

# On compacts possessing strictly plurisubharmonic functions

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*Dedicated to the memory of my teacher Anatoli Georgievich Vitushkin (1931 - 2004).*

**Abstract.** We give a geometric condition on a compact subset of a complex manifold which is necessary and sufficient for the existence of a smooth strictly plurisubharmonic function defined in a neighbourhood of this set.

## 1. Introduction

Plurisubharmonic functions play a central role in complex analysis. Many important and classical results are formulated in terms of these functions, in particular, using the existence of strictly plurisubharmonic functions on a given manifold. For example, Grauert [G] characterized Stein manifolds by existence of smooth strictly plurisubharmonic exhaustion functions. This result was generalized to the case of complex spaces by Narasimhan in [N1] and [N2]. Sibony in [Si1, Theorem 3, p. 362] proved that the existence of a bounded smooth strictly plurisubharmonic function is sufficient for Kobayashi hyperbolicity of a complex manifold. A similar criterion for the existence of Bergman metric on Stein manifolds was established by Chen-Zhang in [CZ, Theorem 1, p. 2998, and observation 2, p. 3002]. Recently Poletsky [P] used manifolds possessing bounded smooth strictly plurisubharmonic functions to develop further the theory of pluricomplex Green functions.

In the present paper we study a question of the existence of smooth strictly plurisubharmonic functions on a given compact set. Smoothness and strict plurisubharmonicity of such functions can be defined as follows.

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**Definition 1.1.** Let  $\mathcal{K}$  be a compact subset of a complex manifold  $\mathcal{M}$ . We say that a function  $\phi$  defined on  $\mathcal{K}$  is smooth and strictly plurisubharmonic if there is a neighbourhood  $\mathfrak{A}$  of  $\mathcal{K}$  in  $\mathcal{M}$  and a smooth strictly plurisubharmonic function  $\varphi$  on  $\mathfrak{A}$  such that  $\varphi|_{\mathcal{K}} = \phi$ .

The next result gives a complete geometric characterization of compacts possessing such functions.

**Main Theorem.** Let  $\mathcal{K}$  be a compact subset of a complex manifold. Then  $\mathcal{K}$  possesses a smooth strictly plurisubharmonic function if and only if  $\mathcal{K}$  does not have 1-pseudoconcave subsets.

1-pseudoconcavity here is understood in the sense of Rothstein [Rot]. By a complex manifold we will always mean a manifold of pure complex dimension which has a Hausdorff topology with a countable basis.

The paper is organized as follows. In Section 2 we recall the basic definitions and the main properties of pseudoconcave sets as well as the construction of a special plurisubharmonic function given in [HST3, Theorem 3.1, part 1]. In Section 3 we provide a constructive way to define the maximal 1-pseudoconcave subset of a given compact set. In Section 4 we prove the Main Theorem and one of its corollaries. Finally, in Section 5 we give some applications of our results and discuss their relation to the other topics.

**Remark.** When our paper was already posted on arXiv and submitted to a journal, Nessim Sibony informed us that an alternative proof of the ‘if’ part of our Main Theorem can be derived from [Si2] and [FS]. Namely, if there are no strictly psh functions on a compact set, then the duality argument in the proof of [Si2, Proposition 2.1] can be extended to produce a non-trivial  $dd^c$ -closed positive current of bidimension  $(1, 1)$  supported in that compact set. (A similar duality argument was earlier used in [HL, Theorem (38)].) The support of such a current is a 1-pseudoconvex set by [FS, Corollary 2.6]. We are grateful to Nessim Sibony for this valuable observation but believe that our direct geometric proof is of independent interest.

## 2. Preliminaries

We recall first the notion of 1-pseudoconvexity in the sense of Rothstein. Let  $\Delta^n := \{z \in \mathbb{C}^n : \|z\|_\infty < 1\}$ , where  $\|z\|_\infty = \max_{1 \leq j \leq n} |z_j|$ . An  $(1, n-1)$  Hartogs figure  $H$  is a set of the form

$$H = \{(z_1, \dots, z_n) \in \Delta^1 \times \Delta^{n-1} : |z_1| < r_1 \text{ or } \|(z_2, \dots, z_n)\|_\infty > r_2\},$$

where  $0 < r_1, r_2 < 1$ , and we write  $\hat{H} := \Delta^n$ .

**Definition 2.1.** Let  $\mathcal{M}$  be a complex manifold of dimension  $n$ . An open set  $\Omega \subset \mathcal{M}$  is called 1-pseudoconvex in  $\mathcal{M}$  if it satisfies the *Kontinuitätssatz* with respect to  $(n-1)$ -polydiscs in  $\mathcal{M}$ , i.e., if for every  $(1, n-1)$  Hartogs figure and every injective holomorphic mapping  $\Phi: \hat{H} \rightarrow \mathcal{M}$  such that  $\Phi(H) \subset \Omega$  one has  $\Phi(\hat{H}) \subset \Omega$ .

This definition was introduced by Rothstein [Rot] in a more general setting of  $q$ -pseudoconvex sets for every  $q = 1, 2, \dots, n-1$ . We restrict our definition to the special case  $q = 1$ , since in the present paper we only need the notion of 1-pseudoconvexity.

Another way to define 1-pseudoconvexity can be described as follows. For an arbitrary  $r \in (0, 1)$  we consider a *spherical hat*

$$\mathbb{S}_r^n := \{z = (z_1, \dots, z_n) \in \mathbb{C}^n : \|z\|^2 = 1, x_1 := \operatorname{Re} z_1 \geq r\}$$

and a *filled spherical hat*

$$\hat{\mathbb{S}}_r^n := \{z = (z_1, \dots, z_n) \in \mathbb{C}^n : \|z\|^2 \leq 1, x_1 := \operatorname{Re} z_1 \geq r\}.$$

**Definition 2.2.** Let  $\mathcal{M}$  be a complex manifold of dimension  $n$ . An open set  $\Omega \subset \mathcal{M}$  is called 1-pseudoconvex in  $\mathcal{M}$  if for every  $r \in (0, 1)$ , every neighbourhood  $U := U(\hat{\mathbb{S}}_r^n) \subset \mathbb{C}^n$  of the filled spherical hat  $\hat{\mathbb{S}}_r^n$  and every injective holomorphic mapping  $\Phi: U \rightarrow \mathcal{M}$  such that  $\Phi(\mathbb{S}_r^n) \subset \Omega$  one has  $\Phi(\hat{\mathbb{S}}_r^n) \subset \Omega$ .

The next statement shows that the above definitions give us the same notion.

**Proposition 2.1.** Let  $\mathcal{M}$  be a complex manifold and  $\Omega \subset \mathcal{M}$  be an open set. Then the following assertions are equivalent:

- (1)  $\Omega$  is 1-pseudoconvex in the sense of Definition 2.1.
- (2)  $\Omega$  is 1-pseudoconvex in the sense of Definition 2.2.

**Proof.** To prove the implication (1)  $\Rightarrow$  (2) we argue by contradiction and assume that there is a domain  $\Omega$  which is 1-pseudoconvex in the sense of Definition 2.1, but not 1-pseudoconvex in the sense of Definition 2.2. Then, in view of Definition 2.2, and after the substitution of  $r$  by  $r - \varepsilon$  with  $\varepsilon > 0$  small enough if necessary, we can assume that  $\Phi(\mathbb{S}_r^n) \subset \Omega$ , but  $\Phi(\operatorname{Int}(\hat{\mathbb{S}}_r^n)) \cap (\mathcal{M} \setminus \Omega) \neq \emptyset$ , where by  $\operatorname{Int}(\hat{\mathbb{S}}_r^n)$  we will denote the interior of the set  $\hat{\mathbb{S}}_r^n$ . Consider now for each  $C \geq 0$  a slightly more general spherical hat

$$\mathbb{S}_{r,C}^n := \{z = (z_1, \dots, z_n) \in \mathbb{C}^n : |z_1 + C|^2 + |z_2|^2 + \dots + |z_n|^2 = C^2 + 2rC + 1, x_1 \geq r\}$$

and a corresponding filled spherical hat

$$\hat{\mathbb{S}}_{r,C}^n := \{z = (z_1, \dots, z_n) \in \mathbb{C}^n : |z_1 + C|^2 + |z_2|^2 + \dots + |z_n|^2 \leq C^2 + 2rC + 1, x_1 \geq r\}$$

and observe that the spherical hats  $\mathbb{S}_{r,C}^n$  depend continuously on the parameter  $C$ , they all contain the "boundary"

$$\partial \mathbb{S}_r^n := \{z = (z_1, \dots, z_n) \in \mathbb{C}^n : \|z\|^2 = 1, x_1 = r\}$$

of the spherical hat  $\mathbb{S}_r^n$  and, moreover, that  $\bigcap_{C>0} \text{Int}(\hat{\mathbb{S}}_{r,C}^n) = \emptyset$ . Hence, there is

$$C_0 := \min\{C : \Phi(\mathbb{S}_{r,C}^n) \cap (\mathcal{M} \setminus \Omega) \neq \emptyset\} > 0.$$

