About the Blaschke products in polydiscs

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Standard interpolations

Basic, classical (standard) interpolations include interpolations of Lagrange, Hermite, Newton, and others.
Lagrange: By the given points $\{w_j\}_{j=1}^m \subset \mathbb{C}$ and the values $c_j \in \mathbb{C}$, find a polynomial $f(z)$ of degree $m-1$ with the property

$$f(w_j) = c_j, \forall j = 1, \ldots, m.$$

Note that the interpolation polynomial is defined in terms of the polynomial $p(z) = (z - w_1) \cdot \ldots \cdot (z - w_m)$ by the formula:

$$f(z) = p(z) \sum_{j=1}^{m} \frac{c_j}{z - w_j} \operatorname*{res}_{w_j} \left(\frac{1}{p} \right).$$

Thus, setting the interpolation nodes as the zero set of the ideal $\langle p \rangle$ provides tools for constructing the interpolation polynomial.

Hermite: By the given points $\{w_j\}_{j=1}^{m} \subset \mathbb{C}$ and the values $c_{j,k} \in \mathbb{C}$, where $j = 1, \ldots, m,\ \ k = 0, \ldots, \mu_j - 1$ find the polynomial $f(z)$ having at given points given values of derivatives up to orders $\mu_j - 1$, that is

$$f^{(k)}(w_j) = c_{J,K}, \forall j = 1, \ldots, m, \forall k = 0, \ldots, \mu_j - 1.$$

In this problem, the corresponding ideal is taken by the generated polynomial

$$p(z) = (z - w_1)^{\mu_1} \cdots (z - w_m)^{\mu_m}$$

Non-standard 1-dimensional interpolation

Problem *Let* $a_{j,k}$ *(*$j = 1, \ldots, m$*;* $k = 0, \ldots, \mu_j - 1$*) and* c *are the given complex numbers. It is necessary to describe the set of all functions* f *that are analytic in the neighborhood of* $\Omega \subset \mathbb{C}$ *of points* $w_1, \ldots, w_m$ *and satisfy the equation:*

$$\sum_{j=1}^{m} \sum_{k=0}^{\mu_j - 1} a_{j,k} f^{(k)}(w_j) = c. \qquad (1)$$

(D. Alpay, P. Jorgensen, I. Lewkowicz, and D. Volok, 2016). Note that if f is the solution of (1), then $f + ph$ is also the solution, where

$$p(z) = \prod_{j=1}^{m} (z - w_j)^{\mu_j}, \quad h \in H(\Omega).$$

In other words, we can work in the quotient ring $H(\Omega)/(p)$ over the ideal generated by the polynomial p.

definition (Ehrenpreis-Palamodov)

Let $I \subset \mathbb{C}[s_1, \ldots, s_n]$ be a primary ideal. The family of linear differential operators with polynomial coefficients $\partial_\ell(\boldsymbol{s}, D)$, $\ell = 1, \ldots, t$ is called Noetherian operators for I if the conditions

$$\partial_\ell(\boldsymbol{s}, D)\varphi(\boldsymbol{s})|_{V(I)} = 0, \quad \forall \ell = 1, \ldots, t$$

is necessary and sufficient for the function $\varphi(\boldsymbol{s})$ to belong to the ideal I.

In the one-dimensional case, an arbitrary polynomial has the form:

$$p(s) = (s - w_1)^{\mu_1} \cdot \ldots \cdot (s - w_k)^{\mu_k},$$

and the ideal generated by it is decomposed into the intersection of the primary components

$$\rho_j = \langle (s - w_j)^{\mu_j} \rangle, \quad j = 1, \ldots, k.$$

A necessary and sufficient condition for a given polynomial φ to belong to the primary component ρ_j is the vanishing of φ by the following operators with constants coefficients:

$$\mathcal{L}_{j,0}, \mathcal{L}_{j,1}, \ldots, \mathcal{L}_{j,\mu_j - 1},$$

где $\mathcal{L}_{i,j}[\varphi(s)] = \left.\frac{d^j \varphi}{ds^j}\right|_{s=w_i}$.

