The Generalized Long Mayer – Vietoris Sequence and Separating Cycles

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Grothendieck residue

Let ω be a meromorphic n-form on an n-dimensional complex-analytic manifold M, and $F_1, \ldots, F_n$ are polar hypersurfaces of ω, $F = F_1 \cup \ldots \cup F_n$. In a sufficiently small neighborhood U_a of an isolated point a of the intersection $Z = F_1 \cap \ldots \cap F_n$ the form ω is given by

$$\omega = \frac{h(z)\, dz_1 \wedge \ldots \wedge dz_n}{f_1(z) \ldots f_n(z)},$$

where $h, f_1, \ldots f_n$ are holomorphic germs at a, $F_k|_{U_a} = \{f_k = 0\}$. The Grothendieck residue of the form ω at the point a is represented by the integral

$$\mathrm{res}_a\, \omega = \frac{1}{(2\pi i)^n} \int\limits_{\gamma^{(a)}} \omega,$$

where $\gamma^{(a)}$ is a *local cycle* at a having the form

$$\gamma^{(a)} = \{z \in U_a \colon |f_1(z)| = \varepsilon_1, \ldots, |f_n(z)| = \varepsilon_n\}.$$

Property of residue

If $h \in \langle f_1, \ldots, f_n \rangle \subset \mathcal{O}_a$ then $\operatorname{res}_a \omega = 0$

It suffices to prove this property for $h = h_k f_k$. In this case the form

$$\omega = \frac{h(z)\, dz_1 \wedge \ldots \wedge dz_n}{f_1(z) \ldots f_n(z)}$$

has only $n-1$ polar hypersurfaces $F_1, \ldots [k] \ldots, F_n$.
At the same time $\gamma^{(a)} \sim 0$ in $U_a \setminus (F_1 \cup \ldots [k] \cdots \cup F_n)$ since $\gamma^{(a)} = \partial \sigma_k$ for the chain

$$\sigma_k = \{|f_1| = \varepsilon_1, \ldots, |f_k| \leq \varepsilon_k, \ldots, |f_n| = \varepsilon_n\}.$$

So by Stokes' theorem $\int\limits_{\gamma^{(a)}} \omega = 0$.

Separating Cycles

Let M be n-dimensional complex manifold and $\mathcal{F} = \{F_1, \dots, F_n\}$ be the set of hypersurfaces in M, $F = F_1 \cup \dots \cup F_n$, and Z_0 be the discrete part of their intersection $Z = F_1 \cap \dots \cap F_n$.

Definition

The cycle $\gamma \in Z_n(X \setminus F)$ separates the set of hypersurfaces $\mathcal{F}$ if

$$\gamma \sim 0 \text{ in } M \setminus (F_1 \cup \dots [k] \dots \cup F_n) \text{ for all } k = 1, \dots, n.$$

The local cycle $\gamma^{(a)}$, $a \in Z_0$, separates the set of polar hypersurfaces of the meromorphic form ω. So if $\gamma \sim \sum t_a \gamma^{(a)}$ then γ is the separating cycle and $\int_\gamma \omega = (2\pi i)^n \sum t_a \operatorname{res}_a \omega$.

Remark

If the integral $\int_\gamma \omega$ of a meromorphic form is represented as the sum of residues, then γ is the separating cycle.

Example 1 (in $\mathbb{C}^2$)

Let $\mathcal{F} = \{F_1, F_2\}$ be the set of complex curves in $\mathbb{C}^2$ such that $Z = F_1 \cap F_2$ is discrete. Consider the open cover $\{U_1, U_2\}$ of $\mathbb{C}^2 \setminus Z$, where $U_j = \mathbb{C}^2 \setminus F_j$. There is the exact long Mayer – Vietoris sequence

$$\cdots \leftarrow H_2(U_1) \oplus H_2(U_2) \xleftarrow{\delta_*} H_2(U_1 \cap U_2) \xleftarrow{\varphi} H_3(U_1 \cup U_2) \longleftarrow \ldots$$

