On compacts possessing strictly plurisubharmonic functions

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Question

Let $\mathcal{M}$ be a complex manifold and $\mathcal{K} \subset \mathcal{M}$ be its compact subset. How to characterize (geometrically) compacts $\mathcal{K}$ which possess a smooth strictly plurisubharmonic function?

Here we need to specify what do we mean by a smooth strictly plurisubharmonic function of a general compact set.

Definition 1. Let $\mathcal{K}$ be a compact subset of a complex manifold $\mathcal{M}$. We say that a function ϕ defined on $\mathcal{K}$ is smooth and strictly plurisubharmonic if there is a neighbourhood $\mathfrak{A}$ of $\mathcal{K}$ in $\mathcal{M}$ and a smooth strictly plurisubharmonic function φ on $\mathfrak{A}$ such that $\varphi|_{\mathcal{K}} = \phi$.

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Motivations

Plurisubharmonic functions play a central role in complex analysis. Many important and classical results are formulated in terms of these functions, in particular, using the existence of strictly plurisubharmonic functions on a given manifold. For example, Grauert [G] characterized Stein manifolds by existence of smooth strictly plurisubharmonic exhaustion functions. This result was generalized to the case of complex spaces by Narasimhan in [N1] and [N2]. Sibony in [Si1, Theorem 3, p. 362] proved that the existence of a bounded smooth strictly plurisubharmonic function is sufficient for Kobayashi hyperbolicity of a complex manifold. A similar criterion for the existence of Bergman metric on Stein manifolds was established by Chen-Zhang in [CZ, Theorem 1, p. 2998, and observation 2, p. 3002]. Recently Poletsky [P] used manifolds possessing bounded smooth strictly plurisubharmonic functions to develop further the theory of pluricomplex Green functions.

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The next result gives a complete geometric characterization of compacts possessing such functions.

Main Theorem. *Let $\mathcal{K}$ be a compact subset of a complex manifold. Then $\mathcal{K}$ possesses a smooth strictly plurisubharmonic function if and only if $\mathcal{K}$ does not have 1-pseudoconcave subsets.*

1-pseudoconcavity here is understood in the sense of Rothstein [Rot]. By a complex manifold we will always mean a manifold of pure complex dimension which has a Hausdorff topology with a countable basis.

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Preliminaries

We recall first the notion of 1-*pseudoconvexity* in the sense of Rothstein. Let $\Delta^n := \{z \in \mathbb{C}^n : \|z\|_\infty < 1\}$, where $\|z\|_\infty = \max_{1\leq j\leq n}|z_j|$. An $(1, n-1)$ Hartogs figure H is a set of the form

$$H = \big\{(z_1, \cdots, z_n) \in \Delta^1 \times \Delta^{n-1} : |z_1| < r_1 \text{ or } \|(z_2, \cdots, z_n)\|_\infty > r_2\big\},$$

where $0 < r_1, r_2 < 1$, and we write $\hat{H} := \Delta^n$.

Definition 2. Let $\mathcal{M}$ be a complex manifold of dimension n. An open set $\Omega \subset \mathcal{M}$ is called 1-pseudoconvex in $\mathcal{M}$ if it satisfies the Kontinuitätssatz with respect to $(n-1)$-polydiscs in $\mathcal{M}$, i.e., if for every $(1, n-1)$ Hartogs figure and every injective holomorphic mapping $\Phi\colon \hat{H} \to \mathcal{M}$ such that $\Phi(H) \subset \Omega$ one has $\Phi(\hat{H}) \subset \Omega$.

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This definition was introduced by Rothstein [Rot] in a more general setting of q-pseudoconvex sets for every $q = 1, 2, \ldots, n-1$. We restrict our definition to the special case $q = 1$, since in the present paper we only need the notion of 1-pseudoconvexity.

