Analytical realization of the strong dual of a space of holomorphic functions with boundary smoothness. Applications.

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On the problem

Let Ω be a convex bounded domain in $\mathbb{C}^n$, $A^\infty(\Omega)$ be the space of functions f holomorphic in Ω and s.t. all their partial derivatives $D^\alpha f$ can be continuously extended on $\overline{\Omega}$ with a topology defined by a system of norms $q_m(f) := \sup\limits_{z\in\Omega, |\alpha|\le m} |(D^\alpha f)(z)|$, $m \in \mathbb{N}$.

Let $\mathcal{H} = \{h_m\}_{m=1}^\infty$ be a family of convex functions $h_m : \mathbb{R}^n \to [0,\infty)$ with $h_m(0) = 0$ s.t. for each $m \in \mathbb{N}$:

i_1). $h_m(x) = h_m(|x_1|, \dots, |x_n|)$, $x = (x_1, \dots, x_n) \in \mathbb{R}^n$;

i_2). $\exists\, a_m > 0$ s.t. $h_m(x) \ge \|x\| \ln(1 + \|x\|) - a_m\|x\| - a_m$, $x \in \mathbb{R}^n$;

i_3). $\lim\limits_{x\to\infty} (h_m(x) - h_{m+1}(x)) = +\infty$;

i_4). $\sup\limits_{\alpha\in\mathbb{Z}_+^n} (h_{m+1}(\alpha + \beta) - h_m(\alpha)) < \infty$ for $\beta \in \mathbb{Z}_+^n$ with $|\beta| = 1$;

i_5). $\forall\, p \in \mathbb{N}\ \exists\, l = l(m,p) \in \mathbb{N}$: $\sum\limits_{\alpha\in\mathbb{Z}_+^n} e^{\max\limits_{|\beta|\le p} h_{m+l}(\alpha+\beta) - h_m(\alpha)} < \infty$.

For each $m \in \mathbb{N}$ let

$$A_m(\Omega) = \{f \in A^\infty(\Omega) : p_m(f) = \sup_{z \in \Omega, \alpha \in \mathbb{Z}_+^n} \frac{|(D^\alpha f)(z)|}{e^{h_m(\alpha)}} < \infty\}. \quad (1)$$

Denote by $A_{\mathcal{H}}(\Omega)$ a projective limit of the spaces $A_m(\Omega)$. $A_{\mathcal{H}}(\Omega)$ is a Fréchet space which is continuosly embedded in the space $A^\infty(\Omega)$. Note that in view of the condition i_4) the space $A_{\mathcal{H}}(\Omega)$ is invariant under differentiation. In view of the condition i_3) the space $A_{m+1}(\Omega)$ is continuosly embedded in $A_m(\Omega)$ for each $m \in \mathbb{N}$.

For each $z \in \mathbb{C}^n$ the function $f_z(\lambda) = e^{\langle \lambda, z \rangle}$ belongs to $A^\infty(\Omega)$. Also, $f_z \in A_{\mathcal{H}}(\Omega)$ (Lemma 1). So for each linear continuous functional Φ on $A^\infty(\Omega)$ ($A_{\mathcal{H}}(\Omega)$) the function $\hat{\Phi}(z) = \Phi(e^{\langle \lambda, z \rangle})$ is correctly defined in $\mathbb{C}^n$. It is called the Laplace transform of Φ.

Our aim is to study the problem of description of the strong dual $A^*_{\mathcal{H}}(\Omega)$ of the space $A_{\mathcal{H}}(\Omega)$ in terms of the Laplace transforms of functionals.

There is some novelty in the setting of the problem. Earlier it was considered for a projective limit $A_{\mathfrak{M}}(\Omega)$ of normed spaces

$$A_m(\Omega) = \{f \in A^\infty(\Omega) : p_m(f) = \sup_{z\in\Omega, \alpha\in\mathbb{Z}_+^n} \frac{|(D^\alpha f)(z)|}{M_{|\alpha|}^{(m)}} < \infty\}, \quad (2)$$

constructed with a help of a family $\mathfrak{M} = \{M^{(m)}\}_{m=1}^\infty$ of logarithmically convex (log.c.) sequences $M^{(m)} = (M_k^{(m)})_{k=0}^\infty$.