where $U_1 \cap U_2 = \mathbb{C}^2 \setminus F$, $U_1 \cup U_2 = \mathbb{C}^2 \setminus Z$,
δ_* is induced by $\delta\colon \gamma \mapsto (\gamma, -\gamma)$,
φ is the connecting homomorphism, $\varphi\colon [\tau] = [\sigma_1 + \sigma_2] \mapsto [\partial \sigma_1]$, σ_j is the chain in U_j, $j = 1, 2$.
If $\gamma \in Z_2(\mathbb{C}^2 \setminus F)$ separates the $\mathcal{F}$, then $[\gamma] \in \ker \delta_*$, so $[\gamma] \in \operatorname{im} \varphi$.
Homology group $H_3(U_1 \cup U_2) = H_3(\mathbb{C}^2 \setminus Z)$ is generated by classes of 3-dimensional spheres S_a of small radius surrounding the point a, $a \in Z$.

Class $[S_a]$ can be represented by the cycle $\partial\Pi_a$, where

$$\Pi_a = \{z \in U_a \colon |f_1(z)| < \varepsilon_1,\ |f_2(z)| < \varepsilon_2\},$$

is the *special analytical polyhedron*. Moreover, the boundary $\partial\Pi_a$ of the polyhedron Π_a is the sum of its 2-dimensional faces

$$\tau_1 = \{|f_1| = \varepsilon_1,\ |f_2| \le \varepsilon_2\}, \quad \tau_2 = \{|f_1| \le \varepsilon_1,\ |f_2| = \varepsilon_2\}.$$

Therefore, $\varphi[S_a] = \varphi[\partial\Pi_a] = \varphi[\tau_1 + \tau_2] = [\partial\tau_1] = [\gamma^{(a)}]$. Since $[\gamma] \in \operatorname{im}\varphi$ there is a cycle $\sigma \sim \sum\limits_{a\in Z} t_a S_a$ such that $[\gamma] = \varphi[\sigma]$ and we get

$$[\gamma] = \varphi[\sigma] = \sum_{a\in Z} t_a \varphi[S_a] = \sum_{a\in Z} t_a [\gamma^{(a)}].$$

So we proved that γ is the separating cycle if and only if $\gamma \sim \sum t_a \gamma^{(a)}$.

Separating cycles in Stein manifolds

Problem

What should be a manifold and a set of hypersurfaces so that any separating cycle can be represented in terms of local cycles?

The problem arose even in the first works on multidimensional residues:

Didon (1873), Picard (1926).

Partial solutions to this problem:

Fantappiè (1931), Martinelli (1955), Sorani (1960), Yuzhakov (1970).

The most complete solution for Stein manifolds: Tsikh (1975).
More generally for Stein manifolds and sets of $N > n$ hypersurfaces:

Tsikh, Yuzhakov (1988), Ulvert (2018).

Example 2 (Borromean rings)

Each circle separates the set of other two circles. For example, the cycle S^1 separates the set $\{S^1, S^1\}$ since

$$S^1 \sim 0 \text{ in } \mathbb{R}^3 \setminus S^1 \text{ and } S^1 \sim 0 \text{ in } \mathbb{R}^3 \setminus S^1.$$

Consider the cover $\{U_1, U_2\}$ of $\mathbb{R}^3$, where $U_1 = \mathbb{R}^3 \setminus S^1$, $U_2 = \mathbb{R}^3 \setminus S^1$.
As it was above in Example 1, there is the exact long Mayer – Vietoris sequence

$$\cdots \leftarrow H_1(U_1) \oplus H_1(U_2) \xleftarrow{\delta_*} H_1(U_1 \cap U_2) \xleftarrow{\varphi} H_2(U_1 \cup U_2) \longleftarrow \ldots,$$

where $U_1 \cap U_2 = \mathbb{R}^3 \setminus \left(S^1 \cup S^1\right)$, $U_1 \cup U_2 = \mathbb{R}^3$. Since $\delta_*[S^1] = 0$ and $H_2(\mathbb{R}^3) \cong 0$, then $[S^1] \in \ker \delta_* = \operatorname{im} \varphi \cong 0$. So

$$S^1 \sim 0 \text{ in } \mathbb{R}^3 \setminus \left(S^1 \cup S^1\right).$$

$\mathfrak{U}$-chains

Let $\mathfrak{U} = \{U_i\}_{i \in I}$ be an open cover of topological space X, where I is an ordered index set.