Another way to define 1-pseudoconvexity can be described as follows. For an arbitrary $r \in (0,1)$ we consider a *spherical hat*

$$\mathbb{S}^n_r := \{z = (z_1, \ldots, z_n) \in \mathbb{C}^n : \|z\|^2 = 1,\ x_1 := \operatorname{Re} z_1 \geq r\}$$

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Definition 3. Let $\mathcal{M}$ be a complex manifold of dimension n. An open set $\Omega \subset \mathcal{M}$ is called 1-pseudoconvex in $\mathcal{M}$ if for every $r \in (0, 1)$, every neighbourhood $U := U(\hat{\mathbb{S}}_r^n) \subset \mathbb{C}^n$ of the filled spherical hat $\hat{\mathbb{S}}_r^n$ and every injective holomorphic mapping $\Phi \colon U \to \mathcal{M}$ such that $\Phi(\mathbb{S}_r^n) \subset \Omega$ one has $\Phi(\hat{\mathbb{S}}_r^n) \subset \Omega$.

The next statement shows that the above definitions give us the same notion.

Proposition 1. Let $\mathcal{M}$ be a complex manifold and $\Omega \subset \mathcal{M}$ be an open set. Then the following assertions are equivalent:

(1) Ω is 1-pseudoconvex in the sense of Definition 2.

(2) Ω is 1-pseudoconvex in the sense of Definition 3.

Now we recall the definition of 1-*pseudoconcavity* for closed sets.

Definition 4. Let $\mathcal{M}$ be a complex manifold and $A \subset \mathcal{M}$ be a closed set. Then A is called 1-pseudoconcave in $\mathcal{M}$ if $\mathcal{M} \setminus A$ is 1-pseudoconvex in $\mathcal{M}$.

Note that more equivalent descriptions of 1-pseudoconvex sets are known. For example, from Theorems 4.2 and 5.1 of Słodkowski [Sl1] it follows, in particular, that a nonempty relatively closed subset A of an open set $V \subset \mathbb{C}^n$ is 1-pseudoconcave in V if and only if plurisubharmonic functions have the local maximum property on A. We will need here an analogous statement in a more general setting of complex manifold.

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Proposition 2. Let $\mathcal{M}$ be a complex manifold and $A \subset \mathcal{M}$ be a closed set. Then the following assertions are equivalent:

(1) For every $\zeta \in A$, there exists a neighbourhood $V \subset \mathcal{M}$ of ζ such that $A \cap V$ is 1-pseudoconcave in V.

(2) A is 1-pseudoconcave in $\mathcal{M}$.

(3) For every $\zeta \in A$, there exists a neighbourhood $V \subset \mathcal{M}$ of ζ such that for every compact set $B \subset V$ and every plurisubharmonic function φ defined in a neighbourhood of B one has $\max_{A\cap B} \varphi \leq \max_{A\cap bB} \varphi$.

Here $\max_{A\cap bK} \varphi$ is meant to be $-\infty$ if $A \cap bK = \emptyset$.

A detailed proof of this statement (which follows the ideas of Słodkowski [Sl1]) can be found in a more general setting of q-pseudoconcave sets in [HST1, Proposition 3.3].

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The following result was proved in [HST2, Theorem 3.1, part 1]. Since it plays an important role in this paper, we will present it here in details for the reader convenience in a slightly different form adapted to the current presentation.

Theorem 1. There exists a domain W in $\mathbb{C}^n$ with coordinates $(z_1, z_2, \ldots, z_n)$, $z_j = x_j + iy_j$, and a smooth plurisubharmonic function $\varphi : W \to [0, +\infty)$ such that

(1) $\Pi_- := \{z \in \mathbb{C}^n : x_1 \leq 0\} \subset W$.

(2) $\varphi = 0$ on Π_-.

(3) $\varphi > 0$ on $W \setminus \Pi_-$.

(4) φ is strictly plurisubharmonic on $W \setminus \Pi_-$.

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Proof. For every $j \in \mathbb{N}$, let $\psi_j \colon \mathbb{B}^n_j(0) \to \mathbb{R}$ be the smooth and strictly plurisubharmonic function defined by

$$\psi_j(z_1, \ldots, z_n) := x_1 - \frac{1}{2^{j-2}} + \frac{1}{j^2 2^{j-1}} \left(y_1^2 + |z_2|^2 + \cdots + |z_n|^2 \right).$$

