Chern Currents of Coherent Sheaves

Richard Lärkäng

Chalmers University of Technology and the University of Gothenburg

15 Jun 2021

Joint work with Elizabeth Wulcan.

Related to project with Lucas Kaufmann Sacchetto and Elizabeth Wulcan.

Introduction

Chern classes of a vector bundle

X a complex manifold, E holomorphic vector bundle on X, D a connection on E.

Chern forms and classes

$c_k(E, D) = \psi_k(\frac{i}{2\pi}\Theta)$, $\Theta = D^2$ curvature of D.

$c_k(E) = [c_k(E, D)] \in H^{2k}(X, \mathbb{R})$
(say, as deRham cohomology of smooth forms or currents)

$c(E, D) = \sum_k c_k(E, D)$.

Chern classes of a vector bundle

X a complex manifold, E holomorphic vector bundle on X, D a connection on E.

Chern forms and classes

$c_k(E, D) = \psi_k(\frac{i}{2\pi}\Theta)$, $\Theta = D^2$ curvature of D.

$c_k(E) = [c_k(E, D)] \in H^{2k}(X, \mathbb{R})$
(say, as deRham cohomology of smooth forms or currents)

$c(E, D) = \sum_k c_k(E, D)$.

Inverse of Chern forms and classes

$c(E, D)^{-1}$ exists since $c(E, D) = 1 + \{nilpotent\}$. (Segre form)
$c(E)^{-1} = [c(E, D)^{-1}]$ (Segre class)

Chern class of a coherent sheaf

Chern class of a coherent sheaf

$\mathcal{F}$ a coherent analytic sheaf on X.

Chern class of a coherent sheaf

$\mathcal{F}$ a coherent analytic sheaf on X. Assume $\mathcal{F}$ has a finite locally free resolution (E, φ):

$$0 \to E_N \xrightarrow{\varphi_N} \dots \xrightarrow{\varphi_1} E_0 \to \mathcal{F} \to 0.$$

Always exist if for example X is projective.

Chern class of a coherent sheaf

$\mathcal{F}$ a coherent analytic sheaf on X. Assume $\mathcal{F}$ has a finite locally free resolution (E, φ):

$$0 \to E_N \stackrel{\varphi_N}{\to} \ldots \stackrel{\varphi_1}{\to} E_0 \to \mathcal{F} \to 0.$$

Always exist if for example X is projective.

Definition (Chern classes and forms of $\mathcal{F}$)

$c(\mathcal{F}) = c(E_0)c(E_1)^{(-1)} \cdots c(E_N)^{(-1)^N}$.

If each E_k is equipped with a connection D_k,
$c(E, D) = c(E_0, D_0)c(E_1, D_1)^{(-1)} \cdots c(E_N, D_N)^{(-1)^N}$.

Chern class of a coherent sheaf

$\mathcal{F}$ a coherent analytic sheaf on X. Assume $\mathcal{F}$ has a finite locally free resolution (E, φ):

$$0 \to E_N \stackrel{\varphi_N}{\to} \dots \stackrel{\varphi_1}{\to} E_0 \to \mathcal{F} \to 0.$$

Always exist if for example X is projective.

Definition (Chern classes and forms of $\mathcal{F}$)

$c(\mathcal{F}) = c(E_0)c(E_1)^{(-1)} \cdots c(E_N)^{(-1)^N}$.

If each E_k is equipped with a connection D_k,
$c(E, D) = c(E_0, D_0)c(E_1, D_1)^{(-1)} \cdots c(E_N, D_N)^{(-1)^N}$.

Let $c_k(E, D)$ denote the part of $c(E, D)$ of degree $2k$.

Motivation for definition of Chern class of a coherent sheaf

$\mathcal{F}$ coherent analytic sheaf with a finite locally free resolution

$$0 \to E_N \stackrel{\varphi_N}{\to} \ldots \stackrel{\varphi_1}{\to} E_0 \to \mathcal{F} \to 0.$$

$$c(\mathcal{F}) = c(E_0)c(E_1)^{(-1)} \cdots c(E_N)^{(-1)^N}.$$

Motivation for definition of Chern class of a coherent sheaf

$\mathcal{F}$ coherent analytic sheaf with a finite locally free resolution

$$0 \to E_N \xrightarrow{\varphi_N} \dots \xrightarrow{\varphi_1} E_0 \to \mathcal{F} \to 0.$$

$c(\mathcal{F}) = c(E_0)c(E_1)^{(-1)} \cdots c(E_N)^{(-1)^N}$.

