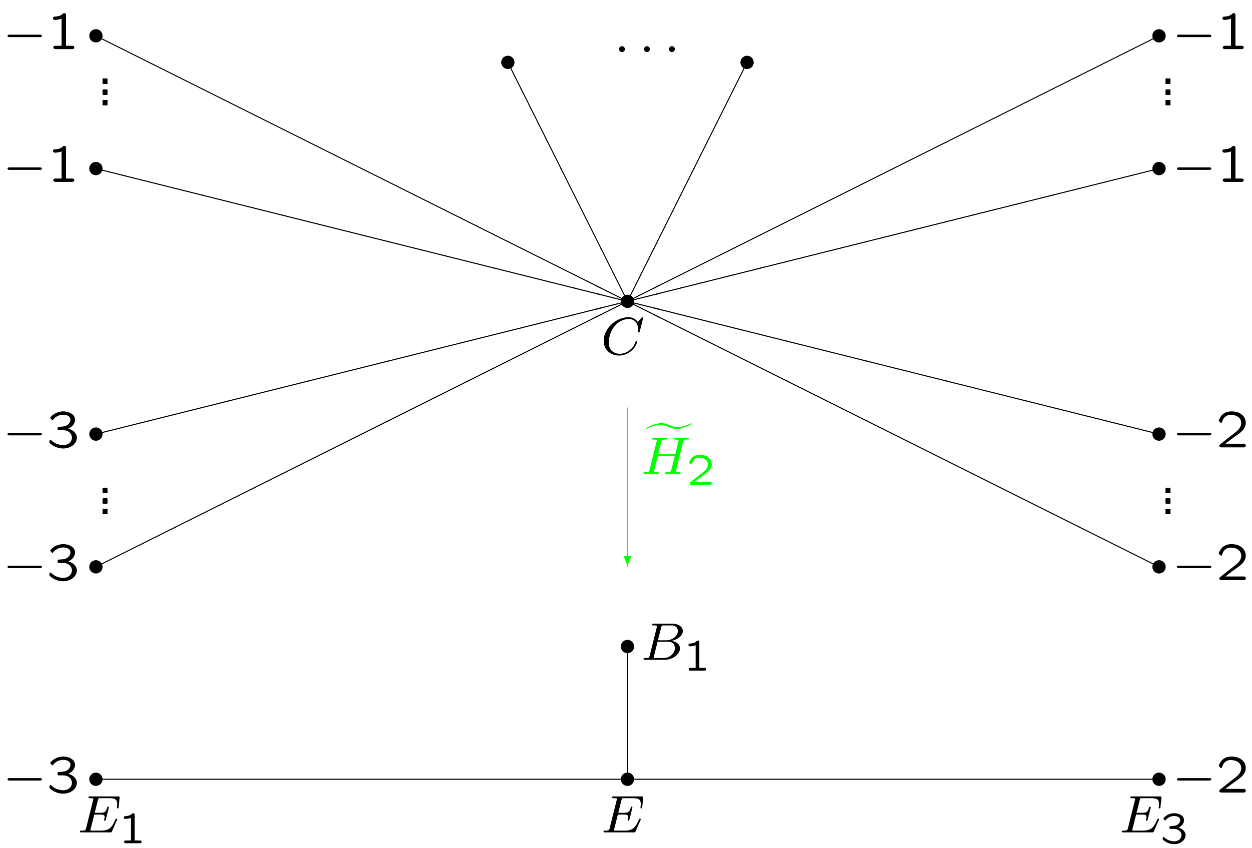

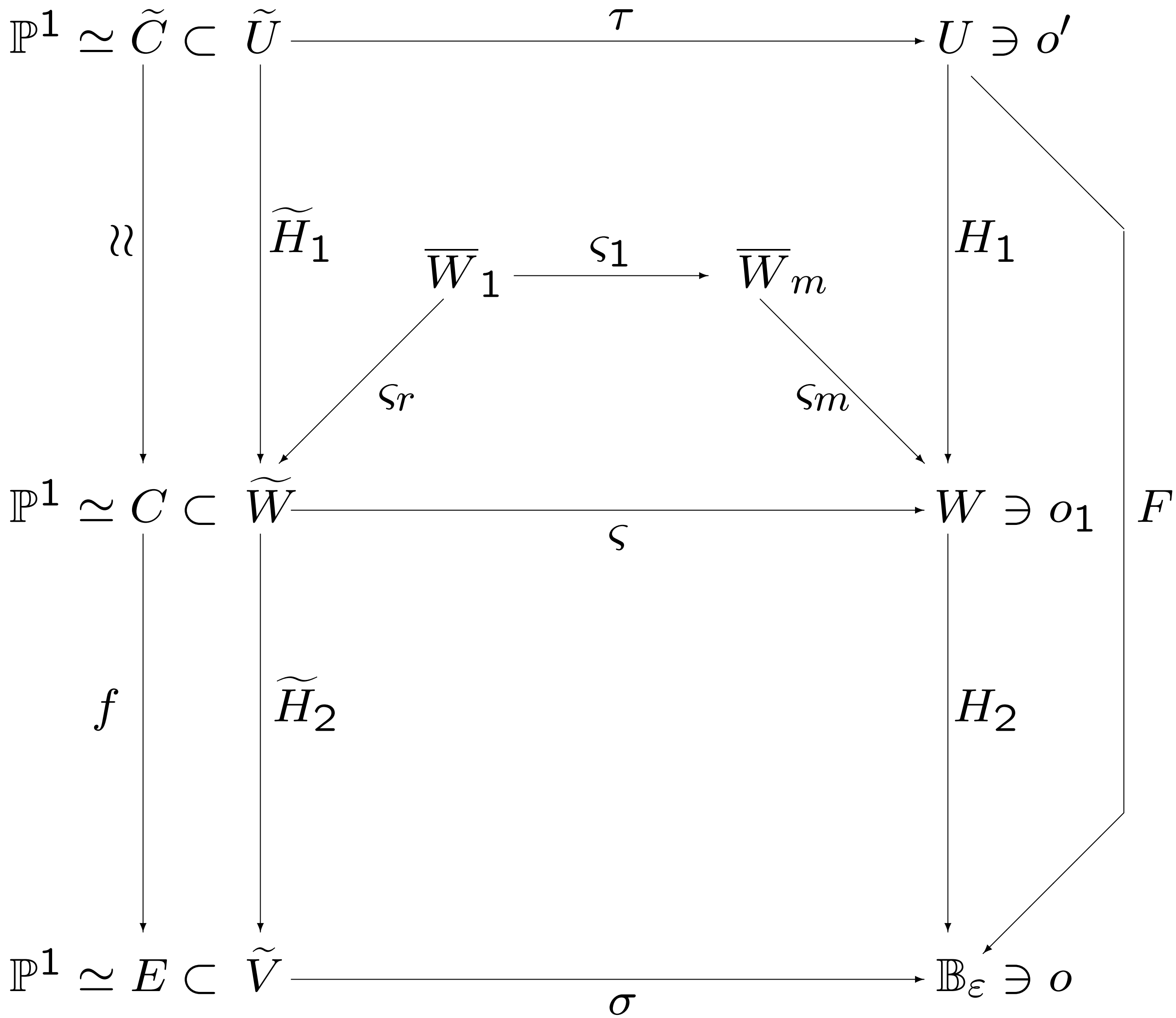
$\mathbb{P}^1 \simeq \widetilde{C} \subset \widetilde{U}$
τ
$U \ni o'$
$\wr$
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ς_1
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ς
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F
f
$\widetilde{H}_2$
H_2
$\mathbb{P}^1 \simeq E \subset \widetilde{V}$
σ
$\mathbb{B}_\varepsilon \ni o$

$(*)$

$$(\mathbb{P}^1, f) \in \mathcal{B}el^0,\ \ \deg f = n,\ \ F \in \beta^{-1}(f) \cap \mathcal{R}^{sm}_{\mathbf{A_2}},\ \deg F = d \Rightarrow$$

$$n = 6k,\ \Pi'(f) = ((\underbrace{2,\dots,2}_{3k}),(\underbrace{3,\dots,3}_{2k})) \Rightarrow d = 6k^2 m_1 m_2,\ \ (m_1, m_2) = 1;$$

$$n = 6k,\ \Pi'(f) = ((1,1,\underbrace{2,\dots,2}_{3k-1}),(\underbrace{3,\dots,3}_{2k})) \Rightarrow d = 12k^2;$$