Definition

A $\mathfrak{U}$-chain in X of multiplicity p and dimension q is an alternating function σ on I^{p+1} with values

$$\sigma(i_0, i_1, \ldots, i_p) \in S_q(U_{i_0} \cap U_{i_1} \cap \ldots \cap U_{i_p}),$$

which vanishes except at a finite number of a points of I^{p+1}.

$\mathfrak{U}$-chains can be identified with elements of the bigraded group

$$C_{p,q} = \bigoplus_{i_0 < i_1 < \ldots < i_p} S_q(U_{i_0} \cap U_{i_1} \cap \ldots \cap U_{i_p}), \quad p, q = 0, 1, \ldots$$

Mayer–Vietoris sequence for the groups of $\mathfrak{U}$-chains

We denote by $S_q^{\mathfrak{U}} = S_q^{\mathfrak{U}}(X)$ the subgroup in $S_q(X)$ generated by singular q-simplices Δ, such that $\operatorname{supp}\Delta \subset U_i$ for some $U_i \in \mathfrak{U}$. The natural inclusion $\iota\colon S_*^{\mathfrak{U}} \to S_*(X)$ is a chain mapping and the homomorphism $\iota_*\colon H(S_*^{\mathfrak{U}}) \to H(X)$ is the isomorphism.
Consider the Čech boundary operator $\delta\colon C_{p,q} \to C_{p-1,q}$

$$(\delta\sigma)(i_0, i_1, \dots, i_{p-1}) = \sum_{i\in I} \sigma(i, i_0, \dots, i_{p-1}),$$

and the operator $\varepsilon\colon C_{0,q} \to S_q^{\mathfrak{U}}$, $\varepsilon\sigma = \sum_{i\in I} \sigma(i)$.

Theorem

The following sequence is exact:

$$0 \longleftarrow S_q^{\mathfrak{U}} \xleftarrow{\varepsilon} C_{0,q} \xleftarrow{\delta} C_{1,q} \xleftarrow{\delta} C_{2,q} \xleftarrow{\delta} \dots.$$

We get the following *extended double complex* for the group of $\mathfrak{U}$-chains which is dual to the familiar Čech–de Rham double complex for differential forms:

$$
\begin{array}{ccccccccc}
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longleftarrow & S_q^{\mathfrak{U}} & \stackrel{\varepsilon}{\longleftarrow} & C_{0,q} & \stackrel{\delta}{\longleftarrow} & C_{1,q} & \stackrel{\delta}{\longleftarrow} & \dots \\
 & & \partial\downarrow & & \partial\downarrow & & \partial\downarrow & & \\
0 & \longleftarrow & S_{q-1}^{\mathfrak{U}} & \stackrel{\varepsilon}{\longleftarrow} & C_{0,q-1} & \stackrel{\delta}{\longleftarrow} & C_{1,q-1} & \stackrel{\delta}{\longleftarrow} & \dots \\
 & & \partial\downarrow & & \partial\downarrow & & \partial\downarrow & & \\
 & & \dots & & \dots & & \dots & & \\
 & & \partial\downarrow & & \partial\downarrow & & \partial\downarrow & & \\
0 & \longleftarrow & S_0^{\mathfrak{U}} & \stackrel{\varepsilon}{\longleftarrow} & C_{0,0} & \stackrel{\delta}{\longleftarrow} & C_{1,0} & \stackrel{\delta}{\longleftarrow} & \dots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