In particular, this problem was studied by B.A. Derjavets (1980-s) in assumption that $\partial\Omega$ is C^2-smooth and the family $\mathfrak{M} = \{M^{(m)}\}_{m=1}^\infty$ of log.c. sequences $M^{(m)} = (M_k^{(m)})_{k=0}^\infty$ is s.t.:

$\beta_1)$ the sequence $(L_k^{(m)} = \frac{M_k^{(m)}}{k!})_{k=0}^\infty$ is increasing and log.c.,

$\beta_2)$ $\sup\limits_{k\in\mathbb{N}} \left(\frac{M_{k+1}^{(m)}}{M_k^{(m)}}\right)^{\frac{1}{k}} < +\infty$; $\beta_3)$ $\lim\limits_{k\to\infty} \left(\frac{M_k^{(m)}}{k!}\right)^{\frac{1}{k}} = +\infty$;

$\beta_4)$ $\lim\limits_{k\to\infty} \frac{Q^k M_k^{(m+1)}}{M_k^{(m)}} = 0 \ \ \forall\, Q > 1$;

β_5) functions $v_m(x) = \inf\limits_{k \in \mathbb{Z}_+} L_k^{(m)} x^{k-1}$ $(x > 0, m \in \mathbb{Z}_+)$ satisfy the condition $\sup\limits_{x>0} \dfrac{v_{m+1}(x)}{x^2 v_m(x)} < \infty$ $(m \in \mathbb{Z}_+)$.

Remark. It was shown in [Musin I.Kh. // Advances in Mathematics Research. 2002. Nova Science Publishers] that conditions of C^2-smoothness of $\partial\Omega$ and $\beta_1), \beta_5)$ are excessive.

In our terms this means that B.A. Derjavets considered a case of the family $\mathcal{H}$ of functions h_m: $h_m(x) = g_m(\sum\limits_{j=1}^{n} |x_j|)$, where $g_m : \mathbb{R} \to [0, \infty)$ is a convex function, $g_m(0) = 0$ s.t. $\forall\, m \in \mathbb{N}$:

1). $g_m(t) = g_m(|t|)$, $t \in \mathbb{R}$;
2). g_m is nondecreasing on $[0, \infty)$;
3). $\forall\, M > 0$ $\exists\, Q > 0$ s.t. $g_m(t) \geq t \ln(t+1) + Mt - Q$, $t \geq 0$;
4). $\exists\, C_m > 0$ s.t. $g_m(t+1) - g_m(t) \leq C_m t$, $t \geq 1$;
5). $\lim\limits_{x \to +\infty} (g_m(t) - g_{m+1}(t) - Mt) = +\infty$ for each $M > 0$;
6). the sequence $\left(\dfrac{\exp(g_m(k))}{k!}\right)_{k=0}^{\infty}$ is logarithmically convex;

In [I.Kh. Musin // Vladikavkaz. Mat. Zh. 22 (3) (2020)] this problem was considered for a case of $\mathcal{H}$ consisting of functions h_m defined by the rule: $h_m : x = (x_1, \dots, x_n) \in \mathbb{R}^n \to h(\sum\limits_{j=1}^{n} |x_j| - m)$ if $\sum\limits_{j=1}^{n} |x_j| > m$; $h_m(x) = 0$ if $\sum\limits_{j=1}^{n} |x_j| \le m$ $(m \in \mathbb{N})$, where
$h : \mathbb{R} \to [0, \infty)$ is a convex function with $h(0) = 0$ such that:
1). $h(t) = h(|t|),\ t \in \mathbb{R}$;
2). h is nondecreasing on $[0, \infty)$;
3). $\exists\, a > 0$ s.t. $h(t) \ge t \ln(t + 1) - at - a,\ t \ge 0$.

All the conditions $i_1) - i_5)$ in this concrete situation are fulfilled.

Note that this case corresponds to the family $\mathfrak{M} = \{M^{(m)}\}_{m=1}^{\infty}$ of sequences $M^{(m)} = (M_k^{(m)})_{k=0}^{\infty}$, where numbers $M_k^{(m)}$ defined by the rule: $M_k^{(m)} = M_{k-m}$ if $k \ge m$ and $M_k^{(m)} = 1$ if $k < m$ and $(M_k)_{k=0}^{\infty}$ is a log.c. sequence such that: $M_0 = 1$,
$\exists Q_1 > 0\ \exists Q_2 > 0\ \ M_k \ge Q_1 Q_2^k k!\ \ (k \in \mathbb{Z}_+)$.

