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Blow-ups for the Horn-Kapranov parametrization of the classical discriminant

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The classical discriminant relates to a polynomial in one variable

$$f(y) = a_0 + a_1 y + \ldots + a_n y^n$$

is an irreducible polynomial,

$$\Delta_n = \Delta_n(a_0, a_1, \ldots, a_n)$$

with integer coefficients that vanishes if and only if $f(y)$ has multiple roots.

The Newton polytope of a polynomial is the convex hull of the exponents set of the monomials.

[Gelfand, Kapranov and Zelevinsky, 1994]:
Newton polytope $\mathcal{N}(\Delta_n)$ *of the indicated discriminant is combinatorially equivalent to an* $(n-1)$*-dimensional cube. Since such a cube has* 2^{n-1} *vertices, it is natural to encode these vertices by all possible subsets of the set* $\{1, 2, \ldots, n-1\}$*.*

Example.

$$f(y)^{red} = 1 + a_1 y + a_2 y^2 + y^3.$$

The discriminant is equal to

$$\Delta_3^{red}(a_1, a_2) = -27 - 4a_1^3 - 4a_2^3 + a_1^2 a_2^2 + 18 a_1 a_2.$$

In the case there are four subdivisions:

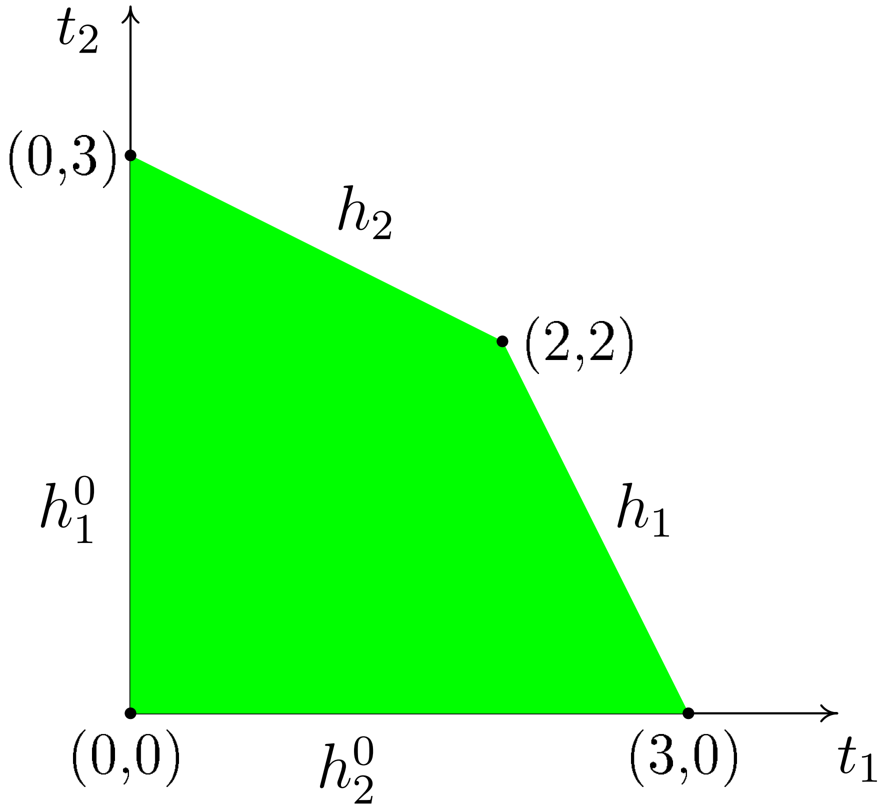
$$I_0 = \varnothing,\ I_1 = \{1\},\ I_2 = \{2\},\ I_3 = \{1, 2\}.$$

with associated vertices

$$(0, 0), (3, 0), (0, 3), (2, 2),$$

and corresponding monomials are

$$-27,\ -4a_1^3,\ -4a_2^3,\ a_1^2 a_2^2.$$



t_2
(0,3)
h_2
(2,2)
h_1^0
h_1
(0,0)
h_2^0
(3,0)
t_1

Since the discriminant is known to be bihomogeneous, the polytope $\mathcal{N}(\Delta_n)$ lies in a plane of $\mathbb{R}^{n+1}$ of codimension 2 defined by two equations

$$\sum_{j=0}^{n} t_j = 2(n-1), \quad \sum_{j=1}^{n} j t_j = n(n-1).$$

Lemma 1. *Within the indicated plane the Newton polytope* $\mathcal{N}(\Delta_n)$ *is defined by* $2(n-1)$ *inequalities*