If  $z_0 \in \mathbb{S}_{r,C_0}^n$  is a point such that  $\Phi(z_0) \in \mathcal{M} \setminus \Omega$ , then we can consider an affine transformation  $\mathbb{L}$  of  $\mathbb{C}^n$  which sends the sphere  $\mathbb{S}^n := \{z = (z_1, \dots, z_n) \in \mathbb{C}^n : \|z\|^2 = 1\}$  to the sphere  $\mathbb{S}_{C_0}^n := \{z = (z_1, \dots, z_n) \in \mathbb{C}^n : |z_1 + C_0|^2 + |z_2|^2 + \dots + |z_n|^2 = C_0^2 + 2rC_0 + 1\}$ , the origin  $O$  to the point  $(-C_0, 0, \dots, 0)$  and the point  $(1, 0, \dots, 0)$  to the point  $z_0$ . If now for  $r' < 1$  close enough to 1 we consider the spherical hat  $\mathbb{S}_{r'}^n$ , the neighbourhood  $\mathbb{L}^{-1}(U(\hat{\mathbb{S}}_{r'}^n))$  of the filled spherical hat  $\hat{\mathbb{S}}_{r'}^n$  and the injective holomorphic mapping  $\Phi \circ \mathbb{L} : \mathbb{L}^{-1}(U(\hat{\mathbb{S}}_{r'}^n)) \rightarrow \mathcal{M}$ , then, by construction, we will get that  $(\Phi \circ \mathbb{L})(1, 0, \dots, 0) \in \mathcal{M} \setminus \Omega$  and for  $\delta > 0$  small enough we will also get that  $(\Phi \circ \mathbb{L})(U_{2\delta}(\mathbb{S}_{r'}^n) \setminus \overline{\mathbb{B}}_1^n(0)) \subset \Omega$ , where  $U_{2\delta}(\mathbb{S}_{r'}^n)$  is the  $2\delta$ -neighbourhood of the set  $\mathbb{S}_{r'}^n$  in  $\mathbb{C}^n$  and  $\mathbb{B}_1^n(0)$  is the ball in  $\mathbb{C}^n$  of radius 1 with center at the origin. Hence, if for  $\delta' > \delta$  close enough to  $\delta$ ,  $r_1 \in (0, \delta')$  close enough to 0 and  $r_2 \in (0, \delta)$  close enough to  $\delta$  we consider the Hartogs figure

$$H_{\delta', \delta, r_1, r_2} = \{(z_1, \dots, z_n) \in \Delta_{\delta'}^1(1+\delta) \times \Delta_{\delta}^{n-1}(0) : |z_1 - (1+\delta)| < r_1 \text{ or } \|(z_2, \dots, z_n)\|_{\infty} > r_2\}$$

and write

$$\hat{H}_{\delta', \delta, r_1, r_2} := \Delta_{\delta'}^1(1+\delta) \times \Delta_{\delta}^{n-1}(0),$$

where  $\Delta_s(a) := \{z \in \mathbb{C} : |z - a| < s\}$ , then we will get that  $H_{\delta', \delta, r_1, r_2} \subset \mathbb{C}^n \setminus \overline{\mathbb{B}}_1^n(0)$ , but  $(1, 0, \dots, 0) \in \hat{H}_{\delta', \delta, r_1, r_2} \cap \overline{\mathbb{B}}_1^n(0)$ . If we now define an affine change of coordinates  $\mathbb{L}'$  in  $\mathbb{C}^n$  by

$$z_1 \rightarrow \delta' z_1 + (1+\delta) =: z'_1, \quad z_j \rightarrow \delta z_j =: z'_j \text{ for } j = 2, 3, \dots, n,$$

then the map  $\Phi \circ \mathbb{L} \circ \mathbb{L}' : \hat{H} \rightarrow \mathcal{M}$  will give us the desired contradiction to the assumption on 1-pseudoconvexity of the domain  $\Omega$  in the sense of Definition 2.1.

To prove the implication (2)  $\Rightarrow$  (1) we will follow the argument used in the proof of Theorem 3.2 in [HST3] and assume, to get a contradiction, that there is a domain  $\Omega$  which is 1-pseudoconvex in the sense of Definition 2.2, but not 1-pseudoconvex in the sense of Definition 2.1. Then for some  $0 < r_1, r_2 < 1$  there exists a  $(1, n-1)$  Hartogs figure  $H = \{(z_1, \dots, z_n) \in \Delta \times \Delta^{n-1} : |z_1| < r_1 \text{ or } \|(z_2, \dots, z_n)\|_{\infty} > r_2\}$  and an injective holomorphic mapping  $\Phi : \hat{H} \rightarrow \mathcal{M}$  such that  $\Phi(H) \subset \Omega$  but  $\Phi(\hat{H}) \cap (\mathcal{M} \setminus \Omega) \neq \emptyset$ . For small  $\varepsilon > 0$ , let  $\varphi : \mathbb{C}_{z_1}^* \times \mathbb{C}_{(z_2, \dots, z_n)}^{n-1} \rightarrow \mathbb{R}$  be the smooth strictly plurisubharmonic function defined by  $\varphi(z) := -\log|z_1| + \varepsilon\|z\|^2$ , and for each  $C \in \mathbb{R}$  let  $G_C$  denote the domain  $G_C := \{\zeta \in \Phi(\hat{H}) : (\varphi \circ \Phi^{-1})(\zeta) < C\}$ . Since for  $C$  large enough the set  $\hat{H} \cap \{z \in \mathbb{C}^n : \varphi < C\}$  contains  $\hat{H} \setminus H$ , and since  $\Phi(\hat{H}) \cap (\mathcal{M} \setminus \Omega) \subset \Phi(\hat{H} \setminus H)$ , we know that for  $C$  large enough  $\Phi(\hat{H}) \cap (\mathcal{M} \setminus \Omega) \subset G_C$ . Let  $C_0 := \inf\{C \in \mathbb{R} : \Phi(\hat{H}) \cap (\mathcal{M} \setminus \Omega) \subset G_C\}$ .

Then the set  $\mathcal{M} \setminus \Omega$  "touches" the strictly pseudoconvex part  $M := bG_{C_0} \cap \Phi(\hat{H})$  of the boundary  $bG_{C_0}$  of  $G_{C_0}$  "from inside". That is  $\Phi(\hat{H}) \cap (\mathcal{M} \setminus \Omega) \subset \bar{G}_{C_0}$ , there is a point  $\zeta_0 \in \Phi(\hat{H}) \cap (\mathcal{M} \setminus \Omega) \cap bG_{C_0}$  and the domain  $G_{C_0}$  is strictly pseudoconvex near the point  $\zeta_0$ . It follows from strict pseudoconvexity of the domain  $G_{C_0}$  near  $\zeta_0$  that there is a bounded strictly convex domain  $G_0 \subset \mathbb{C}^n$ , a point  $z_0 \in bG_0$ , a neighbourhood  $U_0$  of  $z_0$  and an injective holomorphic mapping  $\Psi : U_0 \rightarrow \mathcal{M}$  such that  $\Psi(U_0 \cap G_0) = \Psi(U_0) \cap G_{C_0}$  and  $\Psi(z_0) = \zeta_0$ . Strict convexity of the domain  $G_0 \subset \mathbb{C}^n$  implies that we can find a ball  $\mathbb{B}_R^n(\tilde{C})$  centered at a point  $\tilde{C} \in \mathbb{C}^n$  with radius  $R > 0$  which contains the domain  $G_0$  and such that  $b\mathbb{B}_R^n(\tilde{C}) \cap \bar{G}_0 = z_0$ . If we take an affine change of coordinates  $\mathbb{L}$  in  $\mathbb{C}^n$  which sends the ball  $\mathbb{B}_1^n(0)$  to the ball  $\mathbb{B}_R^n(\tilde{C})$ , the origin  $O$  to the point  $\tilde{C}$  and the point  $(1, 0, \dots, 0)$  to the point  $z_0$  and then consider a translation  $\mathbb{E}_\delta := \mathbb{E} + \delta(1, 0, \dots, 0)$ , where  $\mathbb{E}$  is the identity map of  $\mathbb{C}^n$  and  $\delta > 0$  is small enough, we will see from our construction that the injective holomorphic mapping  $\Psi \circ \mathbb{L} \circ \mathbb{E}_\delta$  applied to a neighbourhood of the filled spherical hat  $\hat{\mathbb{S}}_r^n$  with  $0 < r < 1$  close enough to 1 will give us a contradiction to the assumption of 1-pseudoconvexity of the domain  $\Omega$  in the sense of Definition 2.2.  $\square$

Now we recall the definition of 1-pseudoconcavity for closed sets.