Note that if the restriction of h_m on $[0, \infty)^n$ $(m \in \mathbb{N})$ is nondecreasing in each variable then the condition i_5) can be replaced by the following one:
i_5'). $\forall m, \nu \in \mathbb{N} \quad \exists l = l(m, \nu) \in \mathbb{N}$ s.t. for $\gamma = (1, \ldots, 1) \in \mathbb{Z}_+^n$

$$\sum_{\alpha \in \mathbb{Z}_+^n} \exp(h_{m+l}(\alpha + \nu\gamma) - h_m(\alpha)) < \infty.$$

Thus, we would like to study the above mentioned problem in a more general situation than in [I.Kh. Musin // Vladikavkaz. Mat. Zh. 22 (3) (2020)].

Notations and definitions

For $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}_+^n$, $z = (z_1, \dots, z_n) \in \mathbb{C}^n$ let
$|\alpha| = \alpha_1 + \dots + \alpha_n$, $\alpha! = \alpha_1! \cdots \alpha_n!$, $D^\alpha = \frac{\partial^{|\alpha|}}{\partial z_1^{\alpha_1} \cdots \partial z_n^{\alpha_n}}$.
For $u = (u_1, \dots, u_n)$, $v = (v_1, \dots, v_n) \in \mathbb{C}^n$
$\langle u, v \rangle = u_1 v_1 + \dots + u_n v_n$, $\|u\| = \sqrt{|u_1|^2 + \cdots + |u_n|^2}$.
By λ_m denote the Lebesgue measure in $\mathbb{C}^m$.
For a domain $\mathcal{O} \subseteq \mathbb{C}^n$ $A(\mathcal{O})$ is the space of functions holomorphic in $\mathcal{O}$, $A_c(\Omega)$ is the space of functions holomorphic in Ω and continuous on $\overline{\Omega}$ ($\overline{\Omega}$ is a closure of Ω in $\mathbb{C}^n$) with standard topologies. $A'_{\mathcal{H}}(\Omega)$ is a space of linear continuous functionals on $A_{\mathcal{H}}(\Omega)$, $(A^\infty(\Omega))^*$ $(A_c^*(\Omega)$ is the strong dual of $A^\infty(\Omega)$ $(A_c(\Omega))$.
$H_\Omega(z) = \sup\limits_{\lambda \in \Omega} Re \, \langle \lambda, z \rangle$, $z \in \mathbb{C}^n$, is the support function of Ω.
Put $\ln^+ t = \ln t$ if $t > 1$ and $\ln^+ t = 0$ if $0 \le t \le 1$.
$\mathbb{R}_+^n = \{x = (x_1, \dots, x_n) \in \mathbb{R}^n : x_1 \ge 0, \dots, x_n \ge 0\}$.
The Young-Fenchel conjugate of $g : \mathbb{R}^n \to [-\infty, +\infty]$ is $g^* : \mathbb{R}^n \to [-\infty, +\infty]$ defined by $g^*(x) = \sup\limits_{y \in \mathbb{R}^n} (\langle x, y \rangle - g(y))$.

The main result

For each $m \in \mathbb{N}$ define a function φ_m in $\mathbb{C}^n$ by the rule

$$\varphi_m(z) = h_m^*(\ln^+|z_1|, \dots, \ln^+|z_n|), \; z = (z_1, \dots, z_n) \in \mathbb{C}^n.$$

Since h_m^* is convex in $\mathbb{R}^n$ and takes there finite values, then h_m^* is continuous in $\mathbb{R}^n$. Hence, φ_m is a continuous psh-function in $\mathbb{C}^n$. For each $m \in \mathbb{N}$ introduce the normed space

$$P_m = \left\{ F \in A(\mathbb{C}^n) : \|F\|_m = \sup_{z \in \mathbb{C}^n} \frac{|F(z)|}{\exp(H_\Omega(z) + \varphi_m(z))} < \infty \right\}. \tag{3}$$

The space P_m is continuously embedded in P_{m+1} for each $m \in \mathbb{N}$. Let $P_{\mathcal{H}}$ be an inductive limit of spaces P_m.

Theorem 1

The mapping $L : T \in A_{\mathcal{H}}^*(\Omega) \to \hat{T}$ establishes an isomorphism between the spaces $A_{\mathcal{H}}^*(\Omega)$ and $P_{\mathcal{H}}$.

The proof of this theorem is based on the scheme taken from M. Neymark [Neymark M. // Ark. math., 7 (1969)] and B.A. Taylor [Taylor B.A. // Commun. on pure and appl. mathematics. 1971. **24**:1] and some results from [Musin I.Kh. // Advances in Mathematics Research. 2002. Nova Science Publishers. New York] and Il'dar Kh. Musin, Polina V. Yakovleva. // Central European Journal of Mathematics. **10**:2 (2012)].