$$t_k \geq 0, \quad k = 1, \dots, n-1,$$

$$\sum_{j=1}^{k} (n-k) j t_j + \sum_{j=k+1}^{n-1} k(n-j) t_j \leq nk(n-k),$$

$k = 1, \dots, n-1.$

The face h_k has 2^{n-2} vertices and lies in the hyperplane

$$F_k = \{t \in \mathbb{R}^{n-1} :\ \langle \mu^{(k)}, t \rangle = nk(n-k)\},$$

whose normal vector $\mu^{(k)}$ has the coordinates

$$(n-k)\cdot 1, \ldots, (n-k)\cdot k, k\cdot(n-k-1), \ldots, k\cdot 1.$$

Main objects of interest in this report are truncations of the discriminant Δ to the faces of its Newton polytope. Recall that the *truncation* of a polynomial Δ to a face h of its Newton polytope $\mathcal{N}(\Delta)$ is the sum of all monomials from Δ whose exponents belong to h. This truncation is denoted by $\Delta\big|_h$.

Example. As an example we calculate the truncation of the polynomial

$$f(y) = a_0 + a_1 y + a_2 y^2 + a_3 y^3.$$

The discriminant Δ_3 is equal to

$$\Delta_3 = -27a_3^2a_0^2 + 18a_3a_0a_1a_2 + a_2^2a_1^2 - 4a_2^3a_0 - 4a_1^3a_3.$$

Since inequalities for the face h_1 are the following:

$$t_1 \geq 0, t_2 \geq 0, t_1 + 2t_2 \leq 6, 2t_1 + t_2 = 6,$$

then the truncation $\Delta_3\big|_{h_1}$ is equal to

$$\Delta_3\big|_{h_1} = a_1^2a_2^2 - 4a_1^3a_3.$$

In the report we will give a plan of proof to formulae for the truncations of Δ_n to faces

$$h_K := h_{k_1} \cap \ldots \cap h_{k_p},$$

obtained by intersection p non-coordinate hyper-faces. The multi-index $K = \{k_1, \ldots, k_p\}$ determines a partition of the tuple $\{0, 1, \ldots, n\}$ into $p+1$ subtuples

$$K_i = \{k_i, k_i + 1, \ldots, k_{i+1}\},\ \ i = 0, 1, \ldots, p,$$

assuming that $k_0 = 0$ and $k_{p+1} = n$. Let $l_i := k_{i+1} - k_i$ denote the length of K_i and

$$f_{K_i} := a_{k_i} + a_{k_i+1} y + \ldots + a_{k_{i+1}} y^{l_i}.$$

Main result

Theorem. *In previous notations, the truncation of* Δ_n *to* h_K *admits the following factorization*

$$\Delta_n\big|_{h_K} = a_K^2 \prod_{i=0}^{p} \Delta_{l_i}(f_{K_i}), \tag{1}$$

where $a_K^2 = a_{k_1}^2 \dots a_{k_p}^2$, *and* Δ_{l_i} *are the discriminants of polynomials of degrees* l_i.

Our proof of the theorem is based on Horn-Kapranov parametrization for the reduced discriminant set

$$\nabla_n^{red} := \{x \in \mathbb{C}^{n-1} : \Delta_n^{red}(x) = 0\},$$

This parametrization is n-valued and it is given by

$$x_j = -\frac{ns_j}{\langle \alpha, s\rangle}\left(\frac{\langle \alpha, s\rangle}{\langle \beta, s\rangle}\right)^{\frac{j}{n}}, \quad j = 1, \ldots, n-1, \qquad (2)$$

where β and α are integer vectors

$$(1, 2, \ldots, n-1) \quad and \quad (n-1, n-2, \ldots, 1)$$

respectively, and $s = (s_1 : \ldots : s_{n-1})$ is a parameter from $\mathbb{CP}^{n-2} \setminus \{s : \langle \alpha, s \rangle \langle \beta, s \rangle = 0\}$.

The complete discriminant set $\nabla_n = \{\Delta_n(a) = 0\}$ admits the parametrization

$$\begin{cases} a_0 = s_0, \\ a_j = -\frac{n s_j}{\langle \alpha, s \rangle} s_0 \left(\frac{s_n}{s_0} \frac{\langle \alpha, s \rangle}{\langle \beta, s \rangle} \right)^{\frac{j}{n}}, \quad j = 1, \ldots, n-1, \\ a_n = s_n, \end{cases} \quad (3)$$

Lemma 2. *The truncation of* $\widehat{\Delta}_n$ *to the face* h_K *admits the factorization*

$$\widehat{\Delta}_n|_{h_K} = a_K^2 \prod_{i=0}^{p} \widehat{\Delta}_{l_i}(f_{K_i}),$$

where $a_K^2 = a_{k_1}^2 \dots a_{k_p}^2$*, and* Δ_{l_i} *are the discriminants of polynomials of degrees* l_i*.*

Now we describe the relationship between the reduced discriminant and its truncation to the face $h_K = h_{k_1} \cap \ldots \cap h_{k_p}$. Let $x = (x_1, \ldots, x_{n-1})$ and $\Delta_n^{red}(x)$ is a reduced discriminant.

