CR MANIFOLDS OF INFINITE BLOOM-GRAHAM TYPE

M.A. Stepanova

Multidimensional Residues and Tropical Geometry, Sochi, 2021.

1. Main question

Find a criterion for finite dimensionality of Lie algebra of infinitesimal holomorphic automorphisms in the case of real analytic CR manifolds of uniformly infinite Bloom-Graham type.

This criterion gives the criterion in the general case of real analytic CR manifolds, because the criterion for finite Bloom-Graham type manifolds is known (holomorphic nondegeneracy).

2. Definitions

Let $M \subset \mathbb{C}^N$ be a real analytic submanifold.

Complex tangent space $T_p^c M = T_p M \bigcap i T_p M$ is the maximal complex subspace in $T_p M$.

M is a **CR manifold**, if $\dim T_p^c$ is constant.

A CR submanifold M is *generic*, if $TM + iTM = \mathbb{C}^N$.

$n = \dim_{\mathbb{C}} T^c M$ is **CR dimension**.

$K = \dim_{\mathbb{R}} TM - \dim_{\mathbb{R}} T^c M$ is **codimension**.

$n + K = N$.

(n, K) is **CR type**.

Let $p \in M$ and let M_p be a germ of M at p.

M is **holomorphically nondegenerate**, if at all it's points there is no germ of holomorphic vector field, which is tangent to M.

Let $M \subset \mathbb{C}^{n+K}$ be a real analytic generic submanifold of CR type (n, K), defined in a neighbourhood U of zero, M_p is it's germ at point p.

Let $D_1 = T^c M$ be the distribution of complex tangent spaces on M, defined at a neighbourhood of a point $p \in M$.

Define inductively distributions

$$D_{\nu+1} = [D_\nu, D_1] + D_\nu,$$

where $[\cdot, \cdot]$ is commutator.

Let $D_\nu(p)$ be the value of D_ν at point p. We have the sequence of nested real linear spaces

$$T_p^c M_p = D_1(p) \subseteq D_2(p) \subseteq ... \subseteq D_\nu(p) \subseteq ... \subseteq \mathcal{D}(p) \subseteq T_p M_p,$$

where $\mathcal{D}(p) = \bigcup_{\nu=1}^{\infty} D_\nu(p)$. The sequence is stabilized from some moment.

M has *finite type* at p, if $\mathcal{D}(p) = T_p M$.

If $\mathcal{D}(p) \subsetneqq T_p M$, then M has *infinite type* at p.

Let $d_\nu = \dim D_\nu(p)$.

Consider all ν, for which $d_\nu > d_{\nu-1}$. We obtain a sequence of increasing natural numbers: $2 \le m_1 < ... < m_l$. Let $k_j = d_{m_j} - d_{m_{j-1}}$.

If M has finite type at p, then it's **Bloom-Graham type** is $m = ((m_1, k_1), ..., (m_l, k_l))$; If M has infinite type at p, then it's Bloom-Graham type is $m = ((m_1, k_1), ..., (m_l, k_l), (\infty, d))$, where $d = \dim(T_p M_p) - \dim(\mathcal{D}(p)) = K - k_1 - ... - k_l$. We call the value $d = d(p)$ the *defect* of M at p. For a manifold of finite type $d(p) = 0$.

M has **uniformly infinite type** in U, if it's Bloom-Graham type is infinite for all points $p \in M \cap U$.

If for all $p \in M \cap U$ defect $d(p)$ equals d, then we say that M has **uniformly infinite type of defect** d in U.

Theorem. Let $M \subset \mathbf{C}^N$ be a real-analytic, connected, generic submanifold. If M is holomorphically nondegenerate and has finite type at some point, then $\dim \operatorname{aut} M_p < \infty$ for all $p \in M$.

Reference: M. Bauoendi, P. Ebenfelt, L. Rothschild, "Real submanifolds in complex space and their mappings".

