MULTIDIMENSIONAL RESIDUES AND TROPICAL GEOMETRY

HOLOMORPHIC CONTINUATION OF A FORMAL SERIES ALONG ANALYTIC CURVES

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1. I was initiated to the study of functions holomorphic along analytic curves, by recent works of E. Chirka [2] and a series of papers [3-5] by K.-T. Kim, E. Poletsky, G. Schmalz, J.-Ch. Joo and Ye-Won Cho, the results type whell-known Forelli's theorem[1]: if f is an infinitely smooth function in a neighborhood of a point 0 and the restrictions $f|_l$ are holomorphic in the disc $U(0,1) = l \cap B(0,1)$ for all complex lines $l \ni 0$, then it can be holomorphically extended to the ball $B(0,1) \subset \mathbb{C}^n$.

I will present some of them in a simplified form.

1. E. Chirka (2006). *Let S_τ be a foliation of holomorphic curves passing through the origin. Suppose that the curves are closed and smooth in $B(0,1) \subset \mathbb{C}^2$ and pairwise transversal. If $f(z)$ satisfies the following two conditions:*

(1) $f \in C^\infty\{0\}$,

(2) $f|_{S_\tau}$ *is holomorphic for each* τ.

Then f is holomorphic in $B(0,1) \subset \mathbb{C}^2$.

2. Joo-Kim-Schmalz (2013). *Let Ω be a domain in $\mathbb{C}^n$, containing the origin with a C^1 radial foliation S_τ at the origin. If $f : \Omega \to \mathbb{C}$ satisfies the following two conditions:*

(1) $f \in C^\infty\{0\}$,

(2) $f|_{S_\tau}$ *is holomorphic for each* τ.

Then f is holomorphic in Ω.

The following theorem gives an answer to one of the problems of E. Chirka.

3. Cho-Kim (2020). *Let $U \subset S(0,1) \subset \mathbb{C}^n$ is an open set on the unit sphere and $\mathfrak{I} = \bigcup\{l_w : z = w\xi, w \in U, \xi \in \mathbb{C}, |\xi| < 1\}$ is a pencil of complex disks in $\mathbb{C}^n$. Consider a domain $\Omega \ni 0$ and a pencil $\varphi : \mathfrak{I} \to \Omega$ of*

Riemannian space, where φ is a diffeomorphism. If $f : \Omega \to \mathbb{C}$ satisfies the following two conditions:

(1) $f \in C^{\infty}\{0\}$,

(2) $f|_{\varphi(l_w)}$ *is holomorphic for each* $w \in U$.

Then $\exists r > 0 : f$ *is holomorphic in* $B(0,r) \cup \varphi(\Im)$.

2. The present talk is devoted to an exact description of the domains of holomorphy of functions that are holomorphic on some pencil of analytic curves S_τ. In the descriptions, the pluripotential theory, more precisely the multidimensional Green's function $V(z,K)$ and its propertiesis widely used. We begin with the following essential generalization of Forelli's theorem.

Theorem 1 (cf. [AS]) *Let a set (pencil) of complex lines* $\Im \subset \{z = w\xi : |w| = 1, \xi \in \mathbb{C}\} = \mathbb{P}^{n-1}$ *be given. Let for each complex line* $l \in \Im$ *the restriction* $\sum_{m=0}^{\infty} c_m(w)\,\xi^m$ *of the formal homogeneous power series*

$$f \sim \sum\nolimits_{m=0}^{\infty} c_m(z), \tag{0.1}$$

to l *converges in a disk* $|\xi| < r_l, 0 \le r_l < \infty$. *Here* $c_m(z) = \sum_{|k|=m} c_k z^k, |k| = (k_1, k_2, \dots, k_n), |k| = k_1 + k_2 + \dots + k_n$,
$z^k = z_1^{k_1} z_2^{k_2} \dots z_n^{k_n}$, *is the homogeneous polynomials.*

Then this series converges in an open set

$$G = \left\{ z \in \mathbb{C}^n : |z| \exp V^*\left(\frac{z}{|z|}, E\right) < 1 \right\},$$

where $E = \bigcup_{l \in \Im} l \bigcap B(0, r_l)$ *and* $V^*(\omega, E)$ *is Green function of the set* $E \subset \mathbb{C}^n$.