Choose a smooth function $\chi_j \colon \mathbb{R} \to [0, \infty)$ such that $\chi_j \equiv 0$ on $(-\infty, -1/2^j]$ and such that χ_j is strictly increasing and strictly convex on $(-1/2^j, \infty)$. Set $\widetilde{\varphi}_j := \chi_j \circ \psi_j$. Then $\widetilde{\varphi}_j$ is a smooth plurisubharmonic function on $\mathbb{B}^n_j(0)$ such that $\widetilde{\varphi}_j \equiv 0$ on $\{\psi_j \leq -1/2^j\} \supset \mathbb{B}^n_j(0) \cap \{x_1 \leq 1/2^j\}$ and such that $\widetilde{\varphi}_j$ is strictly plurisubharmonic and positive on $\{\psi_j > -1/2^j\} \supset \mathbb{B}^n_j(0) \cap \{x_1 > 3/2^j\}$. Thus

$$\varphi_j(z) := \begin{cases} \widetilde{\varphi}_j(z)\,, & z \in \mathbb{B}^n_j(0) \cap \{x_1 \geq 1/2^j\} \\ 0 \quad , & z \in \{x_1 < 1/2^j\} \end{cases}$$

is a smooth plurisubharmonic function on $W_j := \mathbb{B}^n_j(0) \cup \{x_1 < 1/2^j\}$ such that φ_j is strictly plurisubharmonic and positive on $\mathbb{B}^n_j(0) \cap \{x_1 > 3/2^j\}$. Observe that $W := \bigcap_{j=1}^{\infty} W_j$ is a connected open neighbourhood of $\{x_1 \leq 0\}$. Then one easily sees that for a sequence $\{\varepsilon_j\}_{j=1}^{\infty}$ of positive numbers that converges to zero fast enough, the function $\varphi := \sum_{j=1}^{\infty} \varepsilon_j \varphi_j$ is smooth and plurisubharmonic on W such that $\varphi \equiv 0$ on $\{x_1 \leq 0\}$ and such that φ is strictly plurisubharmonic and positive on $W \cap \{x_1 > 0\}$, which completes the proof of the theorem.

Corollary 1. For every $r \in (0, 1)$ there is a smooth nonnegative plurisubharmonic function φ_r defined on the domain $\Omega_r := \mathbb{C}^n \setminus \mathbb{S}^n_r$ such that

(1) φ_r is equal to 0 on the set $\Omega_r \setminus \hat{\mathbb{S}}^n_r$.

(2) φ_r is positive and strictly plurisubharmonic in the interior $\mathrm{Int}(\hat{\mathbb{S}}^n_r)$ of the set $\hat{\mathbb{S}}^n_r$.

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Proof. If for each $r \in (0,1)$ we define the function φ_r as

$$\varphi_r(z) := \begin{cases} \varphi((z_1 - r, z_2, \cdots, z_n)), & \text{for } z = (z_1, z_2, \cdots, z_n) \in \mathrm{Int}(\hat{\mathbb{S}}^n_r), \\ 0, & \text{for } z \in \Omega_r \setminus \hat{\mathbb{S}}^n_r, \end{cases}$$

where φ is the function constructed in Theorem 1, then it is straightforward to see that the function φ_r has all the desired properties.

Construction and properties of the set $\mathfrak{n}(\mathcal{K})$

Let $\mathcal{K}' \supset \mathcal{K}''$ be compact sets in a complex manifold $\mathcal{M}$ of dimension n. We say that the set $\mathcal{K}''$ is obtained from the set $\mathcal{K}'$ by a *spherical cut* if there exist $r \in (0,1)$, a neighbourhood $U := U(\hat{\mathbb{S}}^n_r) \subset \mathbb{C}^n$ of the filled spherical hat $\hat{\mathbb{S}}^n_r$ and an injective holomorphic mapping $\Phi\colon U \to \mathcal{M}$ such that $\Phi(\mathbb{S}^n_r) \subset \mathcal{M} \setminus \mathcal{K}'$ and $\mathcal{K}' \setminus \Phi(\mathrm{Int}(\hat{\mathbb{S}}^n_r)) = \mathcal{K}''$.