**Definition 2.3.** Let  $\mathcal{M}$  be a complex manifold and  $A \subset \mathcal{M}$  be a closed set. Then  $A$  is called 1-pseudoconcave in  $\mathcal{M}$  if  $\mathcal{M} \setminus A$  is 1-pseudoconvex in  $\mathcal{M}$ .

Note that more equivalent descriptions of 1-pseudoconvex sets are known. For example, from Theorems 4.2 and 5.1 of Słodkowski [S11] it follows, in particular, that a nonempty relatively closed subset  $A$  of an open set  $V \subset \mathbb{C}^n$  is 1-pseudoconcave in  $V$  if and only if plurisubharmonic functions have the local maximum property on  $A$ . We will need here an analogous statement in a more general setting of complex manifold.

**Proposition 2.2.** Let  $\mathcal{M}$  be a complex manifold and  $A \subset \mathcal{M}$  be a closed set. Then the following assertions are equivalent:

- (1) For every  $\zeta \in A$ , there exists a neighbourhood  $V \subset \mathcal{M}$  of  $\zeta$  such that  $A \cap V$  is 1-pseudoconcave in  $V$ .
- (2)  $A$  is 1-pseudoconcave in  $\mathcal{M}$ .
- (3) For every  $\zeta \in A$ , there exists a neighbourhood  $V \subset \mathcal{M}$  of  $\zeta$  such that for every compact set  $B \subset V$  and every plurisubharmonic function  $\varphi$  defined in a neighbourhood of  $B$  one has  $\max_{A \cap B} \varphi \leq \max_{A \cap bB} \varphi$ .

Here  $\max_{A \cap bK} \varphi$  is meant to be  $-\infty$  if  $A \cap bK = \emptyset$ .

A detailed proof of this statement (which follows the ideas of Słodkowski [S11]) can be found in a more general setting of  $q$ -pseudoconcave sets in [HST2, Proposition 3.3].

The following result was proved in [HST3, Theorem 3.1, part 1]. Since it plays an important role in this paper, we will present it here in details for the reader convenience in a slightly different form adapted to the current presentation.

**Theorem 2.1.** *There exists a domain  $W$  in  $\mathbb{C}^n$  with coordinates  $(z_1, z_2, \dots, z_n)$ ,  $z_j = x_j + iy_j$ , and a smooth plurisubharmonic function  $\varphi : W \rightarrow [0, +\infty)$  such that*

- (1)  $\Pi_- := \{z \in \mathbb{C}^n : x_1 \leq 0\} \subset W$ .
- (2)  $\varphi = 0$  on  $\Pi_-$ .
- (3)  $\varphi > 0$  on  $W \setminus \Pi_-$ .
- (4)  $\varphi$  is strictly plurisubharmonic on  $W \setminus \Pi_-$ .

**Proof.** For every  $j \in \mathbb{N}$ , let  $\psi_j : \mathbb{B}_j^n(0) \rightarrow \mathbb{R}$  be the smooth and strictly plurisubharmonic function defined by

$$\psi_j(z_1, \dots, z_n) := x_1 - \frac{1}{2^{j-2}} + \frac{1}{j^2 2^{j-1}} (y_1^2 + |z_2|^2 + \dots + |z_n|^2).$$

Choose a smooth function  $\chi_j : \mathbb{R} \rightarrow [0, \infty)$  such that  $\chi_j \equiv 0$  on  $(-\infty, -1/2^j]$  and such that  $\chi_j$  is strictly increasing and strictly convex on  $(-1/2^j, \infty)$ . Set  $\tilde{\varphi}_j := \chi_j \circ \psi_j$ . Then  $\tilde{\varphi}_j$  is a smooth plurisubharmonic function on  $\mathbb{B}_j^n(0)$  such that  $\tilde{\varphi}_j \equiv 0$  on  $\{\psi_j \leq -1/2^j\} \supset \mathbb{B}_j^n(0) \cap \{x_1 \leq 1/2^j\}$  and such that  $\tilde{\varphi}_j$  is strictly plurisubharmonic and positive on  $\{\psi_j > -1/2^j\} \supset \mathbb{B}_j^n(0) \cap \{x_1 > 3/2^j\}$ . Thus

$$\varphi_j(z) := \begin{cases} \tilde{\varphi}_j(z), & z \in \mathbb{B}_j^n(0) \cap \{x_1 \geq 1/2^j\} \\ 0, & z \in \{x_1 < 1/2^j\} \end{cases}$$

is a smooth plurisubharmonic function on  $W_j := \mathbb{B}_j^n(0) \cup \{x_1 < 1/2^j\}$  such that  $\varphi_j$  is strictly plurisubharmonic and positive on  $\mathbb{B}_j^n(0) \cap \{x_1 > 3/2^j\}$ . Observe that  $W := \bigcap_{j=1}^{\infty} W_j$  is a connected open neighbourhood of  $\{x_1 \leq 0\}$ . Then one easily sees that for a sequence  $\{\varepsilon_j\}_{j=1}^{\infty}$  of positive numbers that converges to zero fast enough, the function  $\varphi := \sum_{j=1}^{\infty} \varepsilon_j \varphi_j$  is smooth and plurisubharmonic on  $W$  such that  $\varphi \equiv 0$  on  $\{x_1 \leq 0\}$  and such that  $\varphi$  is strictly plurisubharmonic and positive on  $W \cap \{x_1 > 0\}$ , which completes the proof of the theorem.  $\square$

As a consequence of Theorem 2.1 we get the following statement which will be one of the main technical tools in our construction below.

**Corollary 2.1.** *For every  $r \in (0, 1)$  there is a smooth nonnegative plurisubharmonic function  $\varphi_r$  defined on the domain  $\Omega_r := \mathbb{C}^n \setminus \hat{\mathbb{S}}_r^n$  such that*

- (1)  $\varphi_r$  is equal to 0 on the set  $\Omega_r \setminus \hat{\mathbb{S}}_r^n$ .
- (2)  $\varphi_r$  is positive and strictly plurisubharmonic in the interior  $\text{Int}(\hat{\mathbb{S}}_r^n)$  of the set  $\hat{\mathbb{S}}_r^n$ .

**Proof.** If for each  $r \in (0, 1)$  we define the function  $\varphi_r$  as

$$\varphi_r(z) := \begin{cases} \varphi((z_1 - r, z_2, \dots, z_n)), & \text{for } z = (z_1, z_2, \dots, z_n) \in \text{Int}(\hat{\mathbb{S}}_r^n), \\ 0, & \text{for } z \in \Omega_r \setminus \hat{\mathbb{S}}_r^n, \end{cases}$$

where  $\varphi$  is the function constructed in Theorem 2.1, then it is straightforward to see that the function  $\varphi_r$  has all the desired properties.  $\square$

### 3. Construction and properties of the set $\mathfrak{n}(\mathcal{K})$

Let  $\mathcal{K}' \supset \mathcal{K}''$  be compact sets in a complex manifold  $\mathcal{M}$  of dimension  $n$ . We say that the set  $\mathcal{K}''$  is obtained from the set  $\mathcal{K}'$  by a *spherical cut* if there exist  $r \in (0, 1)$ , a neighbourhood  $U := U(\hat{\mathbb{S}}_r^n) \subset \mathbb{C}^n$  of the filled spherical hat  $\hat{\mathbb{S}}_r^n$  and an injective holomorphic mapping  $\Phi: U \rightarrow \mathcal{M}$  such that  $\Phi(\hat{\mathbb{S}}_r^n) \subset \mathcal{M} \setminus \mathcal{K}'$  and  $\mathcal{K}' \setminus \Phi(\text{Int}(\hat{\mathbb{S}}_r^n)) = \mathcal{K}''$ .

Further, for a pair of compact sets  $\mathcal{K}' \supset \mathcal{K}''$  in  $\mathcal{M}$  we say that the set  $\mathcal{K}''$  is obtained from the set  $\mathcal{K}'$  by a *finite sequence of spherical cuts* if there exists a finite decreasing sequence  $\mathcal{K}_1 \supset \mathcal{K}_2 \supset \cdots \supset \mathcal{K}_m$  of compact sets in  $\mathcal{M}$  such that  $\mathcal{K}_1 = \mathcal{K}'$ ,  $\mathcal{K}_m = \mathcal{K}''$  and for each  $j = 2, 3, \dots, m$  the set  $\mathcal{K}_j$  is obtained from the set  $\mathcal{K}_{j-1}$  by a spherical cut.