Auxiliary results

Proposition 1

Let a function $g : \mathbb{R}^n \to \mathbb{R}$, $g(0) = 0$, *be s.t. for some* $a > 0$
$g(x) \geq \|x\| \ln(\|x\| + 1) - a\|x\| - a, \; x \in \mathbb{R}^n$.
Let $b > 0$. *Then for any points* x, y *in* $\mathbb{R}^n$ *s.t.* $\|y - x\| \leq be^{-\|x\|}$

$$|g^*(y) - g^*(x)| \leq be^{2a+b}. \tag{4}$$

In the proof of Proposition 1 it is used that the supremum of the function $g_x(\xi) = \langle \xi, x \rangle - g(\xi)$ taken over $\mathbb{R}^n$ is attained at some point $\xi^* = \xi^*(x)$ such that $\|\xi^*\| \leq e^{2a} e^{\|x\|}$.

From Proposition 1 we have the following corollary.

Corollary 1

Let a function $g : \mathbb{R}^n \to \mathbb{R}$, $g(0) = 0$, *be such that for some* $a > 0$ $g(x) \geq \|x\| \ln(\|x\| + 1) - a\|x\| - a$, $x \in \mathbb{R}^n$. *Let* $b > 0$. *Then for* $x = (x_1, \ldots, x_n), y \in \mathbb{R}^n$ *s.t.* $\|y - x\| \leq be^{-(|x_1| + \cdots + |x_n|)}$

$$|g^*(y) - g^*(x)| \leq be^{2a+b}. \tag{5}$$

The Corollary 1 is used in the proof of the next Proposition.

Proposition 2

Let a function $g : \mathbb{R}^n \to \mathbb{R}$, $g(0) = 0$, *be such that for some* $a > 0$ $g(x) \geq \|x\| \ln(\|x\| + 1) - a\|x\| - a$, $x \in \mathbb{R}^n$. *Let points* $x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n) \in \mathbb{R}^n$ *be such that* $|y_j - x_j| \leq \frac{1}{\prod_{k=1}^{n} (1 + |x_k|)}$, $j = 1, \ldots, n$.

Then

$$|g^*(\ln^+ |y_1|, \ldots, \ln^+ |y_n|) - g^*(\ln^+ |x_1|, \ldots, \ln^+ |x_n|)| \leq 2ne^{2a+2n}.$$

From the Proposition 2 we get the following corollaries.

Corollary 2

Let $g : \mathbb{R}^n \to \mathbb{R}$ *be a function as in the Proposition 2. Let* $z = (z_1, \ldots, z_n), \zeta = (\zeta_1, \ldots, \zeta_n) \in \mathbb{C}^n$ *be such that* $|\zeta_j - z_j| \leq \frac{1}{\prod_{k=1}^{n}(1+|z_k|)}$, $j = 1, \ldots, n$.
Then

$$|g^*(\ln^+|\zeta_1|, \ldots, \ln^+|\zeta_n|) - g^*(\ln^+|z_1|, \ldots, \ln^+|z_n|)| \leq 2ne^{2a+2n}.$$

Corollary 3

Let $z = (z_1, \ldots, z_n), \zeta = (\zeta_1, \ldots, \zeta_n) \in \mathbb{C}^n$ *be such that* $|\zeta_j - z_j| \leq \frac{1}{\prod_{k=1}^{n}(1+|z_k|)}$, $j = 1, \ldots, n$.
Then for any $m \in \mathbb{N}$

$$|\varphi_m(\zeta) - \varphi_m(z)| \leq 2ne^{2a_m+2n}.$$

Corollary 4

Let $z, \zeta \in \mathbb{C}^n$ *be such that* $\|\zeta - z\| \leq \frac{1}{(1+\|z\|)^n}$.
Then $|\varphi_m(\zeta) - \varphi_m(z)| \leq 2ne^{2a_m+2n}, \ m \in \mathbb{N}$.