Proof. If for each $r \in (0,1)$ we define the function φ_r as

$$\varphi_r(z) := \begin{cases} \varphi((z_1 - r, z_2, \cdots, z_n)), & \text{for } z = (z_1, z_2, \cdots, z_n) \in \mathrm{Int}(\hat{\mathbb{S}}^n_r), \\ 0, & \text{for } z \in \Omega_r \setminus \hat{\mathbb{S}}^n_r, \end{cases}$$

where φ is the function constructed in Theorem 1, then it is straightforward to see that the function φ_r has all the desired properties.

Construction and properties of the set $\mathfrak{n}(\mathcal{K})$

Let $\mathcal{K}' \supset \mathcal{K}''$ be compact sets in a complex manifold $\mathcal{M}$ of dimension n. We say that the set $\mathcal{K}''$ is obtained from the set $\mathcal{K}'$ by a *spherical cut* if there exist $r \in (0,1)$, a neighbourhood $U := U(\hat{\mathbb{S}}^n_r) \subset \mathbb{C}^n$ of the filled spherical hat $\hat{\mathbb{S}}^n_r$ and an injective holomorphic mapping $\Phi : U \to \mathcal{M}$ such that $\Phi(\mathbb{S}^n_r) \subset \mathcal{M} \setminus \mathcal{K}'$ and $\mathcal{K}' \setminus \Phi(\mathrm{Int}(\hat{\mathbb{S}}^n_r)) = \mathcal{K}''$.

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Further, for a pair of compact sets $\mathcal{K}' \supset \mathcal{K}''$ in $\mathcal{M}$ we say that the set $\mathcal{K}''$ is obtained from the set $\mathcal{K}'$ by a *finite sequence of spherical cuts* if there exists a finite decreasing sequence $\mathcal{K}_1 \supset \mathcal{K}_2 \supset \cdots \supset \mathcal{K}_m$ of compact sets in $\mathcal{M}$ such that $\mathcal{K}_1 = \mathcal{K}'$, $\mathcal{K}_m = \mathcal{K}''$ and for each $j = 2, 3, \cdots, m$ the set $\mathcal{K}_j$ is obtained from the set $\mathcal{K}_{j-1}$ by a spherical cut.

Then, for a given compact set $\mathcal{K}$ in $\mathcal{M}$, we can consider the family $\mathcal{F}_{\mathcal{K}}$ of compact subsets of $\mathcal{K}$ defined by

$$\mathcal{F}_{\mathcal{K}} := \{\mathcal{K}_\alpha : \mathcal{K}_\alpha \text{ is obtained from } \mathcal{K} \text{ by a finite sequence of spherical cuts}\}_{\alpha \in \mathcal{A}},$$

where $\mathcal{A}$ is a parameter set of this family.

The next statement follows easily from the definition of $\mathcal{F}_{\mathcal{K}}$.

Lemma 1. Let $\mathcal{K}_{\alpha_1}, \mathcal{K}_{\alpha_2}, \cdots, \mathcal{K}_{\alpha_m}$ be a finite set of compacts from $\mathcal{F}_{\mathcal{K}}$, then one also has that $\bigcap_{j=1}^{m} \mathcal{K}_{\alpha_j} \in \mathcal{F}_{\mathcal{K}}$.

Further, for a pair of compact sets $\mathcal{K}' \supset \mathcal{K}''$ in $\mathcal{M}$ we say that the set $\mathcal{K}''$ is obtained from the set $\mathcal{K}'$ by a *finite sequence of spherical cuts* if there exists a finite decreasing sequence $\mathcal{K}_1 \supset \mathcal{K}_2 \supset \cdots \supset \mathcal{K}_m$ of compact sets in $\mathcal{M}$ such that $\mathcal{K}_1 = \mathcal{K}'$, $\mathcal{K}_m = \mathcal{K}''$ and for each $j = 2, 3, \cdots, m$ the set $\mathcal{K}_j$ is obtained from the set $\mathcal{K}_{j-1}$ by a spherical cut.

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Now we can define the set $\mathfrak{n}(\mathcal{K})$ which plays a special role in the present article and will be called in what follows the *nucleus* of $\mathcal{K}$:

$$\mathfrak{n}(\mathcal{K}) := \bigcap_{\mathcal{K}_\alpha \in \mathcal{F}_{\mathcal{K}}} \mathcal{K}_\alpha.$$

The most important for us properties of this set are given in the next statement.