Theorem. Let M be a manifold of uniformly infinite type in U. Then outside a proper analytic set $\dim \operatorname{aut} M_p = 0$ or $\dim \operatorname{aut} M_p = \infty$.

3. Examples

Let $(z, w = u + iv, W = s + it)$ be coordinates in $\mathbb{C}^3$.

Example 1. Consider M, given by
$v = |z|^2, \quad t = 0.$
$\dim \operatorname{aut} M_0 = \infty$, because $2\mathrm{Re}(W^\nu \frac{\partial}{\partial W}) \in \operatorname{aut} M_0$ for any natural ν.

Example 2. Consider M, given by
$v = |z|^2 + (z^2\bar{z} + z\bar{z}^2)u + |z|^4 s, \quad t = 0.$
Direct calculations show that $\dim \operatorname{aut} M_0 = 0$.

4. Notation

Let $z = (z_1, ..., z_n)$;
$w = (w_1, ..., w_l)$, w_j has dimension k_j;
$W = (W_1, ..., W_d)$.
Define weights: $[z] = 1$, $[w_1] = m_1, ..., [w_l] = m_l$,
$[W_1] = ... = [W_\lambda] = \infty$. The corresponding weights are given to conjugate variables.

Let $o(m_j)$ be terms of weight greater than m_j.

Denote $u_j = \operatorname{Re} w_j$, $v_j = \operatorname{Im} w_j$.

Lemma. The following conditions are equivalent:

1) M is a manifold of uniformly infinite type of defect d in U outside a proper analytic set;

2) There exist a coordinate system $(z, w = u + iv, W = s + it)$ (z, w, W are vectors), such that defining equations for M in **reduced form** outside a proper analytic set are

$v_1 = \Phi_1(z, \bar{z}) + o(m_1),$

...

$v_l = \Phi_l(z, \bar{z}, u_1, ..., u_{l-1}) + o(m_l),$

$t = 0.$

5. Reduced form for unformly infinite type

$\approx \Phi_j$ are in standard form in the sense of Bloom and Graham.

More precisely:

$\Phi_1, ..., \Phi_l$ are homogeneous vector forms of weihgts $m_1, ..., m_l$, respectively. Renumber coordinates of forms Φ_j and redesignate them by ϕ_ν. I.e., if $\nu = (k_1 + ... + k_j) + \mu$, then ϕ_ν is the μ-th coordinate of form Φ_{j+1}.

Conditions for Φ_j:

I) Φ_j does not contain monomials of the form

$$c\, z_1^{\alpha_1} \cdot \ldots \cdot z_n^{\alpha_n} \cdot u_1^{\beta_1} \cdot \ldots \cdot u_{j-1}^{\beta_{j-1}}$$

and their conjugates for any multidegree α, β_ν and for $c \neq 0$;

II) it is impossible to compose a polynomial

$$c\, \phi_j\, u_j^{\beta_j} \cdot \ldots \cdot u_l^{\beta_l}$$

with the help of some monomials in ϕ_J for all $j < J$, such that $m_j < m_J$, and for $c \neq 0$;

III) ϕ_j are linearly independent.

6. Criterion of zero dimensionality of $\operatorname{aut} M_p$

We call M_p **real nondegenerate**, if $\dim \operatorname{aut} M_p = 0$, i.e. there exists no germ of nonzero infinitesimal holomorphic automorphism. This definition is analogous to that of holomorphic nondegeneracy.

6. l-nondegeneracy

It is possible to describe holomorphic nondegeneracy for M constructively.

Let $(Z_1, ..., Z_N) = (z, w, W)$ be coordinates, $\rho_j = 0$, $j = 1, ..., K$, be local defining equations of M near p.