Then, for a given compact set  $\mathcal{K}$  in  $\mathcal{M}$ , we can consider the family  $\mathcal{F}_{\mathcal{K}}$  of compact subsets of  $\mathcal{K}$  defined by

$$\mathcal{F}_{\mathcal{K}} := \{ \mathcal{K}_{\alpha} : \mathcal{K}_{\alpha} \text{ is obtained from } \mathcal{K} \text{ by a finite sequence of spherical cuts} \}_{\alpha \in \mathcal{A}},$$

where  $\mathcal{A}$  is a parameter set of this family.

The next statement follows easily from the definition of  $\mathcal{F}_{\mathcal{K}}$ .

**Lemma 3.1.** *Let  $\mathcal{K}_{\alpha_1}, \mathcal{K}_{\alpha_2}, \dots, \mathcal{K}_{\alpha_m}$  be a finite set of compacts from  $\mathcal{F}_{\mathcal{K}}$ , then one also has that  $\bigcap_{j=1}^m \mathcal{K}_{\alpha_j} \in \mathcal{F}_{\mathcal{K}}$ .*

**Proof.** Since the general case by induction can be reduced to the case of two sets  $\mathcal{K}_{\alpha_1}$  and  $\mathcal{K}_{\alpha_2}$ , we will only treat this case here.

First, we observe that by the assumption  $\mathcal{K}_{\alpha_1} \in \mathcal{F}_{\mathcal{K}}$  we get a finite decreasing sequence  $\mathcal{K}'_1 \supset \mathcal{K}'_2 \supset \cdots \supset \mathcal{K}'_{m_1}$  of compact sets in  $\mathcal{M}$  such that  $\mathcal{K}'_1 = \mathcal{K}$ ,  $\mathcal{K}'_{m_1} = \mathcal{K}_{\alpha_1}$  and for each  $j = 2, 3, \dots, m_1$  the set  $\mathcal{K}'_j$  is obtained from the set  $\mathcal{K}'_{j-1}$  by a spherical cut. Similarly, the assumption  $\mathcal{K}_{\alpha_2} \in \mathcal{F}_{\mathcal{K}}$  implies that there is a finite decreasing sequence  $\mathcal{K}''_1 \supset \mathcal{K}''_2 \supset \cdots \supset \mathcal{K}''_{m_2}$  of compact sets in  $\mathcal{M}$  such that  $\mathcal{K}''_1 = \mathcal{K}$ ,  $\mathcal{K}''_{m_2} = \mathcal{K}_{\alpha_2}$  and for each  $j = 2, 3, \dots, m_2$  the set  $\mathcal{K}''_j$  is obtained from the set  $\mathcal{K}''_{j-1}$  by a spherical cut. If now we consider the spherical cuts corresponding to the first sequence with the initial set  $\mathcal{K}$  and then the spherical cuts corresponding to the second sequence, but applied to the initial set  $\mathcal{K}_{\alpha_1}$ , we will get a finite decreasing sequence  $\mathcal{K} = \mathcal{K}'_1 \supset \mathcal{K}''_2 \supset \cdots \supset \mathcal{K}'_{m_1} = \mathcal{K}_{\alpha_1} = \mathcal{K}_{\alpha_1} \cap \mathcal{K} = \mathcal{K}_{\alpha_1} \cap \mathcal{K}'_1 \supset \mathcal{K}_{\alpha_1} \cap \mathcal{K}''_2 \cdots \supset \mathcal{K}_{\alpha_1} \cap \mathcal{K}''_{m_2} = \mathcal{K}_{\alpha_1} \cap \mathcal{K}_{\alpha_2}$  of compact sets in  $\mathcal{M}$  with the property that each set of this sequence is obtained from the previous set by a spherical cut. Since the initial set of this sequence is  $\mathcal{K}$  and the final set is  $\mathcal{K}_{\alpha_1} \cap \mathcal{K}_{\alpha_2}$ , we conclude that  $\mathcal{K}_{\alpha_1} \cap \mathcal{K}_{\alpha_2} \in \mathcal{F}_{\mathcal{K}}$ . This completes the proof of Lemma 3.1.  $\square$

Now we can define the set  $\mathfrak{n}(\mathcal{K})$  which plays a special role in the present article and will be called in what follows the *nucleus* of  $\mathcal{K}$ :

$$\mathfrak{n}(\mathcal{K}) := \bigcap_{\mathcal{K}_{\alpha} \in \mathcal{F}_{\mathcal{K}}} \mathcal{K}_{\alpha}.$$

The most important for us properties of this set are given in the next statement.



**Theorem 3.1.** *The set  $\mathfrak{n}(\mathcal{K})$  is 1-pseudoconcave. Moreover,  $\mathfrak{n}(\mathcal{K})$  is the maximal 1-pseudoconcave subset of the set  $\mathcal{K}$ .*

**Proof.** Assume, to get a contradiction, that the set  $\mathfrak{n}(\mathcal{K})$  is not 1-pseudoconcave. Then, in view of Definition 2.2 and Proposition 2.2, for some  $r \in (0, 1)$  there exist a neighbourhood  $U \subset \mathbb{C}^n$  of the filled spherical hat  $\hat{\mathbb{S}}_r^n$  and an injective holomorphic mapping  $\Phi: U \rightarrow \mathcal{M}$  such that  $\Phi(\mathbb{S}_r^n) \subset \mathcal{M} \setminus \mathfrak{n}(\mathcal{K})$  and  $\Phi(\hat{\mathbb{S}}_r^n) \cap \mathfrak{n}(\mathcal{K}) \neq \emptyset$ . Note that, after the substitution of  $r$  by  $r - \varepsilon$  with  $\varepsilon > 0$  small enough if necessary, but keeping the same  $U$  and  $\Phi$ , we can achieve that  $\Phi(\mathbb{S}_r^n) \subset \mathcal{M} \setminus \mathfrak{n}(\mathcal{K})$  and  $\Phi(\text{Int}(\hat{\mathbb{S}}_r^n)) \cap \mathfrak{n}(\mathcal{K}) \neq \emptyset$ . Let  $\zeta$  be a point of the set  $\Phi(\text{Int}(\hat{\mathbb{S}}_r^n)) \cap \mathfrak{n}(\mathcal{K})$ . Since  $\Phi(\mathbb{S}_r^n) \subset \mathcal{M} \setminus \mathfrak{n}(\mathcal{K})$ , then, in view of compactness of the sets  $\Phi(\mathbb{S}_r^n)$  and  $\mathfrak{n}(\mathcal{K})$ , there is a neighbourhood  $V$  of  $\mathfrak{n}(\mathcal{K})$  such that  $\Phi(\mathbb{S}_r^n) \cap V = \emptyset$ . It follows now from the definition of the set  $\mathfrak{n}(\mathcal{K})$  that there is a finite set of compacts  $\mathcal{K}_{\alpha_1}, \mathcal{K}_{\alpha_2}, \dots, \mathcal{K}_{\alpha_m}$  in the family  $\mathcal{F}_{\mathcal{K}}$  such that  $\bigcap_{j=1}^m \mathcal{K}_{\alpha_j} \subset V$ . By Lemma 3.1 we know that the set  $\tilde{\mathcal{K}} := \bigcap_{j=1}^m \mathcal{K}_{\alpha_j}$  also belongs to the family  $\mathcal{F}_{\mathcal{K}}$ . Since  $\tilde{\mathcal{K}} \subset V$  and  $\Phi(\mathbb{S}_r^n) \cap V = \emptyset$ , we see that  $\Phi(\mathbb{S}_r^n) \cap \tilde{\mathcal{K}} = \emptyset$  and, hence, we can make one more spherical cut to obtain the set  $\tilde{\tilde{\mathcal{K}}} \setminus \Phi(\text{Int}(\hat{\mathbb{S}}_r^n)) =: \tilde{\tilde{\mathcal{K}}}$  from the set  $\tilde{\mathcal{K}}$ . Observe now that, by construction, the set  $\tilde{\tilde{\mathcal{K}}}$  also belongs to the family  $\mathcal{F}_{\mathcal{K}}$  and, moreover, that  $\zeta \notin \tilde{\tilde{\mathcal{K}}}$ . Then, by the definition of the set  $\mathfrak{n}(\mathcal{K})$ , we know that  $\zeta \notin \mathfrak{n}(\mathcal{K})$  which contradicts our choice of the point  $\zeta$ .