Mayer–Vietoris spectral sequence

Consider also the double complex $C = (C_{p,q}; \delta, \partial)$. Based on a double complex C we build a *total complex* TC, formed by a graded group

$$(TC)_n = \bigoplus_{p+q=n} C_{p,q}$$

and a boundary operator $D\colon (TC)_n \to (TC)_{n-1}$, $D = \delta + (-1)^p \partial$. Consider the spectral sequences $\{(E^r_{p,q}; d^r)\}$ of the complex C which corresponds to filtration for TC determined by the formula

$$F_p(TC)_n = \bigoplus_{i \leqslant p} C_{i,n-i}.$$

We have $E^0_{p,q} = C_{p,q}$ and $d^0 = \pm\partial$, so $E^1_{p,q} = H_q(C_{p,*})$ (the vertical homology of the complex C) and the differential $d^1\colon E^1_{p,q} \to E^1_{p-1,q}$ coincides with the mapping induced by the chain mapping $\delta\colon C_{p,*} \to C_{p-1,*}$, i.e. $d^1 = \delta_*$.

Further, the term $E^2_{p,q}$ (the horizontal homology of the vertical homology of the complex C) describes the homology of the complex

$$0 \longleftarrow H_q(C_{0,*}) \xleftarrow{\delta_*} H_q(C_{1,*}) \xleftarrow{\delta_*} H_q(C_{2,*}) \xleftarrow{\delta_*} \dots,$$

Therefore, this spectral sequence (called the Mayer–Vietoris spectral sequence) is a generalization of the long exact Mayer–Vietoris sequence.

In passing to homology of columns of the extended double complex, we get the sequence (for $q = 0, 1, \dots$)

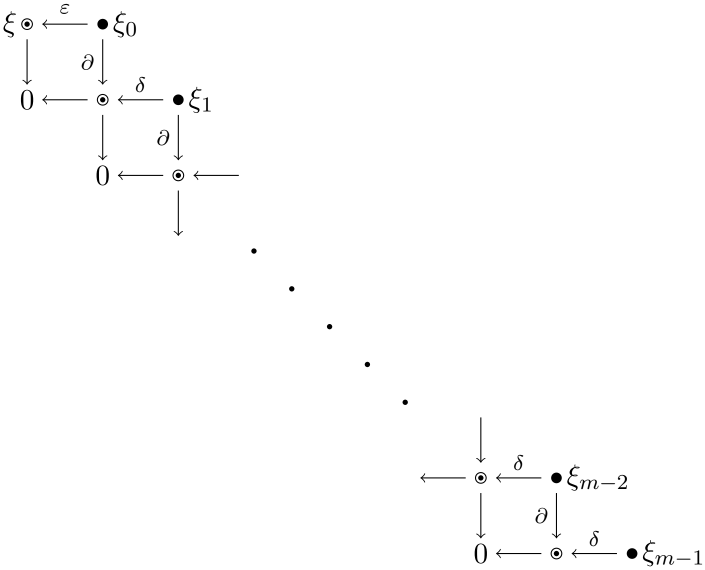
$$0 \longleftarrow H_q(S_*^{\mathfrak{U}}) \xleftarrow{\varepsilon_*} H_q(C_{0,*}) \xleftarrow{\delta_*} H_q(C_{1,*}) \xleftarrow{\delta_*} H_q(C_{2,*}) \xleftarrow{\delta_*} \dots,$$

about which, in general, we can say that it is only semi-exact. The existence of the connecting homomorphism assumes that the open cover $\mathfrak{U}$ of topological space X is finite. We will assume that this covering consists of $m \geq 2$ elements.

Definition (A. Gleason)

The $\mathfrak{U}$-resolution for the cycle $\xi \in Z_r(S_*^{\mathfrak{U}})$ is a sequence $\{\xi_p\}_{p=0}^{m-1}$ of $\mathfrak{U}$-chains, $\xi_p \in C_{p,r-p}$, such that:

1) $\varepsilon\xi_0 = \xi$;
2) $\delta\xi_p = \partial\xi_{p-1}$, $p = 1, \dots, r$.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Remark

A $\mathfrak{U}$-resolution $\{\xi_p\}_{p=0}^{m-1}$ exists for any cycle $\xi \in Z_r(S_*^{\mathfrak{U}})$, wherein $\partial\xi_{m-1} = 0$.