Proposition 3

For any $m \in \mathbb{N}$ *there exists a constant* $l_m > 0$ *such that*

$$h^*_{m+n}(x) \geq h^*_m(x) + \sum_{j=1}^{n} x_j - l_m, \ x = (x_1, \ldots, x_n) \in \mathbb{R}^n_+.$$

From this Proposition we have the following

Corollary 5

$$\varphi_{m+n}(z) \geq \varphi_m(z) + \sum_{j=1}^{n} \ln^+ |z_j| - l_m, \ m \in \mathbb{N} z = (z_1, \ldots, z_n) \in \mathbb{C}^n,$$

where l_m *is a constant from Proposition 3.*

In the proof of the following Proposition the condition i_3) is used.

Proposition 4

Let $m \in \mathbb{N}, c > 0$ be s.t. for $S \in A^*_{\mathcal{H}}(\Omega)$ $|S(f)| \leq cp_m(f)$, Then $S(f) = \sum_{|\alpha|\geq 0} S_\alpha(D^\alpha f)$, $f \in A_{\mathcal{H}}(\Omega)$, where $S_\alpha \in A^*_c(\Omega)$, moreover, $\|S_\alpha\|_{A^*_c(\Omega)} \leq \dfrac{c}{e^{h_m(\alpha)}}$, $\alpha \in \mathbb{Z}^n_+$.

Lemma1

For each $z \in \mathbb{C}^n$ the function $f_z(\lambda) = \exp(\langle \lambda, z \rangle)$ belongs to $A_{\mathcal{H}}(\Omega)$, moreover, $p_m(f_z) = \exp(H_\Omega(z) + \varphi_m(z))$ $\forall\, m \in \mathbb{N}$.

Lemma2

For any $S \in A^*_{\mathcal{H}}(\Omega)$ we have that $\hat{S} \in P_{\mathcal{H}}$.

Three important auxiliary theorems

For each $m \in \mathbb{Z}_+$ let

$$E_m = \left\{ F \in H(\mathbb{C}^n) : N_m(F) = \sup_{z \in \mathbb{C}^n} \frac{|F(z)|}{(1 + \|z\|)^m \exp(H_\Omega(z))} < \infty \right\}.$$

Let E be an inductive limit of the spaces E_m.

Theorem 2

The Laplace transformation $\mathcal{L} : S \in (A^\infty(\Omega))^* \to \hat{S}$ establishes a topological isomorphism between the spaces $(A^\infty(\Omega))^*$ and E.

Theorem 2 is obtained in [Musin I.Kh. // Advances in Mathematics Research. 2002. Nova Science Publishers. New York]. Under assumptions that the boundary of Ω is C^2-smooth this theorem was obtained by B.A. Derjavets [Dissertation ... of candidate of fiz.-mat. nauk. Rostov-on-Don University, 1983]

The next result [I.Kh. Musin, P.V. Yakovleva // CEJM. **10**:2 (2012)] is applied to establish surjectivity of L in the Theorem 1.

Theorem 3

Let $\mathcal{O}$ be a domain of holomorphy in $\mathbb{C}^n$. Let $h \in psh(\mathcal{O})$ and a function $\varphi \in psh(\mathbb{C}^n)$ be such that for some $c_\varphi > 0$ and $\nu > 0$ $|\varphi(z) - \varphi(t)| \le c_\varphi$ if $\|z - t\| \le \frac{1}{(1+\|t\|)^\nu}$. Let for $f \in H(\mathcal{O})$

$$\int_{\mathcal{O}} |f(\zeta)|^2 e^{-2(\varphi(\zeta)+h(\zeta))}\, d\lambda_n(\zeta) < \infty. \tag{6}$$

Then $\exists\ F \in H(\mathbb{C}^n \times \mathcal{O})$ s.t. $F(\zeta,\zeta) = f(\zeta)$ $(\zeta \in \mathcal{O})$ and

$$\int\limits_{\mathbb{C}^n\times\mathcal{O}} \frac{|F(z,\zeta)|^2 e^{-2(\varphi(z)+h(\zeta))}}{(1+\|(z,\zeta)\|)^{2n(\nu+3)}}\, d\lambda_{2n}(z,\zeta) \le$$

$$\le C \int\limits_{\mathcal{O}} |f(\zeta)|^2 e^{-2(\varphi(\zeta)+h(\zeta))}\, d\lambda_n(\zeta),$$

where a positive constant C depends only on n, ν and φ.