Is independent of (E, φ), so consistent with usual definition if $\mathcal{F}$ is a vector bundle.

Motivation for definition of Chern class of a coherent sheaf

$\mathcal{F}$ coherent analytic sheaf with a finite locally free resolution

$$0 \to E_N \xrightarrow{\varphi_N} \dots \xrightarrow{\varphi_1} E_0 \to \mathcal{F} \to 0.$$

$c(\mathcal{F}) = c(E_0)c(E_1)^{(-1)} \cdots c(E_N)^{(-1)^N}$.

Is independent of (E, φ), so consistent with usual definition if $\mathcal{F}$ is a vector bundle.

If $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ is a short exact sequence, then $c(\mathcal{G}) = c(\mathcal{F})c(\mathcal{H})$ (just as for vector bundles).

Existence of Chern currents

If $Z = \operatorname{supp} \mathcal{F}$ has positive codimension, could be natural to find a representative of $c(\mathcal{F})$ with support on Z. Our first result that is possible:

If $Z = \operatorname{supp} \mathcal{F}$ has positive codimension, could be natural to find a representative of $c(\mathcal{F})$ with support on Z. Our first result that is possible:

Theorem

Let $\mathcal{F}$ be a coherent analytic sheaf, (E, φ) a finite locally free resolution of $\mathcal{F}$, D_k connection on E_k.

If $Z = \operatorname{supp} \mathcal{F}$ has positive codimension, could be natural to find a representative of $c(\mathcal{F})$ with support on Z. Our first result that is possible:

Theorem

Let $\mathcal{F}$ be a coherent analytic sheaf, (E, φ) a finite locally free resolution of $\mathcal{F}$, D_k connection on E_k. Then there exist connections $\widehat{D}_k^\epsilon$ on each E_k such that the limit

$$c_\ell^{Res}(E, D) := \lim_{\epsilon \to 0} c_\ell(E, \widehat{D}^\epsilon)$$

exists as a current, which represents $c_\ell(\mathcal{F})$, and has support on $\operatorname{supp} \mathcal{F}$ *for $\ell > 0$.*

If $Z = \operatorname{supp} \mathcal{F}$ has positive codimension, could be natural to find a representative of $c(\mathcal{F})$ with support on Z. Our first result that is possible:

Theorem

Let $\mathcal{F}$ be a coherent analytic sheaf, (E, φ) a finite locally free resolution of $\mathcal{F}$, D_k connection on E_k. Then there exist connections $\widehat{D}_k^\epsilon$ on each E_k such that the limit

$$c_\ell^{Res}(E, D) := \lim_{\epsilon \to 0} c_\ell(E, \widehat{D}^\epsilon)$$

exists as a current, which represents $c_\ell(\mathcal{F})$, and has support on $\operatorname{supp} \mathcal{F}$ for $\ell > 0$.

Can also define products $c_{\ell_1}^{Res}(E, D) \wedge \cdots \wedge c_{\ell_m}^{Res}(E, D)$.

Idea of proof

$\lim_{\epsilon\to 0} c(E, \widehat{D}^\epsilon)$ exists, has support on $Z = \operatorname{supp} \mathcal{F}$.

f section such that $Z(f) = Z$,
χ a cut-off function, $\chi(t) \equiv 0$ for $t \ll 1$, $\chi(t) \equiv 1$ for $t \gg 1$.
$\chi_\epsilon = \chi(|f|^2/\epsilon)$
σ_k the "minimal inverse" (Moore-Penrose inverse) of φ_k.

$$\widehat{D}_k^\epsilon = -\chi_\epsilon \sigma_k D\varphi_k + D_k.$$

$\widehat{D}_k^\epsilon$ essentially appears in work by Baum-Bott (for ϵ fixed).