Theorem

Let $\mathfrak{U} = \{U_i\}$ be a finite open cover of a topological space X, consisting of $m \geq 2$ elements. Then the correspondence of homology classes $[\xi] \mapsto [\xi_{m-1}]$, where $\xi \in Z_r(S_*^{\mathfrak{U}})$ and ξ_{m-1} is the end term of the arbitrary $\mathfrak{U}$-resolution $\{\xi_p\}$ of cycle ξ, defines a connecting homomorphism

$$\varphi \colon H_r(S_*^{\mathfrak{U}}) \to H_{r-m+1}(C_{m-1,*}).$$

Semi-exact long Mayer – Vietoris sequences of homology groups

Theorem

For $m > 2$ the connecting homomorphism φ generates a semi-exact long Mayer – Vietoris sequence of homology groups

$$\ldots \longleftarrow H_{q-m+1}(C_{m-2,*}) \xleftarrow{\delta_*} H_{q-m+1}(C_{m-1,*}) \xleftarrow{\varphi} H_q(S_*^{\mathfrak{U}}) \xleftarrow{\varepsilon_*}$$

$$\xleftarrow{\varepsilon_*} H_q(C_{0,*}) \xleftarrow{\delta_*} \ldots \xleftarrow{\delta_*} H_q(C_{m-1,*}) \xleftarrow{\varphi} H_{q+m-1}(S_*^{\mathfrak{U}}) \longleftarrow \ldots$$

In this sequence there are

$$H_q(S_*^{\mathfrak{U}}) \cong H_q(U_1 \cup \ldots \cup U_m) = H_q(X),$$

$$H_q(C_{p,*}) \cong \bigoplus_{i_0 < i_1 < \ldots < i_p} H_q(U_{i_0} \cap U_{i_1} \cap \ldots \cap U_{i_p}), \quad p = 0, 1, \ldots, m-1.$$

Definition

The cycle $\xi \in Z_q(C_{m-1,*})$ *separates* the set of open subspaces $\{U_1, \ldots, U_m\}$ (the cover $\mathfrak{U}$) if $\delta_*[\xi] = 0$.

We denote by $H_q^{\mathsf{sep}}(C_{m-1,*})$ the subgroup $\ker \delta_* \subset H_q(C_{m-1,*})$ of classes of all separating cycles. We will assume that the codomain of the connecting homomorphism is the subgroup $H_q^{\mathsf{sep}}(C_{m-1,*})$.

Remark

The connecting homomorphism $\varphi\colon H_{q+m-1}(S_*^{\mathfrak{U}}) \to H_q^{\mathsf{sep}}(C_{m-1,*})$ is an epimorphism if and only if the semi-exact long Mayer – Vietoris sequence is exact in term $H_q(C_{m-1,*})$.

Separating cycles in the complex manifold

Let again M be n-dimensional complex-analytic manifold and $\mathcal{F}$ be a set of n hypersurfaces in M.

It is required to find out in which case the given n-cycle γ in $M \setminus F$ is homologically expressed in terms of local cycles $\gamma^{(a)}$, $a \in Z_0$. In view of the above, for this it is necessary the cycle γ separates the given set of hypersurfaces $\mathcal{F}$.

Consider the space $X = M \setminus Z$ and its cover $\mathfrak{U}$ formed by open sets $U_j = M \setminus F_j$, $j = 1, \ldots, n$. Consider the part of the semi-exact long Mayer – Vietoris sequence:

$$\ldots \xleftarrow{\delta_*} H_n(M \setminus F) \xleftarrow{\varphi} H_{2n-1}(S_*^{\mathfrak{U}}) \xleftarrow{\varepsilon_*} \ldots .$$

The subgroup $H_n^{\mathsf{sep}}(M \setminus F) \subset H_n(M \setminus F)$ is the subgroup of classes of all cycles separating the set of hypersurfaces $\mathcal{F}$. Also we denote by $H_n^{\mathsf{loc}}(M \setminus F)$ the subgroup in $H_n(M \setminus F)$ generated by the classes of all local cycles $\gamma^{(a)}$, $a \in Z_0$.