To prove the maximality of  $\mathfrak{n}(\mathcal{K})$  we observe first that, if  $\mathcal{K}_2 \subset \mathcal{K}_1$  are compact sets in  $\mathcal{M}$  such that  $\mathcal{K}_2$  is obtained from  $\mathcal{K}_1$  by a spherical cut, and if  $\mathcal{K}' \subset \mathcal{K}_1$  is a compact set which is 1-pseudoconcave, then, by the definition of 1-pseudoconcave sets, we also have that  $\mathcal{K}' \subset \mathcal{K}_2$ . Applying this argument to a finite sequence of spherical cuts of the set  $\mathcal{K}$ , we see that for an arbitrary 1-pseudoconcave compact subset  $\mathcal{K}'$  of  $\mathcal{K}$  the inclusion  $\mathcal{K}' \subset \mathcal{K}_{\alpha}$  holds true for every  $\mathcal{K}_{\alpha} \in \mathcal{F}_{\mathcal{K}}$ . Thus, by the definition of  $\mathfrak{n}(\mathcal{K})$ , we have that  $\mathcal{K}' \subset \bigcap_{\mathcal{K}_{\alpha} \in \mathcal{F}_{\mathcal{K}}} \mathcal{K}_{\alpha} = \mathfrak{n}(\mathcal{K})$ . This completes the proof of Theorem 3.1.  $\square$

## 4. Proof of the Main Theorem

Now we can complete the proof of the Main Theorem. In order to do this we distinguish two cases.

**Case 1.**  $\mathfrak{n}(\mathcal{K}) \neq \emptyset$ .

In this case we prove that the set  $\mathcal{K}$  does not possess smooth strictly plurisubharmonic functions. Indeed, we argue by contradiction and assume that, in accordance with Definition 1.1, there is a neighbourhood  $\mathfrak{A}$  of  $\mathcal{K}$  in  $\mathcal{M}$  and a smooth strictly plurisubharmonic function  $\varphi$  on  $\mathfrak{A}$ . Since the set  $\mathfrak{n}(\mathcal{K})$  is compact, there is a point  $\zeta_0 \in \mathfrak{n}(\mathcal{K})$  such that  $\varphi(\zeta_0) = \max_{\zeta \in \mathfrak{n}(\mathcal{K})} \varphi(\zeta)$ . Then, in view of strict plurisubharmonicity of  $\varphi$ , in local coordinates near  $\zeta_0$  the function  $\varphi(\zeta_0) - \varepsilon \|\zeta - \zeta_0\|^2$  will still be plurisubharmonic for all sufficiently small  $\varepsilon$ . More precisely, there is a neighbourhood  $V \subset \mathcal{M}$  of  $\zeta_0$ , an injective holomorphic mapping  $\Phi: V \rightarrow \mathbb{C}^n$  which sends the point  $\zeta_0$  to the origine and a number  $\varepsilon_0 > 0$  such that the function  $\varphi(\zeta) - \varepsilon \|\Phi(\zeta)\|^2$  is plurisubharmonic on  $V$  for all  $0 \leq \varepsilon < \varepsilon_0$ .



Since, by Theorem 3.1, the set  $\mathbf{n}(\mathcal{K})$  is 1-pseudoconcave, it follows that if we choose  $V$  small enough, then the property (3) of Proposition 2.2 with  $\mathbf{n}(\mathcal{K})$  on the place of  $A$  holds true. Applying this property to the function  $\varphi(\zeta) - \varepsilon\|\Phi(\zeta)\|^2$  with some  $0 < \varepsilon < \varepsilon_0$  and  $B$  being the closure of a small enough neighbourhood of  $\zeta_0$ , we get, by the choice of  $\zeta_0$ , that

$$\begin{aligned} \varphi(\zeta_0) &= \varphi(\zeta_0) - \varepsilon\|\Phi(\zeta_0)\|^2 = \max_{\zeta \in \mathbf{n}(\mathcal{K}) \cap B} \varphi(\zeta) - \varepsilon\|\Phi(\zeta)\|^2 = \max_{\zeta \in \mathbf{n}(\mathcal{K}) \cap B} (\varphi(\zeta) - \varepsilon\|\Phi(\zeta)\|^2) \\ &\leq \max_{\zeta \in \mathbf{n}(\mathcal{K}) \cap bB} (\varphi(\zeta) - \varepsilon\|\Phi(\zeta)\|^2) < \max_{\zeta \in \mathbf{n}(\mathcal{K}) \cap bB} \varphi(\zeta) \leq \max_{\zeta \in \mathbf{n}(\mathcal{K}) \cap B} \varphi(\zeta) = \varphi(\zeta_0). \end{aligned}$$

This gives the desired contradiction in the Case 1.

**Case 2.**  $\mathbf{n}(\mathcal{K}) = \emptyset$ .

In this case we prove that the set  $\mathcal{K}$  possesses a smooth strictly plurisubharmonic function, i.e., there is a neighbourhood  $\mathfrak{A}$  of  $\mathcal{K}$  in  $\mathcal{M}$  and a smooth strictly plurisubharmonic function  $\varphi$  defined on  $\mathfrak{A}$ . Indeed, since  $\bigcap_{\mathcal{K}_\alpha \in \mathcal{F}_\mathcal{K}} \mathcal{K}_\alpha = \mathbf{n}(\mathcal{K}) = \emptyset$ , and since all the sets  $\mathcal{K}_\alpha \in \mathcal{F}_\mathcal{K}$  are compact, one has finitely many compacts  $\mathcal{K}_{\alpha_1}, \mathcal{K}_{\alpha_2}, \dots, \mathcal{K}_{\alpha_m}$  in  $\mathcal{F}_\mathcal{K}$  such that  $\bigcap_{j=1}^m \mathcal{K}_{\alpha_j} = \emptyset$ . Hence, in view of Lemma 3.1, there is a finite sequence of spherical cuts  $\mathcal{K}_1 \supset \mathcal{K}_2 \supset \dots \supset \mathcal{K}_m$  such that  $\mathcal{K}_1 = \mathcal{K}$  and  $\mathcal{K}_m = \emptyset$ . We use this sequence to construct inductively for each  $j = m-1, m-2, \dots, 1$  a neighbourhood  $\mathfrak{A}_j$  of the compact set  $\mathcal{K}_j$  and a smooth strictly plurisubharmonic function  $\varphi_j$  defined on the set  $\mathfrak{A}_j$ . If we will be able to perform this construction, then, in view of the equality  $\mathcal{K} = \mathcal{K}_1$ , the function  $\varphi := \varphi_1$  defined on  $\mathfrak{A} := \mathfrak{A}_1$  will be a function as desired.

In order to make the first step of our construction we consider the set  $\mathcal{K}_{m-1}$  and observe that, since  $\mathcal{K}_m = \emptyset$ , and since the set  $\mathcal{K}_m$  is obtained from the set  $\mathcal{K}_{m-1}$  by a spherical cut, there exist  $r_{m-1} \in (0, 1)$ , a neighbourhood  $U_{m-1} := U(\hat{\mathbb{S}}_{r_{m-1}}^n) \subset \mathbb{C}^n$  of the filled spherical hat  $\hat{\mathbb{S}}_{r_{m-1}}^n$  and an injective holomorphic mapping  $\Phi_{m-1}: U_{m-1} \rightarrow \mathcal{M}$  such that  $\Phi_{m-1}(\mathbb{S}_{r_{m-1}}^n) \subset \mathcal{M} \setminus \mathcal{K}_{m-1}$  and  $\mathcal{K}_{m-1} \subset \Phi_{m-1}(\text{Int}(\hat{\mathbb{S}}_{r_{m-1}}^n))$ . If we denote by  $\tilde{\varphi}_{m-1}$  the restriction to the set  $U_{m-1}$  of the function  $\varphi_{r_{m-1}}$  provided by Corollary 2.1 with  $r = r_{m-1}$ , then the function  $\varphi_{m-1} := \tilde{\varphi}_{m-1} \circ \Phi_{m-1}^{-1}$  will be smooth and strictly plurisubharmonic on the neighbourhood  $\mathfrak{A}_{m-1} := \Phi_{m-1}(\text{Int}(\hat{\mathbb{S}}_{r_{m-1}}^n))$  of the set  $\mathcal{K}_{m-1}$ .

