# Analytic continuation of diagonals of Laurent series for rational functions

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Consider a Laurent series

$$F(\mathbf{z}) = \sum_{\boldsymbol{\alpha} \in \mathbb{Z}^n} c_{\boldsymbol{\alpha}} \mathbf{z}^{\boldsymbol{\alpha}} \tag{1}$$

for a rational function

$$F(\mathbf{z}) = \frac{P(\mathbf{z})}{Q(\mathbf{z})}$$

of $n$ complex variables.

For irreducible $\boldsymbol{q} = (q_1, \ldots, q_n) \in \mathbb{Z}^n \setminus \{0\}$, the series

$$d_{\boldsymbol{q}}(t) = \sum_{k=-\infty}^{\infty} C_{\boldsymbol{q}\cdot k} t^k$$

is called a *complete* $\boldsymbol{q}$-*diagonal* of the Laurent series (1).

The *amoeba* of a polynomial $Q$ is the image of a hypersurface $Z^{\times}(Q)$ under the logarithmic mapping

$$\Lambda : \left(\mathbb{C}^{\times}\right)^n \to \mathbb{R}^n$$

defined by

$$\Lambda(z) = (\log|z_1|, \ldots, \log|z_n|),$$

where $Z^{\times}(Q)$ is defined in the complex torus $\left(\mathbb{C}^{\times}\right)^n$ by zeroes of $Q$.

**Theorem (P., 2009)**
*Let the Laurent series (1) for a bivariate rational function converge in the domain* $\Lambda^{-1}(E)$*. Then its complete* $\boldsymbol{q}$*-diagonal is an algebraic function.*

For the Laurent series (1) that is convergent in the domain $\Lambda^{-1}(E)$, one has the integral representation

$$d_{\boldsymbol{q}}(t) = \frac{1}{(2\pi\imath)^n} \int_\Gamma \frac{P(\boldsymbol{z})}{Q(\boldsymbol{z})} \frac{\boldsymbol{z}^{\boldsymbol{q}}}{\boldsymbol{z}^{\boldsymbol{q}} - t} \frac{dz_1 \wedge \ldots \wedge dz_n}{z_1 \ldots z_n},$$

where $\Gamma = \Lambda^{-1}(\boldsymbol{y}_2) - \Lambda^{-1}(\boldsymbol{y}_1)$ is the $n$-dimensional cycle in the complement $(\mathbb{C}^\times)^n \setminus (S_1 \cup S_2)$, where $S_1 = Z^\times(Q)$ and $S_2 = Z^\times(\boldsymbol{z}^{\boldsymbol{q}} - t)$.

The *logarithmic Gauss mapping*

$$\gamma_Q : \operatorname{reg} Z^{\times}(Q) \to \mathbb{CP}^{n-1}$$

defined as

$$\gamma_Q(\boldsymbol{z}) = \left( z_1 \frac{\partial Q}{\partial z_1}(\boldsymbol{z}) : \ldots : z_n \frac{\partial Q}{\partial z_n}(\boldsymbol{z}) \right)$$

in regular points $\boldsymbol{z}$ of the hypersurface $Z^{\times}(Q)$.

**Theorem (P., 2021)**
*Let the Laurent series (1) for a rational function of n variables converge in the domain* $\Lambda^{-1}(E)$*, and let* $d_{\boldsymbol{q}}(t)$ *be its complete* $\boldsymbol{q}$*-diagonal. If* $\boldsymbol{q} = \gamma_Q(\boldsymbol{p})$*, where the point* $\boldsymbol{p}$ *is regular for the logarithmic Gauss mapping and* $\Lambda(\boldsymbol{p}) \in \partial E$*, then*

1. *In the case* $n = 2k$ *the point* $t_0 = \boldsymbol{p}^{\boldsymbol{q}}$ *is a branch point of finite order 2 of* $d_{\boldsymbol{q}}(t)$*.*
2. *In the case* $n = 2k + 1$ *the point* $t_0 = \boldsymbol{p}^{\boldsymbol{q}}$ *is a branch point of infinite order (logarithmic branch point) of* $d_{\boldsymbol{q}}(t)$*.*