Idea of proof

$\lim_{\epsilon \to 0} c(E, \widehat{D}^{\epsilon})$ exists, has support on $Z = \operatorname{supp} \mathcal{F}$.

f section such that $Z(f) = Z$,
χ a cut-off function, $\chi(t) \equiv 0$ for $t \ll 1$, $\chi(t) \equiv 1$ for $t \gg 1$.
$\chi_\epsilon = \chi(|f|^2/\epsilon)$
σ_k the "minimal inverse" (Moore-Penrose inverse) of φ_k.

$$\widehat{D}_k^{\epsilon} = -\chi_\epsilon \sigma_k D \varphi_k + D_k.$$

$\widehat{D}_k^{\epsilon}$ essentially appears in work by Baum-Bott (for ϵ fixed).

- $\widehat{D}_{k-1}^{\epsilon} \circ \varphi_k = \varphi_k \circ \widehat{D}_k^{\epsilon}$ where $\chi_\epsilon \equiv 1$.

Idea of proof

$\lim_{\epsilon\to 0} c(E, \widehat{D}^\epsilon)$ exists, has support on $Z = \operatorname{supp}\mathcal{F}$.

f section such that $Z(f) = Z$,
χ a cut-off function, $\chi(t) \equiv 0$ for $t \ll 1$, $\chi(t) \equiv 1$ for $t \gg 1$.
$\chi_\epsilon = \chi(|f|^2/\epsilon)$
σ_k the "minimal inverse" (Moore-Penrose inverse) of φ_k.

$$\widehat{D}_k^\epsilon = -\chi_\epsilon \sigma_k D\varphi_k + D_k.$$

$\widehat{D}_k^\epsilon$ essentially appears in work by Baum-Bott (for ϵ fixed).

- $\widehat{D}_{k-1}^\epsilon \circ \varphi_k = \varphi_k \circ \widehat{D}_k^\epsilon$ where $\chi_\epsilon \equiv 1$. Implies that $\operatorname{supp} c_\ell(E, \widehat{D}^\epsilon) \subseteq \{\chi_\epsilon \not\equiv 1\}$.

Idea of proof

$\lim_{\epsilon\to 0} c(E, \widehat{D}^\epsilon)$ exists, has support on $Z = \operatorname{supp} \mathcal{F}$.

f section such that $Z(f) = Z$,
χ a cut-off function, $\chi(t) \equiv 0$ for $t \ll 1$, $\chi(t) \equiv 1$ for $t \gg 1$.
$\chi_\epsilon = \chi(|f|^2/\epsilon)$
σ_k the "minimal inverse" (Moore-Penrose inverse) of φ_k.

$$\widehat{D}_k^\epsilon = -\chi_\epsilon \sigma_k D\varphi_k + D_k.$$

$\widehat{D}_k^\epsilon$ essentially appears in work by Baum-Bott (for ϵ fixed).

- $\widehat{D}_{k-1}^\epsilon \circ \varphi_k = \varphi_k \circ \widehat{D}_k^\epsilon$ where $\chi_\epsilon \equiv 1$. Implies that $\operatorname{supp} c_\ell(E, \widehat{D}^\epsilon) \subseteq \{\chi_\epsilon \not\equiv 1\}$.
- Existence of limit by "theory of residue currents" (Andersson-Wulcan).

Explicit description of some Chern currents

The fundamental cycle

Let $\mathcal{F}$ be a coherent analytic sheaf, and let Z_i be the irreducible components of $Z = \operatorname{supp} \mathcal{F}$.

The fundamental cycle

Let $\mathcal{F}$ be a coherent analytic sheaf, and let Z_i be the irreducible components of $Z = \operatorname{supp} \mathcal{F}$.

Definition

The fundamental cycle of $\mathcal{F}$ is

$$[\mathcal{F}] = \sum m_i [Z_i],$$

where m_i is the geometric multiplicity of Z_i in $\mathcal{F}$.

One definition of m_i is that generically on Z_i, $\mathcal{F}$ is locally a free $\mathcal{O}_{Z_i}$-module of rank m_i.