Theorem 4

Let $\mathcal{O}$ be a domain of holomorphy in $\mathbb{C}^n$. Let $h \in psh(\mathcal{O})$ and a function $\varphi \in psh(\mathbb{C}^n)$ be such that for some $c_\varphi > 0$ and $\nu > 0$ $|\varphi(z) - \varphi(t)| \le c_\varphi$ if $\|z - t\| \le \frac{1}{(1+\|t\|)^\nu}$. Let a function $S \in H(\mathbb{C}^n \times \mathcal{O})$ be such that $S(\zeta, \zeta) = 0$ for $\zeta \in \mathcal{O}$ and

$$|S(z,\zeta)| \le e^{\varphi(z)+h(\zeta)}, \ z \in \mathbb{C}^n, \zeta \in \mathcal{O}.$$

Then there exist functions $S_1, \dots, S_n \in H(\mathbb{C}^n \times \mathcal{O})$ such that:

a) $S(z,\zeta) = \sum_{j=1}^{n} S_j(z,\zeta)(z_j - \zeta_j), \ (z,\zeta) \in \mathbb{C}^n \times \mathcal{O};$

b) $\int\limits_{\mathbb{C}^n \times \mathcal{O}} \frac{|S_j(z,\zeta)|^2}{e^{2(\varphi(z)+h(\zeta)+m\ln(1+\|(z,\zeta)\|))}} \, d\lambda_{2n}(z,\zeta) < \infty \ (j = 1, \dots, n)$ *for some $m > 0$ not depending on S.*

Theorem 4 [I.Kh. Musin, P.V. Yakovleva // CEJM. **10**:2 (2012)] is applied to establish injectivity of L in the Theorem 1.

Properties of spaces $A_{\mathcal{H}}(\Omega)$, $A^*_{\mathcal{H}}(\Omega)$ and $P_{\mathcal{H}}$.

Definition 1

(M^*)-space is a l.c.s. F which is the projective limit of a sequence of normed spaces F_k with linear continuous mappings $g_{mk} : F_k \to F_m$, $m < k$, s.t. $g_{k,k+1}$ is compact for each $k \in \mathbb{N}$.

Using Montel's theorem and the condition i_3) it can be proved that $A_{\mathcal{H}}(\Omega)$ is the (M^*)-space. Thus, $A_{\mathcal{H}}(\Omega)$ is a (FS)-space.

Definition 2

Let $(E_m)_{m\in\mathbb{N}}$ be a sequence of Banach spaces such that E_m is continuously embedded in E_{m+1} for each $m \in \mathbb{N}$ and $E = \cup_{m\in\mathbb{N}} E_m$. If for each $m \in \mathbb{N}$ there is $k > m$ s.t. the embedding map of E_m in E_k is compact, then the countable locally convex inductive limit of spaces $E := \varinjlim E_m$ is called a (DFS)-space.

With a help of Montel's theorem and Corollary 5 it can be shown that the embeddings $j_m : P_m \to P_{m+n}$ are compact for each $m \in \mathbb{N}$. Hence, the space $P_{\mathcal{H}}$ is a (DFS)-space.
The space $A^*_{\mathcal{H}}(\Omega)$ as the strong dual of the Fréchet-Schwartz space $A_{\mathcal{H}}(\Omega)$ is a (DFS)-space.

Sketch of the proof of Theorem 1.

Let us only show that L is surjective. Let $F \in P_{\mathcal{H}}$. Then $F \in P_m$ for some $m \in \mathbb{N}$. Hence,

$$\int_{\mathbb{C}^n} \frac{|F(\zeta)|^2 e^{-2H_\Omega(\zeta)+\varphi_m(\zeta))}}{(1+\|\zeta\|)^{2n+1}}\, d\lambda_n(\zeta) < \infty. \qquad (7)$$

From this using Corollary 5 we get that

$$\int_{\mathbb{C}^n} |F(\zeta)|^2 e^{-2(H_\Omega(\zeta)+\varphi_{m+n(n+1)}(\zeta))}\, d\lambda_n(\zeta) < \infty. \qquad (8)$$

Note that functions H_Ω and $\varphi_{m+n(n+1)}$ are plurisubharmonic in $\mathbb{C}^n$ and for some $C_\Omega > 0$

$$|H_\Omega(u) - H_\Omega(v)| \le C_\Omega,\ \ u, v \in \mathbb{C}^n : \|u - v\| \le 1. \qquad (9)$$