Noetherian operators in the multidimensional case

For an arbitrary $n \geq 1$, the ρ_j of the zero-dimensional polynomial ideal $\langle p_1, \ldots, p_n \rangle$ are assigned to the roots of w_j. The Noetherian operators for ρ_j are sets of differential operators with constant coefficients

$$\mathcal{L}_{w_j,\ell}(\partial/\partial s)|_{w_j}, \ell \in A_{w_j}.$$

Here A_{w_j} is a finite subsets in $\mathbb{N}^n$.

Problem (Alpay D., Yger A., 2019):

Let $\boldsymbol{p}^{-1}(0) = \{w_1, \ldots, w_m\}$, *U be an open subset of* $\mathbb{C}^n$, *containing* $\boldsymbol{p}^{-1}(0)$. *Let* $a_{j,\ell}$ $(j = 1, \ldots, m,\ \ell \in A_{w_j})$ *together with the number c be given complex numbers. It is necessary to describe the space of holomorphic functions* $f : U \to \mathbb{C}$, *with the property:*

$$\sum_{j=1}^{m} \sum_{\ell \in A_{w_j}} a_{j,\ell} \mathcal{L}_{w_j,\ell}[f](w_j) = c. \tag{2}$$

Basis in $\mathbb{C}[\boldsymbol{s}]/(\boldsymbol{p})$

One of the tools for solving the interpolation problem is the monomial basis

$$\mathcal{B} = \{\boldsymbol{s}_k^{\beta}; k = 0, \ldots, N(\boldsymbol{p}) - 1\}$$

in the factor space $\mathbb{C}[\boldsymbol{s}]/(\boldsymbol{p})$. In fact, this quotient is the space of remainders when dividing polynomials by the ideal $(\boldsymbol{p})$. The $\mathcal{B}$ basis is constructed using the Grobner basis for the ideal $(\boldsymbol{p})$.

Grothendieck residue

definition Let $h, p_1, \ldots, p_n$ be a holomorphic functions in the neighborhood U_a of a point $a \in \mathbb{C}^n$, and the map $\boldsymbol{p} = (p_1, \ldots, p_n)$ has an isolated zero at the point a. *Local Grothendieck residue* of the meromorphic form $\omega = h\, dz/(p_1 \cdots p_n)$ at the point a is called the integral

$$\operatorname*{res}_a \omega = (2\pi i)^{-n} \int\limits_{\Gamma_a} \frac{h(z)\, dz_1 \wedge \ldots \wedge dz_n}{p_1(z) \cdots p_n(z)}.$$

In the case of a finite set of zeros of map $\boldsymbol{p} = (p_1, \ldots, p_n)$ the global Grothendieck residue is defined as the sum of all local residues

$$\operatorname{Res}[\omega] = \sum_a \operatorname*{res}_a \omega.$$

Theorem(Alpay, Yger) Пусть $\{w_1, \dots, w_m\} = \boldsymbol{p}^{-1}(0)$, U be an open subset in $\mathbb{C}^n$, containing $\boldsymbol{p}^{-1}(0)$. Let the sequence

$$\boldsymbol{a} = \{a_{j,\boldsymbol{\ell}}, j = 1, \dots, m, \boldsymbol{\ell} \in A_{w_j}\}$$

and the complex number c are given. Let's define the germs in $\mathcal{O}_{\mathbb{C}^n, w_j}$

$$h^{\boldsymbol{a}}_{w_j}(\boldsymbol{s}) = \sum_{\boldsymbol{\lambda} \in A_{w_j}} a_{j,\boldsymbol{\lambda}} (\boldsymbol{s} - w_j)^{\boldsymbol{\lambda}} / \boldsymbol{\lambda}!,$$

which make up the set $\boldsymbol{h}^{\boldsymbol{a}}_{\boldsymbol{w}} = [h^{\boldsymbol{a}}_{w_1}, \dots, h^{\boldsymbol{a}}_{w_m}]$. Then:

- either the coordinate system $(\alpha_0[\boldsymbol{h}^{\boldsymbol{a}}_{\boldsymbol{w}}], \dots, \alpha_{N(\boldsymbol{p})-1}[\boldsymbol{h}^{\boldsymbol{a}}_{\boldsymbol{w}}])$ is null system, then the problem has no solutions in the case $c \neq 0$, and any function $f \in H(U)$ is the solution in the case $c = 0$;

- either the coordinate system $(\alpha_0[\boldsymbol{h}_{\boldsymbol{w}}^{\boldsymbol{a}}], \ldots, \alpha_{N(\boldsymbol{p})-1}[\boldsymbol{h}_{\boldsymbol{w}}^{\boldsymbol{a}}])$ isn't null system, then $f \in H(U)$ satisfies the condition (2) if and only if

$$\left[\alpha_0[f], \ldots, \alpha_{N(\boldsymbol{p})-1}[f]\right] \cdot \boldsymbol{Q}_{\boldsymbol{p}}[\mathcal{B}] \begin{bmatrix} \alpha_0([\boldsymbol{h}_{\boldsymbol{w}}^{\boldsymbol{a}}]) \\ \vdots \\ \alpha_{N(p)-1}([\boldsymbol{h}_{\boldsymbol{w}}^{\boldsymbol{a}}]) \end{bmatrix} = c,$$

where

$$\boldsymbol{Q}_{\boldsymbol{p}}[\mathcal{B}] = \mathrm{Res}\left[\frac{\boldsymbol{s}^{\beta_{k_1}+\beta_{k_2}}\, d\boldsymbol{s}}{p_1(\boldsymbol{s}) \cdots p_n(\boldsymbol{s})}\right]_{0 \le k_1, k_2 \le N(\boldsymbol{p})-1}.$$

Local case of non-standard problem

Let $p = (p_1, \ldots, p_n)$ be an ideal with isolated null at the origin. Denote $\{\mathcal{L}_{0,j}\}$ as a set of Noetherian operators for the primary component of this ideal at zero. We consider the problem of constructing a holomorphic germ at the origin with the condition

$$\sum_j a_j \mathcal{L}_{0,j}[f](0) = c, \tag{3}$$

where $\{a_j\}$ and c are given numbers.

Example

Let's consider the following system in $\mathbb{C}^3$:

$$\begin{aligned} p_1 &= z_1^3 - z_2 z_3 \\ p_2 &= z_2^3 - z_1 z_3 \\ p_3 &= z_3^3 - z_1 z_2 \end{aligned}$$

This map $\boldsymbol{p}(\boldsymbol{z}) : \mathbb{C}^3 \to \mathbb{C}^3$ has a discrete zero set consisting 17 isolated zeros:

$$\begin{aligned} \{&(0,0,0),(1,1,1),(1,-1,-1),(1,-i,i),(1,i,-i),\\ &(-1,-1,1),(-1,1,-1),(-1,i,i),(-1,-i,-i),\\ &(i,-i,1),(i,i,-1),(i,-1,i),(i,1,-i),\\ &(-i,i,1),(-i,-i,-1),(-i,1,i),(-i,-1,-i)\} \end{aligned}$$

proposition

Set of differential operators:

$$\{\mathcal{L}_{0,j}\}_{j=1}^{11} = \Big\{\partial^0, \frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2}, \frac{\partial}{\partial z_3}, \frac{\partial^2}{\partial z_1^2}, \frac{\partial^2}{\partial z_2^2}, \frac{\partial^2}{\partial z_3^2},$$
$$\left(\frac{\partial^3}{\partial z_1^3} + 6\frac{\partial^2}{\partial z_2 \partial z_3}\right), \left(\frac{\partial^3}{\partial z_2^3} + 6\frac{\partial^2}{\partial z_1 \partial z_3}\right), \left(\frac{\partial^3}{\partial z_3^3} + 6\frac{\partial^2}{\partial z_1 \partial z_2}\right),$$
$$\left(\frac{\partial^4}{\partial z_1^4} + \frac{\partial^4}{\partial z_2^4} + \frac{\partial^4}{\partial z_3^4} + 24\frac{\partial^3}{\partial z_1 \partial z_2 \partial z_3}\right)\Big\}$$

is a set of Noetherian operators for the ideal:

$$I_0(\boldsymbol{p}) = \{(z_1^3 - z_2 z_3)h_1 + (z_2^3 - z_1 z_3)h_2 + (z_3^3 - z_1 z_2)h_3\},$$