We have

$$H_n^{\mathsf{loc}}(M \setminus F) \subset H_n^{\mathsf{sep}}(M \setminus F).$$

Problem

What should be a manifold and a set of hypersurfaces so that

$$H_n^{\mathsf{sep}}(M \setminus F) = H_n^{\mathsf{loc}}(M \setminus F)?$$

Let us show that $H_n^{\mathsf{loc}}(M \setminus F) \subset \operatorname{im} \varphi$, where

$$\varphi\colon H_{2n-1}(M \setminus Z) \to H_n^{\mathsf{sep}}(M \setminus F).$$

It suffices to show that each generator $[\gamma^{(a)}]$, $a \in Z_0$, of the group $H_n^{\mathsf{loc}}(M \setminus F)$ have preimage in $H_{2n-1}(M \setminus Z)$. For a fixed point $a \in Z_0$, consider the $(2n-1)$-dimensional sphere S_a centred at the point a of a small radius. The class $[S_a]$ can be represented as a cycle $\partial \Pi_a$, where

$$\Pi_a = \{z \in U_a \colon |f_i(z)| < \varepsilon_i,\ i = 1, \ldots, n\}.$$

Moreover, the boundary $\partial\Pi_a$ of the polyhedron Π_a is the sum of its $(n-1)$-dimensional faces

$$\tau_j = \{|f_1| \le \varepsilon_1, \ldots, |f_j| = \varepsilon_j, \ldots, |f_n| \le \varepsilon_n\}, \quad j = 1, \ldots, n,$$

taken with suitable orientation, at that $\operatorname{supp} \tau_j \subset U_j$. Therefore, $\partial\Pi_a \in Z_{2n-1}(S_*^{\mathfrak{U}})$, and for the cycle $\partial\Pi_a$ can be built the $\mathfrak{U}$-resolution $\{\xi_p\}$. It is directly verified that terms of the resolution can be taken as follows:

$$\xi_p(i_0, i_1, \ldots, i_p) = \pm\tau_{i_0} \cap \tau_{i_1} \cap \ldots \cap \tau_{i_p}.$$

Moreover, the final term $\xi_{n-1} = \xi_{n-1}(1, \ldots, n)$ of the resolution is the local cycle $\gamma^{(a)}$. So $\varphi[S_a] = \varphi[\partial\Pi_a] = [\gamma^{(a)}]$, as required to prove. It also follows from the last reasoning that if the group $H_{2n-1}(M \setminus Z)$ is generated by the classes of cycles S_a, $a \in Z_0$, in particular if $H_{2n-1}(M) \cong 0$ and $Z = Z_0$, then $H_n^{\mathsf{loc}}(M \setminus F) = \operatorname{im}\varphi$.

Let $H_{2n-1}(M) \cong 0$ and let the intersection $Z = F_1 \cap \ldots \cap F_n$ be discrete.

Theorem

The groups $H_n^{\mathsf{sep}}(M \setminus F)$ and $H_n^{\mathsf{loc}}(M \setminus F)$ are coincide if and only if the corresponding semi-exact long Mayer – Vietoris sequence

$$\ldots \xleftarrow{\delta_*} H_n(M \setminus F) \xleftarrow{\varphi} H_{2n-1}(S_*^{\mathfrak{U}}) \xleftarrow{\varepsilon_*} \ldots$$

is exact in the term $H_n(M \setminus F)$.

Theorem

The groups $H_n^{\mathsf{sep}}(M \setminus F)$ and $H_n^{\mathsf{loc}}(M \setminus F)$ are coincide if the following condition is hold:

$$H_{2n-2}(C_{0,*}) \cong H_{2n-3}(C_{1,*}) \cong \ldots \cong H_{n+1}(C_{n-3,*}) \cong 0.$$

Theorem (Tsikh)

Let M be a Stein manifold of dimension n. Then

$$H_n^{\mathsf{sep}}(M \setminus F) = H_n^{\mathsf{loc}}(M \setminus F)$$

for any set $\{F_1, \ldots, F_n\}$ of hypersurfaces in M.