Corollary 1. (Generalization of Forelli's theorem) *Let the pencil* $\Im$ *is not* $R-$ *polar. If* f *is an infinitely smooth function in a neighborhood of the point* 0 *and the restrictions* $f|_l$ *holomorphically continues to the disk* $U(0,r_l), 0 \le r_l < \infty$ *for all complex lines* $l \in \Im$, *then* f *can be holomorphically continued into the domain*

$$G = \left\{ z \in \mathbb{C}^n : |z| \exp V^*\left(\frac{z}{|z|}, E\right) < 1 \right\}.$$

S k e t c h o f t h e p r o o f. The condition of infinite smoothness of the function f reduces the proof of Forelli's theorem to the proof of holomorphy of the formal homogeneous power series

$$f(z) \sim \sum_{i,j=0}^{\infty} c_{i,j}(z,\bar{z}) = \sum_{i=0}^{\infty} c_{i,0}(z) + \sum_{i=0,j=1}^{\infty} c_{i,j}(z,\bar{z}), \tag{0.2}$$

where

$$c_{i,j}(z,\bar{z}) = \sum_{|I|=i,\ |J|=j} c_{IJ} z^I \bar{z}^J,\ \ I=(i_1,i_2,...,i_n),\ |I|=i_1+i_2+...+i_n,$$

$$J=(j_1,j_2,...,j_n),\ \ |J|=j_1+j_2+...+j_n,\ z^I=z_1^{i_1}z_2^{i_2}...z_n^{i_n},\ \ \bar{z}^J=\bar{z}_1^{j_1}\bar{z}_2^{j_2}...\bar{z}_n^{j_n},$$

is homogeneous polynomials of variables $(z,\bar{z})$ of bedegree (i,j).

The conditions of the homomorphy of the restrictions $f|_l$ means that for each $l=\{z=w\xi\}\in\Im$ the $\sum_{i,j=0}^{\infty} c_{i,j}(w\xi,,\bar{w}\bar{\xi})$ of the series $\sum_{i,j=0}^{\infty} c_{i,j}(z,\bar{z})$ converges. Moreover,

$$\sum_{i=0,j=1}^{\infty} c_{i,j}(w\xi,\bar{w}\bar{\xi})\equiv 0, \tag{0.3}$$

and the sum $\sum_{i=0}^{\infty} c_{i,j}(w\xi,\bar{w}\bar{\xi})=\sum_{i=0}^{\infty} c_{i,0}(w\xi)$ is holomorphic in the disk $U(0,r_l), 0\le r_l<\infty$.

Now applying Theorem 0.1 we take the proof of the Corollary 1.

S k e t c h o f t h e p r o o f o f T h e o r e m 1. We give the proof in the simple case, $0<r\le 1\ \forall l\in\Im$. We fix the numbers $N\in\mathbb{N}, r>0, 0<\varepsilon<r$ and put $\Im_r=\{l\in\Im: r_l\ge r\}$. Denote $F_{N,r,\varepsilon}=\{w\in S(0,1): l=\{z=w\xi\}\in\Im_r, |\sum_{m=0}^{\infty} c_m(w)\xi^m|\le N\forall|\xi|\le r-\varepsilon\}$. Then by Cauchy's inequalities $|c_m(w)|\le\frac{N}{(r_l-\varepsilon)^m}, w\in F_{N,r,\varepsilon}, m=0,1,2,\dots$, i.e. $|c_m((r_l-\varepsilon)w)|\le N, w\in F_{N,r,\varepsilon}, m=0,1,2,\dots$.Since $F_{N,r,\varepsilon}$ is circular compact, then $|c_m(z)|\le N, z=w\xi, |\xi|\le r_l-\varepsilon, w\in F_{N,r,\varepsilon}, m=0,1,2,\dots$. This implies, $|c_m(z)|\le N\forall z\in E_{N,r,\varepsilon}, m=0,1,2,\dots$, where $E_{N,r,\varepsilon}=\{z=w\xi: w\in F_{N,r,\varepsilon}, |\xi|\le r_l-\varepsilon\}$. By Bernstein-Walsh inequality $|c_m(z)|\le N\exp mV^*(z,E_{N,r,\varepsilon}), z\in\mathbb{C}^n, m=0,1,2,\dots$. In particular,

$$\left|c_m\left(\frac{z}{|z|}\right)\right|\le N\exp mV^*\left(\frac{z}{|z|},E_{N,r,\varepsilon}\right), z\in\mathbb{C}^n\setminus\{0\}, m=0,1,2,\dots,$$

which is equivalent to the inequality

$$|c_m(z)|\le N\left[|z|\exp V^*\left(\frac{z}{|z|},E_{N,r,\varepsilon}\right)\right]^m, z\in\mathbb{C}^n\setminus\{0\}, m=0,1,2,\dots.$$