Now we proceed with the inductive step of our construction. Assume that we have already constructed a neighbourhood  $\mathfrak{A}_j$  of the compact set  $\mathcal{K}_j$  and a smooth strictly plurisubharmonic function  $\varphi_j$  defined on the set  $\mathfrak{A}_j$ . After shrinking the set  $\mathfrak{A}_j$  if necessary, we can assume that  $\mathfrak{A}_j$  has a smooth boundary and the function  $\varphi_j$  is smooth on  $\overline{\mathfrak{A}}_j$ , hence there is a smooth extension, which we denote by  $\varphi'_j$ , of  $\varphi_j$  to the whole of  $\mathcal{M}$ . Further, since the set  $\mathcal{K}_j$  is obtained from the set  $\mathcal{K}_{j-1}$  by a spherical cut, there exist  $r_{j-1} \in (0, 1)$ , a neighbourhood  $U_{j-1} := U(\hat{\mathbb{S}}_{r_{j-1}}^n) \subset \mathbb{C}^n$  of the filled spherical hat  $\hat{\mathbb{S}}_{r_{j-1}}^n$  and an injective holomorphic mapping  $\Phi_{j-1}: U_{j-1} \rightarrow \mathcal{M}$  such that  $\Phi_{j-1}(\mathbb{S}_{r_{j-1}}^n) \subset \mathcal{M} \setminus \mathcal{K}_{j-1}$  and  $\mathcal{K}_{j-1} \setminus \Phi_{j-1}(\text{Int}(\hat{\mathbb{S}}_{r_{j-1}}^n)) = \mathcal{K}_j$ . If we denote by  $\tilde{\varphi}_{j-1}$  the restriction to the set  $U_{j-1}$  of the function  $\varphi_{r_{j-1}}$  provided by Corollary 2.1 with  $r = r_{j-1}$ , then the function  $\tilde{\varphi}_{j-1}' := \tilde{\varphi}_{j-1} \circ \Phi_{j-1}^{-1}$  will be smooth and strictly plurisubharmonic on the neighbourhood

$\Phi_{j-1}(\text{Int}(\hat{\mathbb{S}}_{r_{j-1}}^n))$  of the set  $\mathcal{K}_{j-1} \setminus \mathfrak{A}_j$ . Moreover, if we denote by  $\varphi''_{j-1}$  the extension of the function  $\tilde{\varphi}''_{j-1}$  by zero to the rest of the domain  $\mathcal{W}_{j-1} := \mathcal{M} \setminus \Phi_{j-1}(\mathbb{S}_{r_{j-1}}^n)$ , then, in view of the construction of the function  $\varphi_{r_{j-1}}$  (see Corollary 2.1), the function  $\varphi''_{j-1}$  will be smooth and plurisubharmonic on  $\mathcal{W}_{j-1}$ , equal to zero on  $\mathcal{W}_{j-1} \setminus \Phi_{j-1}(\text{Int}(\hat{\mathbb{S}}_{r_{j-1}}^n))$  and positive and strictly plurisubharmonic on  $\Phi_{j-1}(\text{Int}(\hat{\mathbb{S}}_{r_{j-1}}^n))$ . Finally, since  $\mathcal{K}_{j-1}$  is a subset of the open set  $\mathfrak{A}_j \cup \Phi_{j-1}(\text{Int}(\hat{\mathbb{S}}_{r_{j-1}}^n))$ , we can choose for the set  $\mathfrak{A}_{j-1}$  an open neighbourhood of the compact  $\mathcal{K}_{j-1}$  such that  $\mathfrak{A}_{j-1} \subset \subset \mathfrak{A}_j \cup \Phi_{j-1}(\text{Int}(\hat{\mathbb{S}}_{r_{j-1}}^n))$  and then we can define the function  $\varphi_{j-1} := \varphi'_j + C\varphi''_{j-1}$  with  $C > 0$  being chosen so large that  $\varphi_{j-1}$  is strictly plurisubharmonic not only on the set  $\mathfrak{A}_{j-1} \cap \mathfrak{A}_j$ , where  $\varphi'_j$  is strictly plurisubharmonic and  $C\varphi''_{j-1}$  is plurisubharmonic, but also on the set  $\mathfrak{A}_{j-1} \setminus \mathfrak{A}_j \subset \subset \Phi_{j-1}(\text{Int}(\hat{\mathbb{S}}_{r_{j-1}}^n))$ , where  $\varphi''_{j-1}$  is strictly plurisubharmonic, and hence, for large enough  $C$ ,  $\varphi'_j + C\varphi''_{j-1} = \varphi_{j-1}$  also is. This proves our argument by induction and hence also the existence of a smooth strictly plurisubharmonic functions on  $\mathcal{K}$  in the Case 2. The proof of the Main Theorem is now completed.  $\square$

Note, that using the same argument as in the Case 2 above (i.e. taking a finite sequence of spherical cuts  $\mathcal{K}_1 \supset \mathcal{K}_2 \supset \dots \supset \mathcal{K}_m$  and then applying Corollary 2.1) in the situation when  $\mathfrak{n}(\mathcal{K}) \neq \emptyset$ , and choosing as a starting point of our inductive construction the function which is identically equal to zero in a neighbourhood of  $\mathfrak{n}(\mathcal{K})$ , we can obtain the following result.

**Theorem 4.1.** *Let  $\mathcal{K}$  be a compact subset of a complex manifold  $\mathcal{M}$  and  $\mathcal{V}$  be a neighbourhood of the set  $\mathfrak{n}(\mathcal{K})$  in  $\mathcal{M}$ . Then there is a neighbourhood  $\mathfrak{A}$  of  $\mathcal{K}$  in  $\mathcal{M}$  and a smooth nonnegative plurisubharmonic function  $\varphi$  defined on  $\mathfrak{A}$  such that:*

- (1)  $\varphi$  is positive and strictly plurisubharmonic on  $\mathfrak{A} \setminus \bar{\mathcal{V}}$ .
- (2)  $\varphi|_{\mathfrak{n}(\mathcal{K})} \equiv 0$ .

## 5. Applications and open questions

Now we will discuss some applications of our results and their connection to the other topics which naturally divides this section by the content into four subsections.

**1. Kobayashi hyperbolicity.** For the first application we need the next statement which gives a link between Kobayashi hyperbolicity and the existence of bounded strictly plurisubharmonic functions. It is just a slight reformulation of Theorem 3 on p. 362 in [Si1].

**Theorem 5.1.** *Let  $M$  be a complex manifold which has a bounded continuous strictly plurisubharmonic function. Then  $M$  is Kobayashi hyperbolic.*

As a direct consequence of this statement and our Main Theorem we get the following result.

**Theorem 5.2.** *Let  $\mathcal{K}$  be a compact subset of a complex manifold  $\mathcal{M}$ . If  $\mathcal{K}$  does not have 1-pseudoconcave subsets, then there is a neighbourhood  $\mathfrak{A}$  of  $\mathcal{K}$  in  $\mathcal{M}$  which is Kobayashi hyperbolic.*

**2. The core of a compact.** The next topic, which is naturally connected to the content of this paper, is related to the notion of the *core* of a complex manifold. This notion was introduced and systematically studied by Harz-Shcherbina-Tomassini in [HST2] - [HST5]. Further results on the foliated structure of the core were obtained by Poletsky-Shcherbina in [PS] and Ślodkowski in [Sl2].

Recall first the definition of the core of a manifold which was given in [HST2] - [HST3]:

**Definition 5.1.** *Let  $\mathcal{M}$  be a complex manifold. Then the set*

$$\mathfrak{c}(\mathcal{M}) := \left\{ \zeta \in \mathcal{M} : \text{every smooth plurisubharmonic function on } \mathcal{M} \text{ that is} \right. \\ \left. \text{bounded from above fails to be strictly plurisubharmonic in } \zeta \right\}$$

*is called the core of  $\mathcal{M}$ .*

In the same vein we can define a notion of the core in the setting of the present paper.

**Definition 5.2.** *Let  $\mathcal{K}$  be a compact subset of a complex manifold  $\mathcal{M}$ . Then the set*

$$\mathfrak{c}(\mathcal{K}) := \left\{ \zeta \in \mathcal{K} : \text{every function which is smooth and plurisubharmonic on a neighbour-} \right. \\ \left. \text{hood of } \mathcal{K} \text{ in } \mathcal{M} \text{ fails to be strictly plurisubharmonic in } \zeta \right\}$$

*is called the core of  $\mathcal{K}$ .*

This definition obviously implies that the set  $\mathfrak{c}(\mathcal{K})$  is compact. Since, by Theorem 4.1, for each point  $\zeta \notin \mathfrak{n}(\mathcal{K})$  there is a smooth plurisubharmonic function defined in a neighbourhood of  $\mathcal{K}$  which is strictly plurisubharmonic in  $\zeta$ , the following property holds true:

**Theorem 5.3.** *Let  $\mathcal{K}$  be a compact subset of a complex manifold  $\mathcal{M}$ . Then  $\mathfrak{c}(\mathcal{K}) \subset \mathfrak{n}(\mathcal{K})$ .*

We do not know if the reverse statement holds true or not:

**Question 1.** Let  $\mathcal{K}$  be a compact subset of a complex manifold  $\mathcal{M}$ . Is it always true that  $\mathfrak{n}(\mathcal{K}) \subset \mathfrak{c}(\mathcal{K})$ ?