$$n = 6k+1,\ \Pi'(f) = ((1,\underbrace{2,\dots,2}_{3k}),(1,\underbrace{3,\dots,3}_{2k})) \Rightarrow d = (6k+1)^2;$$

$$n = 6k+2,\ \Pi'(f) = ((\underbrace{2,\dots,2}_{3k+1}),(1,1,\underbrace{3,\dots,3}_{2k})) \Rightarrow d = 6(3k+1)^2;$$

$$n = 6k+3,\ \Pi'(f) = ((1,\underbrace{2,\dots,2}_{3k+1}),(\underbrace{3,\dots,3}_{2k+1})) \Rightarrow d = 3(2k+1)^2(2m+1);$$

$$n = 6k+4,\ \Pi'(f) = ((\underbrace{2,\dots,2}_{3k+2}),(1,\underbrace{3,\dots,3}_{2k+1})) \Rightarrow d = 2(3k+2)^2 m.$$

Description of $\mathcal{R}_T^{sm} \cap \beta^{-1}(\mathcal{B}el_2^0)$.

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$F \in \mathcal{R}_{\mathbf{E}_7}^{sm} \cup (\bigcup_{k=1}^{\infty} \mathcal{R}_{\mathbf{A}_{2k+1}}^{sm}) \cup (\bigcup_{k=1}^{\infty} \mathcal{R}_{\mathbf{D}_k}^{sm})$ **and**

$\beta(F) = f \in \mathcal{B}el_2^0$, $\deg f = n$ $\Rightarrow$

F **is equivalent to one of the following covers**

(in all cases $m_1, m_2 \geq 1$, $\mathrm{GCD}(m_1, m_2) = 1$**):**

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$F \in \mathcal{R}^{sm}_{\mathbf{E}_7}: \quad u = z^{3m_1},\ v = z^{2m_1} + w^{m_2}, \quad m_2 > 1;$

$$F \in \mathcal{R}^{sm}_{\mathbf{A_0}}: \quad u = z^{m_1},\ v = w, \qquad \beta(F) = id;$$

$$F \in \mathcal{R}^{sm}_{\mathbf{A_1}}: \quad u = z^{nm_1},\ v = w^{nm_2}, \qquad n \geq 1;$$

$$F \in \mathcal{R}^{sm}_{\mathbf{A_{2k+1}}}: \quad u = (z^m + w^{m_0})^n,\ v = w, \qquad n, m, m_0 > 1,\ k+1 = nm_0;$$

$$F \in \mathcal{R}^{sm}_{\mathbf{A_{2k+1}}}: \quad u = z^{(k+1)m_1},\ v = z^{m_1} + w^{m_2},\ m_2 > 1,\ n = k+1;$$

$$F \in \mathcal{R}^{sm}_{\mathbf{A_{2k+1}}}: \quad u = (\omega_j z^{m_1} - w^{m_2})^{k+1},\ v = z^{m_1} - w^{m_2},$$

$$\omega_j = exp(2\pi ji/k+1),\ 1 \leq j \leq k = n-1;$$

$$F \in \mathcal{R}^{sm}_{\mathbf{D_4}}: \quad u = z^{m_1 n},\ v = (z^{m_1} + w^{m_2})^n, \qquad n \geq 2;$$

$$F \in \mathcal{R}^{sm}_{\mathbf{D_4}}: \quad u = (z^{m_1} - w^{m_2})^n,\ v = (z^{m_1} - \omega_j w^{m_2})^n,$$

$$n \geq 2,\ \omega_j = exp(2\pi ji/n),\ 1 \leq j \leq n-1;$$

$$F \in \mathcal{R}^{sm}_{\mathbf{D_{2k+2}}}: \quad u = (z^{m_1} - w^{m_2})^{n_1}, \quad v = z^{m_1 n},$$

$$n = n_1 k \geq 1:$$

$$F \in \mathcal{R}^{sm}_{\mathbf{D_{2k+3}}}: \quad u = z^{2m_1},\ v = z^{m_1(2k+1)} + w^{m_2},$$

$$m_2 > 1,\ n = 2,\ \mathrm{GCD}(2k+1, m_2) = 1;$$

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