So applying Theorem 3 with $\nu = 1$ we can find a function $\Phi \in H(\mathbb{C}^{2n})$ such that $\Phi(z, z) = F(z)$ for $z \in \mathbb{C}^n$ and for some $c > 0$ not depending on F

$$\int\limits_{\mathbb{C}^{2n}} \frac{|\Phi(z,\zeta)|^2 e^{-2(H_K(Im\, z)+\varphi_{m+n(n+1)}(\zeta))}}{(1+\|(z,\zeta)\|)^{8n}}\ d\lambda_{2n}(z,\zeta) \le$$

$$\le c\int_{\mathbb{C}^n} |F(\zeta)|^2 e^{-2(H_\Omega(\zeta)+\varphi_{m+n(n+1)}(\zeta))}\ d\lambda_n(\zeta).$$

Since $|\Phi|^2 \in psh(\mathbb{C}^{2n})$, then for any $R > 0$,
$z, \zeta = (\zeta_1, \dots, \zeta_n) \in \mathbb{C}^n$

$$|\Phi(z,\zeta)|^2 \le \frac{1}{\lambda_{2n}(R)} \int\limits_{B_R(z,\zeta)} |\Phi(t,u)|^2\ d\lambda_{2n}(t,u),$$

where $B_R(z,\zeta)$ is a closed ball in $\mathbb{C}^{2n}$ of a radius R with the center at the point (z,ζ), $\lambda_{2n}(R)$ is a volume of $B_R(z,\zeta)$. From this inequality putting $R = \frac{1}{2n\prod\limits_{k=1}^{n}(1+|\zeta_k|)}$ and using the inequality (9) and Corollary 3 in a standard way we get that for some $c_1 > 0$

$$|\Phi(z,\zeta)| \le c_1(1+\|z\|)^{4n}(1+\|\zeta\|)^{4n+n^2} e^{H_\Omega(z)+\varphi_{m+n(n+1)}(\zeta)},\ (z,\zeta) \in \mathbb{C}^{2n}.$$

Now using Corollary 5 we have for some $c_2 > 0$

$$|\Phi(z,\zeta)| \le c_2(1+\|z\|)^{4n} e^{H_\Omega(z)+\varphi_{m+n(n^2+5n+1)}(\zeta)}, \ (z,\zeta) \in \mathbb{C}^{2n}. \quad (10)$$

We expand $\Phi(z,\zeta)$ in a power series in ζ: $\Phi(z,\zeta) = \sum\limits_{|\alpha|\ge 0} \Phi_\alpha(z)\zeta^\alpha$.

By the Cauchy formula for any $\alpha \in \mathbb{Z}_+^n$, positive numbers $r_1, \dots, r_n$ we have that

$$C_\alpha(z) = \frac{1}{(2\pi i)^n} \int_{|\zeta_1|=r_1} \dots \int_{|\zeta_n|=r_n} \frac{\Phi(z,\zeta)}{\zeta_1^{\alpha_1+1} \dots \zeta_n^{\alpha_n+1}} \, d\zeta_1 \dots d\zeta_n, \ z \in \mathbb{C}^n.$$

Thus, $C_\alpha \in H(\mathbb{C}^n)$. Using (10) and since the restriction of $\varphi_{m+\nu}$ on $[0,\infty)^n$ is nondecreasing in each variable, we have that

$$|C_\alpha(z)| \le \frac{c_2(1+\|z\|)^{4n} e^{H_\Omega(z)+\varphi_{m+\nu}(r)}}{r^\alpha}, \quad (11)$$

where $r = (r_1, \dots, r_n) \in \mathbb{R}^n$ (all components are positive), $\nu = n(n^2+5n+1)$.

From this we get that for any $\alpha \in \mathbb{Z}_+^n, z \in \mathbb{C}^n$

$$|C_\alpha(z)| \le \frac{c_2(1+\|z\|)^{4n} e^{H_\Omega(z))}}{e^{h_{m+\nu}(\alpha)}}. \tag{12}$$

Therefore, the set $\left\{e^{h_{m+\nu}(\alpha)} C_\alpha\right\}_{\alpha \in \mathbb{Z}_+^n}$ is bounded in E_{4n}. Hence, it is bounded in E. Since the spaces $(A^\infty(\Omega))^*$ and E are isomorphic (by Theorem 2), then there exist functionals $S_\alpha \in (A^\infty(\Omega))^*$ such that $\hat{S}_\alpha = C_\alpha$ and the set $\mathcal{A} = \left\{e^{h_{m+\nu}(\alpha)} S_\alpha\right\}_{\alpha \in \mathbb{Z}_+^n}$ is bounded in $(A^\infty(\Omega))^*$. From this we conclude that there exist numbers $l \in \mathbb{Z}_+$ and $c_3 > 0$ such that for any $\alpha \in \mathbb{Z}_+^n$

$$|S_\alpha(f)| \le \frac{c_3 q_l(f)}{e^{h_{m+\nu}(\alpha)}}\ ,\ f \in A^\infty(\Omega). \tag{13}$$