One of the most important properties of the core  $\mathfrak{c}(\mathcal{M})$ , proved in [HST3], is its 1-pseudoconcavity. The other crucial property of  $\mathfrak{c}(\mathcal{M})$ , proved in [HST4] for manifolds of dimension 2 and in [PS] and [Sl2] for the general case, claims that  $\mathfrak{c}(\mathcal{M})$  can be decomposed as a disjoint union of pseudoconcave sets such that every smooth bounded

plurisubharmonic function on  $\mathcal{M}$  is constant on each of these sets. We do not know if similar statements for the core  $\mathfrak{c}(\mathcal{K})$  hold true or not:

**Question 2.** Let  $\mathcal{K}$  be a compact subset of a complex manifold  $\mathcal{M}$ . Is it always true that  $\mathfrak{c}(\mathcal{K})$  is 1-pseudoconcave?

**Question 3.** Let  $\mathcal{K}$  be a compact subset of a complex manifold  $\mathcal{M}$ . Is it always true that  $\mathfrak{c}(\mathcal{K})$  can be decomposed as a disjoint union of 1-pseudoconcave sets  $\{E_\alpha\}_{\alpha \in \mathcal{A}}$  such that  $\varphi|_{E_\alpha} \equiv \text{const}$  for each  $\alpha \in \mathcal{A}$ ?

**3. On the structure of the nucleus.** Part (3) of Proposition 2.2 above suggests that 1-pseudoconcave sets resemble in a sense complex analytic varieties. For this reason one can ask a question if such sets and, in particular, in view of Theorem 3.1, the nucleus  $\mathfrak{n}(\mathcal{K})$  of a given compact set  $\mathcal{K}$  in a complex manifold  $\mathcal{M}$ , will have some kind of analytic structure. The answer to this question is, in general, negative even if the dimension of  $\mathcal{M}$  is equal to 2. A corresponding example can be obtained if we choose for  $\mathcal{M}$  the 2-dimensional complex projective space  $\mathbb{P}^2$  with the projective coordinates  $[z : w : \zeta]$  and for  $\mathcal{K}$  the compactification by the point  $[1 : 0 : 0]$  of the Wermer type set  $\mathcal{E} \subset \mathbb{C}_{z,w}^2 \subset \mathbb{P}_{z,w,\zeta}^2$  constructed in [HST1]. In this case, by the properties of  $\mathcal{E}$  established in Theorem 1.1 of [HST1], the set  $\mathcal{K}$  is 1-pseudoconcave, hence,  $\mathfrak{n}(\mathcal{K}) = \mathcal{K}$ , and, moreover, it has no analytic subsets of positive dimension.

Since the answer to the question above is, in general, negative, one can restrict the question to the case when the compact set  $\mathcal{K}$  has more structure, for example, to the case when it is a smooth real hypersurface in  $\mathcal{M}$ . It is not difficult to see that even in this case the nucleus  $\mathfrak{n}(\mathcal{K})$  of  $\mathcal{K}$  does not need to contain any analytic subvarieties. Indeed, if we choose the set  $\mathcal{K}$  to be a  $C^2$ -small generic perturbation of the surface  $\{[z_1 : z_2 : z_3 : z_4] \in \mathbb{P}^3 : |z_1|^2 + |z_2|^2 - |z_3|^2 - |z_4|^2 = 0\}$  in  $\mathbb{P}^3$ , then the Levi form of  $\mathcal{K}$  has the signature  $(1, 1)$ , hence  $\mathcal{K}$  is 1-pseudoconcave. However, by genericity of the perturbation, we see that  $\mathcal{K}$  can not have any holomorphic disc inside, even locally.

Surprisingly, in the case when the dimension of  $\mathcal{M}$  is equal to 2 and  $\mathcal{K}$  is a hypersurface (even not necessarily smooth) in  $\mathcal{M}$ , the set  $\mathfrak{n}(\mathcal{K})$  will have a very special structure:

**Theorem 5.4.** *Let  $\mathcal{M}$  be a complex manifold of dimension 2 and let  $\mathcal{K}$  be a continuous hypersurface of the graph type in  $\mathcal{M}$ . Then locally the set  $\mathfrak{n}(\mathcal{K})$  is a disjoint union of holomorphic discs.*

Here  $\mathcal{K}$  is a continuous hypersurface of the graph type in  $\mathcal{M}$  means that for every point  $\zeta \in \mathcal{K}$  there is a neighbourhood  $\Omega$  in  $\mathcal{M}$  and local holomorphic coordinates  $(z, w)$  in  $\Omega$  such that  $\mathcal{K} \cap \Omega = \{(z, w) \in B_r^3(0) \times \mathbb{R}_v : v = h(z, u)\} =: \Gamma_h$  – the graph of a continuous function  $h : B_r^3(0) \rightarrow \mathbb{R}_v$ , where  $B_r^3(0) := \{(z, u) \in \mathbb{C}_z \times \mathbb{R}_u : |z|^2 + u^2 < r^2\}$  and  $w = u + iv$ .

**Proof.** Let  $\zeta$  be a point of  $\mathfrak{n}(\mathcal{K})$ . Since, by our assumptions,  $\mathcal{K}$  is a continuous hypersurface of the graph type in  $\mathcal{M}$ , we can choose a neighbourhood  $\Omega$  of  $\zeta$  as described

above. Moreover, without restriction of generality we can assume, maybe after shrinking  $\Omega$  if necessary, that in local holomorphic coordinates  $(z, w)$  the set  $\Omega$  has the form  $\Omega = B_r^3(0) \times (-a, a) \subset B_r^3(0) \times \mathbb{R}_v \subset \mathbb{C}_{z,w}^2$  for some  $r, a > 0$ . Since, by Theorem 3.1, the set  $\mathbf{n}(\mathcal{K})$  is 1-pseudoconcave, and since, by part (1) of Proposition 2.2, 1-pseudoconcavity is a local property, and taking into account that the statement of the theorem also has a local nature, it will be enough to restrict our consideration to the set  $\Omega$ .

Consider now the domains  $\Omega_- := \{(z, w) \in \Omega : v < h(z, u)\}$  and  $\Omega_+ := \{(z, w) \in \Omega : v > h(z, u)\}$  and note that the hulls of holomorphy of these domains are single-sheeted and have the form  $\Omega_\ominus := \{(z, w) \in \Omega : v < h_-(z, u)\}$  and  $\Omega_\oplus := \{(z, w) \in \Omega : v > h_+(z, u)\}$ , respectively, where  $h_- \geq h$  is upper semicontinuous and  $h_+ \leq h$  is lower semicontinuous in  $B_r^3(0)$  (the proof of this rather elementary fact can be found, for example, in Lemma 1 of [C1]). Since  $\mathbf{n}(\mathcal{K}) \cap \Omega$  is 1-pseudoconcave, and since the dimension of  $\mathcal{M}$  is equal to 2, we conclude that the domain  $\mathcal{W} := \Omega \setminus \mathbf{n}(\mathcal{K})$  is pseudoconvex. The inclusion  $\mathbf{n}(\mathcal{K}) \subset \mathcal{K}$  implies that  $\Omega_- \subset \mathcal{W}$  and  $\Omega_+ \subset \mathcal{W}$  and, hence, by pseudoconvexity of  $\mathcal{W}$ , that  $\Omega_\ominus \subset \mathcal{W}$  and  $\Omega_\oplus \subset \mathcal{W}$ . Therefore, one also has the inclusion

$$\mathbf{n}(\mathcal{K}) \cap \Omega \subset b\Omega_\ominus \cap b\Omega_\oplus \cap \Omega =: E. \quad (1)$$

The results of [Sh] (see also Theorem 1 in [C2]) tell us now that the set  $E = \bigcup_{\alpha \in \mathcal{A}} E_\alpha$  is the disjoint union of complex analytic discs  $\{E_\alpha\}_{\alpha \in \mathcal{A}}$  which are closed in  $\Omega$ . Hence, to finish the proof of the theorem it is enough to prove the following statement:

**Claim.** *If for some  $\alpha_0 \in \mathcal{A}$  one has that  $E_{\alpha_0} \setminus \mathbf{n}(\mathcal{K}) \neq \emptyset$ , then  $E_{\alpha_0} \cap \mathbf{n}(\mathcal{K}) = \emptyset$ .*

This Claim, in view of inclusion (1), will imply that  $\mathbf{n}(\mathcal{K}) \cap \Omega = \bigcup_{\alpha \in \mathcal{B}} E_\alpha$  for some subset  $\mathcal{B}$  of  $\mathcal{A}$  and, hence, will prove the theorem.