It follows that the homogeneous series $\sum_{m=0}^{\infty} c_m(z)$ converges in

$$G_{N,r,\varepsilon}=\left\{z\in\mathbb{C}^n: |z|\exp V^*\left(\frac{z}{|z|},E_{N,r,\varepsilon}\right)<1\right\}.$$

Tending first $N\to\infty$ and then $\varepsilon\to 0$ we obtain the uniformly convergence of the series $\sum_{m=0}^{\infty} c_m(z)$ inside the open set $G_r=\left\{z\in\mathbb{C}^n: |z|\exp V^*\left(\frac{z}{|z|},E_r\right)<1\right\}$. As $r\downarrow 0$, the set E_r increasing converges to E. Consequently, $V^*\left(\frac{z}{|z|},E_r\right)\downarrow$

$V^*\left(\frac{z}{|z|}, E\right)$ and the series $\sum_{m=0}^{\infty} c_m(z)$ uniformly converges inside open set

$$G = \left\{ z \in \mathbb{C}^n : |z| \exp V^*\left(\frac{z}{|z|}, E\right) < 1 \right\},$$

the sum of which coincides with $f(z)$ in the neighborhood of 0. □

3. Theorem 2. *Let an arbitrary set $E \subset \mathbb{C}^n$ be given. Suppose the series of homogeneous polynomials $f(z) = \sum_{m=0}^{\infty} c_m(z)$, where $c_m(z)$ are homogeneous polynomials of degrees m, converge at each point $z^0 \in E, f(z^0) = \sum_{m=0}^{\infty} c_m(z^0)$. Then this series converges uniformly inside the ball*

$$B\left(0, \frac{1}{\exp \gamma(E)}\right) = \left\{ z \in \mathbb{C}^n : | z |< \frac{1}{\exp \gamma(E)} \right\}. \tag{0.4}$$

Here $\gamma(E) = \overline{\lim_{w \to z}} \left[V^*(z, E) - \ln |z|\right]$ is the Robin constant of the set E.

Corollary 2. (sf. [9]) *Under the conditions of Theorem 2, if the set E is not pluripolar in $\mathbb{C}^n$, then the formal series $\sum_{m=0}^{\infty} c_m(z)$ has holomorphic sum $f(z) = \sum_{m=0}^{\infty} c_m(z)$, at least in the nonempty ball $|z| < \exp^{-1}\gamma(E)$.*

P r o o f o f t h e t h e o r e m 4.5. Fix a numbers $0 < \varepsilon < 1, N \in \mathbb{N}$. Put

$$E_{\varepsilon,N} = \left\{ z \in E : |c_m(z)|^{1/m} \leq 1 + \varepsilon, m \geq N \right\}. \tag{0.5}$$

It is clear that $E_{\varepsilon,N} \subset E_{\varepsilon,N+1}, N = 1, 2, ...$, and since the series $\sum_{m=0}^{\infty} c_m(z)$ converges at each point $z^0 \in E$, then $E = \bigcup_{m=1}^{\infty} E_{\varepsilon,N}$. From continuity, the inequality in (0.5) is true up to the closure $\overline{E}_{\varepsilon,N}$, i.e. $\|c_m(z)\|_{\bar{E}_{\varepsilon,N}}^{1/m} \leq 1 + \varepsilon, m \geq N$. By Bernstein-Walsh inequality

$$|c_m(z)|^{1/m} \leq (1+\varepsilon) \exp V\left(z, \bar{E}_{\varepsilon,N}\right), z \in \mathbb{C}^n, m \geq N. \tag{0.6}$$

It follows that for a fixed radius $R > 0$

$$|c_m(z)|^{1/m} \leq (1+\varepsilon) \max_{|z|=R} \exp V^*\left(z, \bar{E}_{\varepsilon,N}\right), z \in \partial B(0, R), m \geq N,$$

and for arbitrary $z \in \mathbb{C}^n$, from here we have

$$|c_m(z)|^{1/m} \leq \left| c_m\left(\frac{|z|}{R} \cdot \frac{Rz}{|z|}\right) \right|^{1/m} \leq \frac{|z|}{R} \left| c_m\left(\frac{Rz}{|z|}\right) \right|^{1/m}$$

$$\leq (1+\varepsilon) |z| \frac{\max_{|z|=R} \exp V^*\left(z, \bar{E}_{\varepsilon,N}\right)}{R}, m \geq N.$$

Using this inequality, as $R \to \infty$ we get

$$|c_m(z)|^{1/m} \leq (1+\varepsilon) |z| \exp \gamma\left(\bar{E}_{\varepsilon,N}\right), z \in \mathbb{C}^n, m \geq N. \tag{0.7}$$