Main result: A description of current $c_\ell^{Res}(E, D)$ in basic situations:

Main result: A description of current $c_\ell^{Res}(E, D)$ in basic situations:

Theorem

Assume that $\mathcal{F}$ is a coherent sheaf of pure codimension $p > 0$ with a finite locally free resolution (E, φ) with connections D_k such that $(D_k)_{(0,1)} = \bar{\partial}$. Then

$$c_p^{Res}(E, D) = (-1)^{p-1}(p-1)![\mathcal{F}].$$

Main result: A description of current $c_\ell^{Res}(E, D)$ in basic situations:

Theorem

Assume that $\mathcal{F}$ is a coherent sheaf of pure codimension $p > 0$ with a finite locally free resolution (E, φ) with connections D_k such that $(D_k)_{(0,1)} = \bar{\partial}$. Then

$$c_p^{Res}(E, D) = (-1)^{p-1}(p-1)![\mathcal{F}].$$

Well-known at class level when $\mathcal{F}$ pushforward of vector bundle on subvariety. Our proof via theory of residue currents associated to a complex.

Main result: A description of current $c_\ell^{Res}(E, D)$ in basic situations:

Theorem

Assume that $\mathcal{F}$ is a coherent sheaf of pure codimension $p > 0$ with a finite locally free resolution (E, φ) with connections D_k such that $(D_k)_{(0,1)} = \bar{\partial}$. Then

$$c_p^{Res}(E, D) = (-1)^{p-1}(p-1)![\mathcal{F}].$$

Well-known at class level when $\mathcal{F}$ pushforward of vector bundle on subvariety. Our proof via theory of residue currents associated to a complex.

Other products of degree $0 < d \leq p$ vanish.

Residue currents associated to a complex

Let (E, φ) be a generically exact complex of holomorphic Hermitian vector bundles

$$0 \to E_N \stackrel{\varphi_N}{\to} \dots \stackrel{\varphi_1}{\to} E_0.$$

Residue currents associated to a complex

Let (E, φ) be a generically exact complex of holomorphic Hermitian vector bundles

$$0 \to E_N \stackrel{\varphi_N}{\to} \dots \stackrel{\varphi_1}{\to} E_0.$$

Andersson-Wulcan defined an associated residue current R^E, where $R^E = \sum_k R_k$, and R_k is a $\mathrm{Hom}(E_0, E_k)$-valued $(0, k)$-current.

Residue currents associated to a complex

Let (E, φ) be a generically exact complex of holomorphic Hermitian vector bundles

$$0 \to E_N \stackrel{\varphi_N}{\to} \dots \stackrel{\varphi_1}{\to} E_0.$$

Andersson-Wulcan defined an associated residue current R^E, where $R^E = \sum_k R_k$, and R_k is a $\mathrm{Hom}(E_0, E_k)$-valued $(0, k)$-current.

Example

Given a section s of a line bundle L, the residue current associated to the complex

$$0 \to -L \stackrel{[s]}{\to} \mathcal{O} \to 0$$

is the current $[\bar\partial(\frac{1}{s})]$, where $\bar\partial(\frac{1}{s}) = \lim_{\epsilon\to 0} \bar\partial\chi(|s|^2/\epsilon)\frac{1}{s}$.

Our proof that

$$c_p^{Res}(E, D) = (-1)^{p-1}(p-1)![\mathcal{F}] \qquad (*)$$

boils down to proving the following theorem.

Our proof that

$$c_p^{Res}(E,D) = (-1)^{p-1}(p-1)![\mathcal{F}] \qquad (*)$$

boils down to proving the following theorem.

Theorem

Assume that $\mathcal{F}$ is a coherent sheaf of codimension $p > 0$ and that (E,φ) is a finite locally free resolution of $\mathcal{F}$ with connections D_k such that $(D_k)_{(0,1)} = \bar{\partial}$. Then

$$c_p^{Res}(E,D) = \frac{(-1)^{p-1}}{(2\pi i)^p p}\,\mathrm{tr}(D\varphi_1 \cdots D\varphi_p R_p^E).$$

Our proof that

$$c_p^{Res}(E, D) = (-1)^{p-1}(p-1)![\mathcal{F}] \qquad (*)$$

boils down to proving the following theorem.