**Proof of the Claim.** If  $E_{\alpha_0} \setminus \mathbf{n}(\mathcal{K}) \neq \emptyset$ , then we can take a continuous function  $h^* : B_r^3(0) \rightarrow \mathbb{R}_v$  such that  $h^* \geq h$  on  $B_r^3(0)$ ,  $h^* = h$  on  $\pi_{z,u}(\mathbf{n}(\mathcal{K}) \cap \Omega)$  and  $h^* > h$  on  $\pi_{z,u}(E_{\alpha_0} \setminus \mathbf{n}(\mathcal{K}))$ , where  $\pi_{z,u} : \mathbb{C}_{z,w}^2 \rightarrow \mathbb{C}_z \times \mathbb{R}_u$  is the canonical projection, and then, as above, we consider the domains  $\Omega_-^* := \{(z, w) \in \Omega : v < h^*(z, u)\}$  and  $\Omega_+^* := \{(z, w) \in \Omega : v > h^*(z, u)\}$  and their hulls of holomorphy which have the form  $\Omega_\ominus^* := \{(z, w) \in \Omega : v < h_-^*(z, u)\}$  and  $\Omega_\oplus^* := \{(z, w) \in \Omega : v > h_+^*(z, u)\}$ , respectively, where  $h_-^* \geq h^*$  is upper semicontinuous and  $h_+^* \leq h^*$  is lower semicontinuous in  $B_r^3(0)$ . Since  $h^* = h$  on  $\pi_{z,u}(\mathbf{n}(\mathcal{K}) \cap \Omega)$ , we have that  $\mathbf{n}(\mathcal{K}) \cap \Omega \subset \Gamma_{h^*}$ , and then, by pseudoconcavity of  $\mathbf{n}(\mathcal{K})$ , we can conclude again from the results of [Sh] that

$$\mathbf{n}(\mathcal{K}) \cap \Omega \subset b\Omega_\ominus^* \cap b\Omega_\oplus^* \cap \Omega =: E^*,$$

where  $E^* = \bigcup_{\alpha \in \mathcal{A}^*} E_\alpha^*$  is the disjoint union of complex analytic discs which are closed in  $\Omega$ . Now, the property  $h^* > h$  on  $\pi_{z,u}(E_{\alpha_0} \setminus \mathbf{n}(\mathcal{K})) \neq \emptyset$  tells us that the disc  $E_{\alpha_0}$  does not belong to the family  $\{E_\alpha^*\}_{\alpha \in \mathcal{A}^*}$ , which implies that  $E_{\alpha_0} \subset \Omega_\ominus^* \subset \mathcal{W}$  and, hence, also that  $E_{\alpha_0} \cap \mathbf{n}(\mathcal{K}) = \emptyset$ . This proves the Claim and completes the proof of Theorem 5.4.  $\square$

**Remark.** Note, that in the case when the dimension of  $\mathcal{M}$  is larger than 2, a statement analogous to the Theorem 5.4 is not known and very difficult to get even for  $\mathcal{K}$  being the boundary of a smooth pseudoconvex domain  $\mathcal{D} \subset \mathcal{M}$ . The problems here are related, in particular, to the jump of the rank of the Levi form and absence of a version of Frobenius theorem for distributions of varying dimension. This kind of difficulties is also present in the following, slightly reformulated here, old problem of Rossi [Ros, Conjecture 5.12 on p. 489] which is, to the best of our knowledge, still open:

**Conjecture.** *Let  $\mathcal{D}$  be a pseudoconvex domain with smooth boundary in a complex manifold  $\mathcal{M}$  of dimension at least 3. Let  $\mathfrak{B}$  be the set of all weakly pseudoconvex points in  $b\mathcal{D}$  and  $\text{Int}(\mathfrak{B})$  is the interior of  $\mathfrak{B}$  in  $b\mathcal{D}$ . Then for each point  $\zeta \in \text{Int}(\mathfrak{B})$  there is a variety  $\mathcal{V} \subset b\mathcal{D}$  of dimension at least one passing through the point  $\zeta$ .*

## References

- [CZ] B.-Y. Chen and J. H. Zhang, *The Bergman metric on a Stein manifold with a bounded plurisubharmonic function*, Trans. Amer. Math. Soc. **354** (2002), no. 8, 2997 - 3009.
- [C1] E. M. Chirka, *Approximation by polynomials on star-like subsets of  $\mathbb{C}^n$* , (Russian) Mat. Zametki **14** (1973), no. 1, 55 - 60.
- [C2] E. M. Chirka, *Levi and Trépreau theorems for continuous graphs*, (Russian) Tr. Mat. Inst. Steklova **235** (2001), Anal. i Geom. Vopr. Kompleks. Analiza, 272 - 287; translation in Proc. Steklov Inst. Math. **235** (2001), no. 4, 261 - 276.
- [FS] J.E. Fornæss and N. Sibony, *Oka's inequality for currents and applications*, Math. Ann. **301** (1995), no. 3, 399 - 419.
- [G] H. Grauert, *On Levi's problem and the imbedding of real-analytic manifolds*, Ann. Math. **68** (1958), no. 2, 460 - 472.
- [HL] R. Harvey and H.B. Lawson, Jr., *An intrinsic characterization of Kähler manifolds*, Invent. Math. **74** (1983), no. 2, 169 - 198.
- [HST1] T. Harz, N. Shcherbina and G. Tomassini, *Wermer type sets and extension of CR functions*, Indiana Univ. Math. J. **61** (2012), no. 1, 431 - 459.
- [HST2] T. Harz, N. Shcherbina and G. Tomassini, *On defining functions and cores for unbounded domains*, arXiv:1405.2250v3.
- [HST3] T. Harz, N. Shcherbina and G. Tomassini, *On defining functions and cores for unbounded domains I*, Math. Z. **286** (2017), no. 3-4, 987 - 1002.
- [HST4] T. Harz, N. Shcherbina and G. Tomassini, *On defining functions and cores for unbounded domains II*, J. Geom. Anal. **30** (2020), no. 3, 2293 - 2325.
- [HST5] T. Harz, N. Shcherbina and G. Tomassini, *On defining functions and cores for unbounded domains III*, submitted for publication.

- [N1] R. Narasimhan, *The Levi problem for complex spaces*, Math. Ann. **142** (1961), no. 4, 355 - 365.
- [N2] R. Narasimhan, *The Levi Problem for Complex Spaces II*, Math. Ann. **146** (1962), no. 3, 195 - 216.
- [P] E.A. Poletsky, *Pluricomplex Green functions on complex manifolds*, J. Geom. Anal. **30** (2020), no. 2, 1396 - 1410.
- [PS] E.A. Poletsky and N. Shcherbina, *Plurisubharmonically separable complex manifolds*, Proc. Amer. Math. Soc. **147** (2019), no. 6, 2413 - 2424.
- [R] R. Richberg, *Stetige streng pseudokonvexe Funktionen*, Math. Ann. **175** (1967), no. 4, 257 - 286.
- [Ros] H. Rossi, *Holomorphically convex sets in several complex variables*, Ann. of Math. (2) **74** (1961), no. 3, 470 - 493.
- [Rot] W. Rothstein, *Zur Theorie der analytischen Mannigfaltigkeiten im Räume von  $n$  komplexen Veränderlichen. (German)*, Math. Ann. **129** (1955), no. 1, 96 - 138.
- [Sh] N. Shcherbina, *On the polynomial hull of a graph*, Indiana Univ. Math. J. **42** (1993), no. 2, 477 - 503.
- [Si1] N. Sibony, *A class of hyperbolic manifolds*, Recent developments in several complex variables (Proc. Conf., Princeton Univ., Princeton, N. J., 1979), 357 - 372, Ann. of Math. Stud., **100**, Princeton Univ. Press, Princeton, N. J., 1981.
- [Si2] N. Sibony, *Levi problem in complex manifolds*, Math. Ann. **371** (2018), no. 3-4, 1047 - 1067.
- [Sl1] Z. Słodkowski, *Local maximum property and  $q$ -plurisubharmonic functions in uniform algebras*, J. Math. Anal. Appl. **115** (1986), no. 1, 105-130.
- [Sl2] Z. Słodkowski, *Pseudoconcave decompositions in complex manifolds*, Advances in complex geometry, 239 - 259, Contemp. Math., **735**, Amer. Math. Soc., Providence, RI, 2019.

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