The normal space to the face h_K is generated by vectors $\mu^{(k_1)}, \ldots, \mu^{(k_p)}$. Let μ be the $p \times (n-1)$-matrix composed by the rows $\mu^{(k_j)}$. Denote by $\mu_{(1)}, \ldots, \mu_{(n-1)}$ the columns of μ.

In the complex algebraic torus $(\mathbb{C} \setminus 0)^{n-1}$ with coordinates x consider a complex p-dimensional surface $x = \tau^{\mu}$, $\tau \in (\mathbb{C} \setminus 0)^{p}$ where the matrix monomial τ^{μ} means the usual monomial map $\tau \to x$:

$$x_j = \tau^{\mu(j)} = \tau_1^{\mu_{(j)}^{(1)}} \dots \tau_p^{\mu_{(j)}^{(p)}}, \quad j = 1, \dots, n-1.$$

Define the function

$$H_K(\tau;x) := \tau_1^{d_1}\dots\tau_p^{d_p}\cdot\Delta_n^{red}\left(\frac{x_1}{\tau^{\mu_{(1)}}},\dots,\frac{x_{n-1}}{\tau^{\mu_{(n-1)}}}\right), \qquad (4)$$

where d_i is the weighted degree of Δ_n^{red} respective the weight $\mu_{(i)}$, i.e. the maximum among the scalar products $\langle\lambda,\mu^{(i)}\rangle$, when λ runs over all exponents of monomials from $\Delta_n^{red}(x)$.

Lemma 3. *When* $\tau \to 0$, *the function* $H_K(\tau; x)$ *converges to the truncation of the discriminant to* h_K :

$$H_K(\tau; x) \underset{\tau \to 0}{\longrightarrow} \Delta_n^{red}(x)\big|_{h_K}.$$

Remark, at the points of the union of hyperplanes

$$\{\langle\alpha, s\rangle = 0\} \cup \{\langle\beta, s\rangle = 0\} \subset \mathbb{CP}^{n-2},$$

where $s_j \neq 0$, (2) does not take finite value. However at uncertainly points, where some coordinate functions s_j vanish simultaneously with at least one of the forms $\langle\alpha, s\rangle$, $\langle\beta, s\rangle$, this parametrization gives limit positions for the discriminant set.

According to the theory of correspondences these limit positions are interpreted as blow-ups in the space $\mathbb{CP}_s^{n-2} \times \mathbb{C}_x^{n-1}$, which contains the graph of the map (2).

It suffices to prove the assertion of the theorem for the discriminant

$$\Delta_n^{red}(x_1, \ldots, x_{n-1}) = \Delta_n(1, x_1, \ldots, x_{n-1}, 1):$$

$$\Delta_n^{red}(x)\big|_{h_K} = a_K^2 \Delta_{l_0}(1, x_1, \ldots, x_{k_1}) \left[\prod_{i=1}^{p-1} \Delta_{l_i}(f_{K_i})\right] \times \Delta_{l_p}(x_{k_p}, \ldots, x_{n-1}, 1) \qquad (5)$$

where the first and the last discriminants Δ_{l_0} and Δ_{l_p} correspond to the "half-reduced"polynomial.

Lemma 4. *In equality (5) the zero set of the truncation* $\Delta_n^{red}\big|_{h_K}$ *on the left-hand side contains the zero of the product on the right-hand side.*

To prove the Lemma 4, we examen the zero sets

$$Z_\tau = \{x \in \mathbb{C}^{n-1} : \; H_K(\tau; x) = 0\}, \quad \tau \neq 0$$

of the function (4). According to (2), these sets admit the parametrization

$$x_j = -\tau^{\mu(j)} \frac{n s_j}{\langle \alpha, s\rangle} \left(\frac{\langle \alpha, s\rangle}{\langle \beta, s\rangle} \right)^{\frac{j}{n}}, \quad j = 1, \ldots, n-1. \quad (6)$$

We have to prove that the zero set of the truncation to h_K for the reduced n-discriminant contains the zero sets of two extreme (in the right-hand side of (5)) semi-reduced discriminants and of each complete discriminant Δ_{l_i}, $1 \le i \le p-1$:

1) $K_0 = \{0, 1, \ldots, k_1\}$,

2) $K_i = \{k_i, k_i + 1, \ldots, k_{i+1}\}, \quad 1 \le i \le p-1$,

3) $K_p = \{k_p, k_p + 1, \ldots, k_n\}$.