Theorem

Assume that $\mathcal{F}$ is a coherent sheaf of codimension $p > 0$ and that (E, φ) is a finite locally free resolution of $\mathcal{F}$ with connections D_k such that $(D_k)_{(0,1)} = \bar{\partial}$. Then

$$c_p^{Res}(E, D) = \frac{(-1)^{p-1}}{(2\pi i)^p p} \operatorname{tr}(D\varphi_1 \cdots D\varphi_p R_p^E).$$

$(*)$ follows by this theorem and a previous results of ours:

$$\frac{1}{(2\pi i)^p p!} \operatorname{tr}(D\varphi_1 \cdots D\varphi_p R_p^E) = [\mathcal{F}].$$

Formula for Chern forms of a complex

(E, φ) a complex of vector bundles of length N, E_k connection with curvature Θ_k.

Formula for Chern forms of a complex

(E, φ) a complex of vector bundles of length N, E_k connection with curvature Θ_k.
Let

$$e_\ell = (\frac{i}{2\pi})^p \sum_{k=0}^{N} (-1)^k \operatorname{tr} \Theta_k^\ell.$$

(Equals the so-called *Chern character* up to a factor p!)

Formula for Chern forms of a complex

(E, φ) a complex of vector bundles of length N, E_k connection with curvature Θ_k.
Let

$$e_\ell = (\frac{i}{2\pi})^p \sum_{k=0}^{N} (-1)^k \operatorname{tr} \Theta_k^\ell.$$

(Equals the so-called *Chern character* up to a factor $p!$)

$$c_p(E, D) = \frac{(-1)^{p-1}}{p} e_p + \tilde{Q}_p(e_1, \dots, e_{p-1}).$$

Outline of proof that $c_p^{Res}(E, D) = C_p \, \mathrm{tr}(D\varphi_1 \cdots D\varphi_p R_p^E)$:

$$c_p(E, \widehat{D}^{\epsilon})$$

Outline of proof that $c_p^{Res}(E, D) = C_p \operatorname{tr}(D\varphi_1 \cdots D\varphi_p R_p^E)$:

$$c_p(E, \widehat{D}^\epsilon) = C_p e_p + \dots$$

Recall: $e_p = (\frac{i}{2\pi})^p \sum_{k=0}^N (-1)^k \operatorname{tr} \Theta_k^p$

Outline of proof that $c_p^{Res}(E,D) = C_p \operatorname{tr}(D\varphi_1 \cdots D\varphi_p R_p^E)$:

$$\begin{aligned} c_p(E, \widehat{D}^\epsilon) &= C_p e_p + \dots \\ &= C_p \operatorname{tr} \bar\partial \chi_\epsilon \wedge D\varphi_1 \cdots D\varphi_p \sigma_p \bar\partial \sigma_{p-1} \cdots \bar\partial \sigma_1 + \dots \end{aligned}$$

Recall: $e_p = (\frac{i}{2\pi})^p \sum_{k=0}^N (-1)^k \operatorname{tr} \Theta_k^p$

(C_p changes between the lines)

Outline of proof that $c_p^{Res}(E,D) = C_p \operatorname{tr}(D\varphi_1 \cdots D\varphi_p R_p^E)$:

$$\begin{aligned} c_p(E, \widehat{D}^\epsilon) &= C_p e_p + \dots \\ &= C_p \operatorname{tr} \bar\partial \chi_\epsilon \wedge D\varphi_1 \cdots D\varphi_p \sigma_p \bar\partial \sigma_{p-1} \cdots \bar\partial \sigma_1 + \dots \\ &\to C_p \operatorname{tr} D\varphi_1 \cdots D\varphi_p R_p^E \end{aligned}$$

Recall: $e_p = (\frac{i}{2\pi})^p \sum_{k=0}^N (-1)^k \operatorname{tr} \Theta_k^p$

(C_p *changes between the lines*)

Thank you for listening!