Define a functional T on $A_{\mathcal{H}}(\Omega)$ by the rule:

$$T(f) = \sum_{|\alpha|\geq 0} S_\alpha(D^\alpha f), \ f \in A_{\mathcal{H}}(\Omega). \tag{14}$$

Show that T is a linear continuous functional on $A_{\mathcal{H}}(\Omega)$. Using the inequality (3) we have that for any $f \in A_{\mathcal{H}}(\Omega)$, $\alpha \in \mathbb{Z}_+^n$ and $s \in \mathbb{N}$

$$|S_\alpha(D^\alpha f)| \leq c_3 p_{m+s}(f) e^{\max\limits_{|\beta|\leq l} h_{m+s}(\alpha+\beta) - h_{m+\nu}(\alpha)}. \tag{15}$$

Using the conditions i_5) we chose a positive integer $s > \nu$ so that the series $\sum\limits_{|\alpha|\geq 0} e^{\max\limits_{|\beta|\leq l} h_{m+s}(\alpha+\beta) - h_{m+\nu}(\alpha)}$ converges. Then for any $f \in A_{\mathcal{H}}(\Omega)$ the series in the right of (14) is absolutely converging. Moreover, for chosen s there exists a constant $c_4 > 0$ not depending on $f \in A_{\mathcal{H}}(\Omega)$ such that $|T(f)| \leq c_4 p_{m+s}(f)$. Hence, the linear functional T is correctly defined and continuous. Besides that $\hat{T} = F$. Thus, L is surjective.

Let P_{ij} be polynomials ($i = 1, \ldots, m; j = 1, \ldots, r$) and $P = (P_{ij})_{\substack{i=\overline{1,m};\\ j=\overline{1,r}}}$. Consider an operator $\vec{P}(D)$ acting from $A^m_{\mathcal{H}}(\Omega)$ into $A^r_{\mathcal{H}}(\Omega)$ by the rule:

$$\vec{P}(D)\begin{pmatrix} f_1 \\ \vdots \\ f_m \end{pmatrix} = \begin{pmatrix} \sum\limits_{j=1}^{m} P_{1j}(D)f_j \\ \vdots \\ \sum\limits_{j=1}^{m} P_{rj}(D)f_j \end{pmatrix}$$

Consider the set of all r-tuples $Q = (Q_1, \ldots, Q_r)$ with polynomial components such that

$$P_{1j}(z)Q_1(z) + \cdots P_{rj}(z)Q_r(z) = 0, \; j = 1, \ldots, m.$$

It is known that this set has a finite number of generators. Let $Q^{(l)} = (Q^{(l)}_1, \ldots, Q^{(l)}_r)$ be its generators ($l = 1, \ldots, s$).

Theorem 5

The equation $\vec{P}(D)\vec{f} = \vec{g}$ *is solvable in* $A^m_{\mathcal{H}}(\Omega)$ *for* $\vec{g} \in A^r_{\mathcal{H}}(\Omega)$ *iff* $\sum\limits_{i=1}^{r} Q_i^{(l)}(D) g_i = 0,\ l = 1, \ldots, s.$

The proof of Theorem 5 is based on next Theorem by L. Hörmander.

Theorem 6

Given the system $P \in M[p \times q]$ *(with polynomial components) there is an integer* N *such that for psh in* $\mathbb{C}^n$ *functions* φ *and* $-\ln d$ *such that* $0 < d \leq 1,\ d(z+\zeta) \leq 2d(\zeta)$, *if* $|\Re z_j| \leq 1$, $|\Im z_j| \leq 1 \ \ (j = 1, \ldots, n)$; $|\varphi(z+\zeta) - \varphi(\zeta)| \leq C_0$, *if* $|z_j| \leq d(\zeta)$ $(j = 1, \ldots, n)$, *and all* $u \in (A(\mathbb{C}^n))^q$ *one can find* $v \in (A(\mathbb{C}^n))^q$ *with* $Pu = Pv$ *and* $\int \|v\|^2 e^{-\varphi_N}\, d\lambda \leq K \int \|Pu\|^2 e^{-\varphi}\, d\lambda$, *where* $\varphi_N(z) = \varphi(z) - N \ln d(z) + N \ln(1 + \|z\|^2)$, K *is independent on* u, φ, d.