1) $K_0 = \{0, 1, \ldots, k_1\}$

In view of (6) one has

$$x_j = (-1)^{1-\frac{j}{k_1}} \frac{n^{1-\frac{j}{k_1}} s_j}{\langle \alpha, s \rangle} \left(\frac{x_{k_1} \langle \alpha, s \rangle}{s_{k_1}} \right)^{j/k_1},$$

$j = 1, \ldots, k_1 - 1.$

Let us define in the projective space with homogeneous coordinates $(s_1 : \ldots : s_{n-1})$ the plane σ' by equations

$$\langle \beta, s \rangle = 0, \; s_{k_1+1} = \ldots = s_{n-1} = 0.$$

Computations give

$$x_j\big|_{\sigma'} = -\frac{k_1 s_j}{\langle \alpha', s' \rangle}\left(\frac{s_{k_1}\langle \alpha', s' \rangle}{\langle \beta', s' \rangle}\right)^{j/k_1},$$

$j = 1, \ldots, k_1 - 1;\ x_{k_1} = s_{k_1}$.

This parametrization of semi-reduced discriminant set of the equation $1 + x_1 y + \ldots + x_{k_1} y^{k_1} = 0$.

Thus, we conclude that, under the projection onto the subspace of coordinates $x_1, \ldots, x_{k_1}$ the limit (as $\tau \to 0$) set of the left-hand side of (5) is mapped into the discriminant set

$\{\Delta_{l_0}(1, x_1, \ldots, x_{k_1}) = 0\}$.

2) $K_i = \{k_i, k_i + 1, \ldots, k_{i+1}\}, \quad 1 \le i \le p-1,$

Computations give

$$x_{k_i+t} = x_{k_i} \frac{s_{k_i+t}}{s_{k_i}} \Big(\frac{x_{k_{i+1}}}{x_{k_i}} \frac{s_{k_i}}{s_{k_{i+1}}} \Big)^{\frac{t}{l_i}}.$$

Consider the restriction of x_{k_i+t} on the plane

$$\sigma'' = \{\langle \alpha, s\rangle = \langle \beta, s\rangle = 0, \quad s_j = 0 \ \ \forall j \notin K_i\}.$$

Finally, for $t = 1, \ldots, l_i - 1$ we get

$$x_{k_i+t} = -\frac{l_i s_{k_i+t}}{\langle \alpha'', s'' \rangle} s_{k_i} \left(\frac{s_{k_{i+1}}}{s_{k_i}} \frac{\langle \alpha'', s'' \rangle}{\langle \beta'', s'' \rangle} \right)^{t/l_i},$$

$x_{k_i} = s_{k_i}$, $x_{k_{i+1}} = s_{k_{i+1}}$, where

$$s'' = (s_{k_i}, \ldots s_{k_{i+1}}),$$

$$\beta'' = (1, 2, \ldots, l_i - 1), \quad \alpha'' = (l_i - 1, \ldots, 2, 1).$$

By (3) under the projection onto subspace of coordinates $x_{k_i}, x_{k_i+1}, \dots, x_{k_{i+1}}$ the limit zero set of the left-hand side of (5) is mapped into discriminant set $\{\Delta_{l_i}(f_{K_i}) = 0\}$.

Equality (5) is proved as follows. By Lemma 1, the Newton polytopes of the polynomials on the left and right-hand sides of (5) coincide. In thise case, intersection theory and Lemma 4 imply that these polynomials have identical zero sets. However, by Lemma 1, their extremal parts coincide, so the polynomials coincide as well.

Theorem. *The discriminant of the polynomial*

$$x_0 + x_1 y^{n_1} + \ldots + x_s y^{n_s} + x_{s+1} y^n$$

can be written down as

$$(-1)^{\frac{n(d-1)}{2}} d^n \cdot (x_0 x_{s+1})^{d-1}$$

$$\times \Big[\Delta(x_0 + x_1 z^{m_1} + \ldots + x_s z^{m_s} + x_{s+1} z^m) \Big]^d,$$

where $m_k := \frac{n_k}{d}$, $m = \frac{n}{d}$, $d = GCD(n_1, \ldots, n_s, n)$.