Theorem 2. The set $\mathfrak{n}(\mathcal{K})$ is 1-pseudoconcave. Moreover, $\mathfrak{n}(\mathcal{K})$ is the maximal 1-pseudoconcave subset of the set $\mathcal{K}$.

Proof of the Main Theorem

In order to complete the proof of the Main Theorem we distinguish two cases.

Case 1. $\mathfrak{n}(\mathcal{K}) \neq \emptyset$.

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Applications and open questions

1. Kobayashi hyperbolicity.

As a direct consequence of our Main Theorem and the mentioned above result of Sibony we get the following statement.

Theorem 3. Let $\mathcal{K}$ be a compact subset of a complex manifold $\mathcal{M}$. If $\mathcal{K}$ does not have 1-pseudoconcave subsets, then there is a neighbourhood $\mathfrak{U}$ of $\mathcal{K}$ in $\mathcal{M}$ which is Kobayashi hyperbolic.

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2. The core of a compact.

The notion of the core of a complex manifold was introduced and systematically studied by Harz-Shcherbina-Tomassini in [HST1] - [HST4]. Further results on the foliated structure of the core were obtained by Poletsky-Shcherbina in [PS] and Słodkowski in [Sl2].

In the same vein we can define a notion of the core in the setting of the present paper.

Definition 5. Let $\mathcal{K}$ be a compact subset of a complex manifold $\mathcal{M}$. Then the set

$$\mathfrak{c}(\mathcal{K}) := \{\zeta \in \mathcal{K} : \text{every function which is smooth and plurisubharmonic on a neighbourhood of } \mathcal{K} \text{ in } \mathcal{M} \text{ fails to be strictly plurisubharmonic in } \zeta\}$$

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This definition obviously implies that the set $\mathfrak{c}(\mathcal{K})$ is compact. Since, by the argument used in the proof of the Main Theorem in Case 2, for each point $\zeta \notin \mathfrak{n}(\mathcal{K})$ there is a smooth plurisubharmonic function defined in a neighbourhood of $\mathcal{K}$ which is strictly plurisubharmonic in ζ, the following property holds true:

Theorem 4. Let $\mathcal{K}$ be a compact subset of a complex manifold $\mathcal{M}$. Then $\mathfrak{c}(\mathcal{K}) \subset \mathfrak{n}(\mathcal{K})$.

We do not know if the reverse statement holds true or not:

Question 1. Let $\mathcal{K}$ be a compact subset of a complex manifold $\mathcal{M}$. Is it always true that $\mathfrak{n}(\mathcal{K}) \subset \mathfrak{c}(\mathcal{K})$?

3. On the structure of the nucleus.

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In general it is difficult to give a reasonable description of the nucleus of a compact set $\mathcal{K} \subset \mathcal{M}$ even in the case when $\mathcal{K}$ is a smooth submanifold of $\mathcal{M}$. Surprisingly, in the case when the dimension of $\mathcal{M}$ is equal to 2 and $\mathcal{K}$ is a hypersurface (even not necessarily smooth) in $\mathcal{M}$, the set $\mathfrak{n}(\mathcal{K})$ will have a very special structure:

Theorem 5. Let $\mathcal{M}$ be a complex manifold of dimension 2 and let $\mathcal{K}$ be a continuous hypersurface of the graph type in $\mathcal{M}$. Then locally the set $\mathfrak{n}(\mathcal{K})$ is a disjoint union of holomorphic discs.

Here $\mathcal{K}$ *is a continuous hypersurface of the graph type in* $\mathcal{M}$ means that for every point $\zeta \in \mathcal{K}$ there is a neighbourhood Ω in $\mathcal{M}$ and local holomorphic coordinates (z, w) in Ω such that $\mathcal{K} \cap \Omega = \{(z, w) \in B_r^3(0) \times \mathbb{R}_v : v = h(z, u)\} =: \Gamma_h$ – the graph of a continuous function $h : B_r^3(0) \to \mathbb{R}_v$, where $B_r^3(0) := \{(z, u) \in \mathbb{C}_z \times \mathbb{R}_u : |z|^2 + u^2 < r^2\}$ and $w = u + iv$.

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