It suffices to prove this theorem for the following assumptions:
1) $H_{2n-1}(M) \cong 0$; 2) $M \setminus F_j$, $j = 1, \ldots, n$, are the Stein manifolds; 3) the intersection $Z = F_1 \cap \ldots \cap F_n$ is discrete. It remains to note that all possible intersections of the sets $U_j = M \setminus F_j$ are also Stein manifolds. The triviality of the required homology groups follows from the fact that for an arbitrary Stein manifold X the homology groups $H_q(X)$ are trivial for $q > \dim X$.

Remark

For the Stein manifold M and an arbitrary set of hypersurfaces $\mathcal{F}$ in M the connecting homomorphism $\varphi\colon H_{2n-1}(M \setminus Z) \to H_n^{\mathsf{sep}}(M \setminus F)$ is an isomorphism.

Combinatorial coefficients in the G-Kh formula

[A. G. Khovanskii, Leonid Monin, "The resultant of developed systems of Laurent polynomials", Mosc. Math. J., 17:4 (2017)]

Let $f_1, \dots, f_n$ be Laurent polynomials with developed Newton polyhedra $\Delta_1, \dots, \Delta_n$, $\Delta = \Delta_1 + \dots + \Delta_n$. Let F be a hypersurface in $\mathbb{T}^n = (\mathbb{C} \setminus 0)^n$ defined by the equation $f_1 = \dots = f_n = 0$. We denote by T^n_A the toric cycle corresponding to the vertex A of Δ.

Theorem (Topological theorem of Gelfond & Khovanskii)

In $\mathbb{T}^n \setminus F$ the sum of the local cycles $\gamma^{(a)}$ over all roots a of the system $f_1 = \dots = f_n = 0$ is homologous to the cycle $(-1)^n \sum k_A T^n_A$, where the sum is taken over all vertices A of Δ and k_A is the combinatorial coefficient at the vertex A.

Let Σ_j be the dual fan of the Newton polyhedron Δ_j, $j = 1, \ldots, n$. So $\{\Sigma_1, \ldots, \Sigma_n\}$ is a set of tropical hypersurfaces in $\mathbb{R}^n$,

$$\Sigma = \Sigma_1 \cup \ldots \cup \Sigma_n.$$

The condition of the developing of the set of polyhedra $\{\Delta_j\}$ implies

$$\Sigma_1 \cap \ldots \cap \Sigma_n = \{O\},$$

where O is the origin. Each vertex A of the polyhedron Δ corresponds to the connected component C_A of $\mathbb{R}^n \setminus \Sigma$. Let t_A be an arbitrary point in C_A. So the homology group $H_0(\mathbb{R}^n \setminus \Sigma)$ is generated by classes of 0-dimensional cycles t_A, $A \in \mathrm{Vert}\,\Delta$. Consider the space $\mathbb{R}^n \setminus \{O\}$ and its cover $\mathfrak{U}$ formed by open sets $U_j = \mathbb{R}^n \setminus \Sigma_j$, $j = 1, \ldots, n$. There exists a polyhedron $\tilde{\Delta}$ combinatorially equivalent to the polyhedron Δ such that $\partial\tilde{\Delta} \in Z_{n-1}(S_*^{\mathfrak{U}})$.

Proposition

$$\varphi([\partial\tilde{\Delta}]) = \sum_{A \in \mathrm{Vert}\,\Delta} k_A [t_A],$$

where φ is the connecting homomorphism in the corresponding semi-exact long Mayer – Vietoris sequence

$$\ldots \xleftarrow{\delta_*} H_0(\mathbb{R}^n \setminus \Sigma) \xleftarrow{\varphi} H_{n-1}(S_*^{\mathfrak{U}}) \xleftarrow{\varepsilon_*} \ldots,$$

$$H_{n-1}(S_*^{\mathfrak{U}}) \cong H_{n-1}(\mathbb{R}^n \setminus \{O\}) \cong \mathbb{Z},$$

$$\operatorname{im} \varphi = H_0^{\mathsf{sep}}(\mathbb{R}^n \setminus \Sigma) \cong \mathbb{Z}.$$

БЛАГОДАРЮ ЗА ВНИМАНИЕ