CR vector fields are tangent to M fields of the form $\sum_{j=1}^{N} a_j \frac{\partial}{\partial \bar{Z}_j}$

Let $L_1, ..., L_n$ be a basis of CR vector fields near $p \in M$. Then M is **l-nondegenerate** at p, if

$$\text{span}\{L^\alpha\left(\frac{\partial \rho_j}{\partial z}\right)(p, \bar{p}) : \ j \in \{1, ..., K\}, |\alpha| \leq l\} = \mathbb{C}^N.$$

Theorem. The following conditions are equivalent.
(i) M is holomorphically nondegenerate.
(ii) There exists $p_0 \in M$ and $l_0 > 0$ such that M is l_0-nondegenerate at p_0.
(iii) There exists V, a proper real analytic subset of M and an integer $l \leq n$, such that M is l-nondegenerate at every $p \in M \backslash V$.

Reference: M. Bauoendi, P. Ebenfelt, L. Rothschild, "Real submanifolds in complex space and their mappings".

Theorem. The following conditions are equivalent.

(i) M is real nondegenerate.

(ii) There exists V, a proper real analytic subset of M and some finite set of integers J, such that M is J-nondegenerate at every $p \in M \setminus V$.

J-nondegeneracy

Let $X = 2\,\mathrm{Re}(f(z,w,W)\frac{\partial}{\partial z} + g(z,w,W)\frac{\partial}{\partial w} + h(z,w,W)\frac{\partial}{\partial W})$, where f, g, h are vectors of germs of holomorphic functions. Let $f = (f_1, ..., f_n)$; $g = (g_1, ..., g_l)$, $g_j = (g_{j1}, ..., g_{jk_j})$; $h = (h_1, ..., h_d)$. We write the tangency condition of X to M:

$$\operatorname{Im} g_j = 2\operatorname{Re}\Big(f\frac{\partial}{\partial z}\Phi_j(z,\bar z,u_1,...,u_{j-1}) + g\frac{\partial}{\partial w}\Phi_j(z,\bar z,u_1,...,u_{j-1}) +$$

$$f\frac{\partial}{\partial z}F_j(z,\bar z,u,s) + g\frac{\partial}{\partial w}F_j(z,\bar z,u,s) + h\frac{\partial}{\partial W}F_j(z,\bar z,u,s)\Big), 1 \le j \le l,$$

$$\operatorname{Im} h_j = 0,\ 1 \le j \le d,$$

$$w = u + i(\Phi + F),\ W = s + i(\Psi + G).$$

When there are no solutions?

It can be verified constructively.

Using coefficients in Taylor series of (f, g, h), we can write $d+1$ matrices. And then give the answer in terms of ranks of these matrices. $j_0, ..., j_d$ are sizes of the matrices.

$J = (j_0, j_1, ..., j_d)$

We obtain the following criterion of real nondegeneracy at points outside a proper analytic subset:

Theorem. Real nondegeneracy at p is equivalent to J-nondegeneracy for some J at this point.

Thank you!

Список литературы

[1] V.K. Beloshapka, CR-Manifolds of Finite Bloom–Graham Type: the Method of Model Surface, Russian Journal of Mathematical Physics volume 27, pages 155–174 (2020).

[2] Th. Bloom and I. Graham, "On "Type" Conditions for Generic Real Submanifolds of C^n ", Invent. Math. 40, 217–243 (1977).

[3] M.S. Baouendi, P. Ebenfelt, L.P. Rothschild, Real Submanifolds in Complex Space and Their Mappings, Princeton Univ. Press, Princeton, NJ, 1999.

[4] S.S. Chern, J.K. Moser, "Real hypersurfaces in complex manifold", Acta Math., 133:3–4 (1974), 219–271.

[5] В. К. Белошапка, "Вещественные подмногообразия комплексного пространства: их полиномиальные модели, автоморфизмы и проблемы классификации", УМН, 57:1(343) (2002), 3–44; Russian Math. Surveys, 57:1 (2002), 1–41.

[6] V.K. Beloshapka, "Can a stabilizer be eight-dimensional?", Russian Journal of Mathematical Physics, 2012, Volume 19, Issue 2, pp 135–145.

[7] М. А. Степанова, "Модификация теоремы Блума–Грэма: введение весов в комплексном касательном пространстве", Тр. ММО, 79, № 2, МЦНМО, М., 2018, 237–246