где $h_1, h_2, h_3 \in \mathcal{O}_0$.

theorem (1)

If the coordinates of the germ $h_0^{\boldsymbol{a}}(\boldsymbol{s})$ aren't all equal to zero, then the holomorphic function $f(\boldsymbol{s})$ satisfies problem 2, if and only if the following condition on the coefficients of this function in the factor space is satisfied $H(U_0)/(\sum_1^n H(U_0)p_j)_{loc}$:

$$\left(a_1 + \frac{27}{24}a_{11}\right)\alpha_1[f] + \frac{7a_8}{6}\alpha_2[f] + \frac{7a_9}{6}\alpha_3[f] + \frac{7a_{10}}{6}\alpha_4[f] + \frac{a_5}{2}\alpha_5[f] + \frac{a_6}{2}\alpha_6[f] + \frac{a_7}{2}\alpha_7[f] + a_4\alpha_8[f] + a_3\alpha_9[f] + a_2\alpha_{10}[f] + a_1\alpha_{11}[f] = -c. \tag{4}$$

That is, the coordinate vector of the function f in the factor space lies in the given affine hyperplane $\Pi^{\boldsymbol{a}} \in \mathbb{C}^{11}$

definition

A Blaschke product is a function of the form:

$$B(z) = e^{i\alpha} z^K \prod_{n\geq 1}^{\infty} \frac{|z_n|}{z_n} \frac{z_n - z}{1 - \bar{z}_n z}, \tag{5}$$

in which $\alpha \in \mathcal{R}, K \in \mathcal{N}_0$ and $\{z_1, z_2, \ldots, z_n, \ldots\}$ is a sequence in the open unit disk without origin $D\backslash\{0\}$ that satisfies the Blaschke condition:

$$\sum_n (1 - |z_n|) < \infty$$

definition

A finite Blaschke product is a function of the form:

$$B(z) = \prod_{k \geq 1}^{n} \frac{z_k - z}{1 - \bar{z}_k z}, \tag{6}$$

where $\{z_1, z_2, ..., z_n\}$ – a finite set in the open unit disk D.

The definition of Blaschke product allowed us to solve important problems of the interpolation theory in the unit disk. For example, Blaschke's theorem states that a sequence $\{z_k\}$ in a disk is a null set for a holomorphic function bounded in D if and only if the sequense satisfies the Blaschke condition.

In addition to bounded functions, similar descriptions could be applicable for functions from Hardy classes.
Each factor $\frac{z_k - z}{1 - \bar{z}_k z}$ of the product (6) is a rational function of the form:

$$b_k = \frac{p(z)}{q(z)} = z\frac{\overline{q(1/\bar{z})}}{q(z)}.$$

In the case when q has real coefficients, b_k can be represented as:

$$b_k = z\frac{q(1/z)}{q(z)}.$$

By the analog of the Blaschke factor in $\mathbb{C}^3$, we mean the triple of special inner rational functions in the unit polydisk of $\mathbb{C}^3$. We construct inner rational functions using the Lee-Yang polynomial. To do this, we fix an arbitrary symmetric $n * n$ matrix (a_{jk}) with real coefficients satisfying the condition $0 < |a_j k| < 1$. The corresponding Lee-Yang polynomial is constructed from the given matrix as follows:

$$f(z_1, z_2, \ldots, z_n) = \sum_J \prod_{j \in J} \left(z_j \prod_{k \notin J} a_{jk} \right),$$

where J runs through the set of all subsets of the set $\{1, 2, \ldots, n\}$.

In view of the following expression:

$$f(z_1, z_2, \ldots, z_n) = z_1 z_2 \ldots z_n f(1/z_1, 1/z_2, \ldots, 1/z_n)$$

the amoeba of the polynomial f is symmetric with respect to the origin. Moreover, the following theorem is valid:

theorem (M. Passare, A. Tsikh)

Let A be an amoeba of the Lee-Yang polynomial, then the closed positive and negative orthants $\pm\mathbb{R}^n_+$ intersect the amoeba A only at the origin:

$$\mathbb{R}^n_+ \cap A = -\mathbb{R}^n_+ \cap A = \{0\}$$

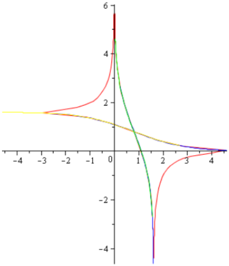

Рис.: in the case n=2

Let's consider the Lee-Yang polynomial of three variables. It is associated with the following matrix:

$$(a_{jk}) = \begin{pmatrix} a_{11} & a & b \\ a & a_{22} & c \\ b & c & a_{33} \end{pmatrix},$$

where $\{a_{11}, a_{22}, a_{33}, a, b, c\} \in (-1,1)\backslash\{0\}$. The corresponding Lee-Yang polynomial will look like:

$$f = (z_1 z_2 z_3 + bcz_1 z_2 + abz_2 z_3 + acz_1 z_3) + (abz_1 + acz_2 + bcz_3 + 1).$$

Denote the left bracket as f_1, the right bracket as f_2 and fix some point (z_1^0, z_2^0, z_3^0) from the skeleton $\Delta = \{|z_j| = 1, j = 1,2,3\}$ of the polydisk D. Consider the following set of functions:

$$\begin{matrix} p_1 = f_1(z_1^0, z_2, z_3) & p_2 = f_1(z_1, z_2^0, z_3) & p_3 = f_1(z_1, z_2, z_3^0) \\ q_1 = f_2(z_1^0, z_2, z_3) & q_2 = f_2(z_1, z_2^0, z_3) & q_3 = f_2(z_1, z_2, z_3^0) \end{matrix}$$

definition

Mapping $\left(\frac{p_1}{q_1}, \frac{p_2}{q_2}, \frac{p_3}{q_3}\right)$ we call the three-dimensional analog of the Blaschke factor if the zeros of the denominators q_i do not intersect the open unit polydisk D^3.(in this case, we will call q_i admissible)

definition

A space $H^p(D^n)$ of holomorphic in the polydisk D^n functions f with norm

$$||f||_p = \sup_{r\in(0,1)} \left\{ \int\limits_{w\in T^n} |f(rw)|^p dm_n \right\}^{1/p},$$

where the measure $dm_n = \frac{1}{(2\pi i)^n} \frac{dw_1}{w_1} \wedge \ldots \wedge \frac{dw_n}{w_n}$, is called a Hardy space of order p.

definition

A function $g \in H^{\infty}(D^n)$ is called inner if its radial limit values satisfy the condition $|g^*(w)| = 1$ almost everywhere on T^n.

theorem (2)

Functions p_j/q_j in the definition of the Blaschke factor are inner functions in the polycircle D^3.

To describe the admissible denominators q_i, consider the polar set of cube $K = [-1, 1]^3$, which consists of solutions of inequality $|x| + |y| + |z| \leq 1$ (in the conjugate space).

theorem (3)

The denominators q_i are admissible if and only if $(ab, ac, bc) = (x, y, z)$ lie in the polar set and satisfy the system of inequalities:

$$|x| > |yz|, |y| > |xz|, |z| > |xy|.$$

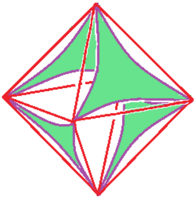

Рис.: triple (x, y, z) corresponding to valid denominators on the bound of the polar set

Thank you for your attention!

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