It follows from (0.7) that the series $f(z)=\sum_{m=0}^{\infty}c_m(z)$ converges uniformly in the ball

$$B\left(0,\frac{1}{(1+\varepsilon)\exp\gamma(E_{\varepsilon,N})}\right)=\left\{|z|<\frac{1}{(1+\varepsilon)\exp\gamma(E_{\varepsilon,N})}\right\}. \tag{0.8}$$

Tending first $N\to\infty$, and then $\varepsilon\to 0$ we get from (0.8) that the series $\sum_{m=0}^{\infty}c_m(z)$ converges uniformly in the ball

$$B\left(0,\frac{1}{\exp\gamma(E)}\right)=\left\{|z|<\frac{1}{\exp\gamma(E)}\right\}.$$

The theorem is proved.

3. Holomorphy along analytic curves. Using Theorem 2, we prove the following curvilinear variation of Forelli's Theorem.

Theorem 3. *Let the domain $0\in\Omega\subset\mathbb{C}^n$ be fibered by a smooth radial family of analytic curves $A_\lambda=\{z=p_\lambda(\xi)\},\lambda\in\mathbb{P}^{n-1}$, at the point 0, where $p_\lambda(\xi)=\left(p_\lambda^1(\xi),p_\lambda^2(\xi),...,p_\lambda^n(\xi)\right)$ is a holomorphic vector function in the unit disk $U=\{|\xi|<1\}:p_\lambda(\xi)=a_1(\lambda)\xi+a_2(\lambda)\xi^2+...,a_k(\lambda)\in C^1(\mathbb{P}^{n-1}),\ k=1,2,...,\ a_1(\lambda)\neq 0,\ \Omega=\bigcap_\lambda A_\lambda$. If the function $f\in C^\infty(V)$, where $V\subset\mathbb{C}^n$ is some neighborhood of $0\in\mathbb{C}^n$, has the property that each restriction $f|_{A_\lambda},\ \lambda\in\mathbb{P}^{n-1}$, that is defined in the neighborhood $V\bigcap A_\lambda$, holomorphically continues to the whole A_λ, then f continues holomorphically to Ω.*

P r o o f. Expand, as above, the infinitely smooth function f in the Taylor formal series

$$f(z)\sim\sum_{i,j=0}^{\infty}c_{i,j}(z,\bar z)=\sum_{i=0}^{\infty}c_{i,0}(z)+\sum_{i=0,j=1}^{\infty}c_{i,j}(z,\bar z), \tag{0.9}$$

where

$$c_{i,j}(z,\bar z)=\sum_{|I|=i,\ |J|=j}c_{IJ}z^I\bar z^J,\ \ I=(i_1,i_2,...,i_n),\ |I|=i_1+i_2+...+i_n,$$

$$J=(j_1,j_2,...,j_n),\ \ |J|=j_1+j_2+...+j_n,\ z^I=z_1^{i_1}z_2^{i_2}...z_n^{i_n},\ \ \bar z^J=\bar z_1^{j_1}\bar z_2^{j_2}...\bar z_n^{j_n},$$

is homogeneous polynomials of variables $(z,\bar z)$ of bedegree (i,j). The conditions of the theorem means that for each $\lambda\in\mathbb{P}^{n-1}$ the restriction $\sum_{i,j=0}^{\infty}c_{i,j}(p_\lambda(\xi),\bar p_\lambda(\xi))$ of the series $\sum\limits_{i,j=0}^{\infty}c_{i,j}(z,\bar z)$ to the A_λ converges. Moreover,

$$\sum_{i=0,j=1}^{\infty}c_{i,j}(p_\lambda(\xi),\bar p_\lambda(\xi))\equiv 0, \tag{0.10}$$

and the sum $\sum\limits_{i=0}^{\infty} c_{i,0}\left(p_\lambda(\xi), \bar{p}_\lambda(\xi)\right)$ is holomorphic in the unit disk $U: \ |\xi| < 1$ and in a neighborhood of zero coincides with $f|_{A_\lambda} = f\left(p_\lambda(\xi)\right).$

Identity (0.10) for all $\lambda \in \mathbb{P}^{n-1}$ is equivalent to identities $c_{i,j}(z, \bar{z}) \equiv 0, \ i = 0, 1, 2, ..., \ j = 1, 2, ...$ This follows from the fact that $\Omega = \bigcap_\lambda A_\lambda$ and according to (0.10) the series $\sum\limits_{i=0, j=1}^{\infty} c_{i,j}(z, \bar{z})$ converges in Ω, its sum is identically equal to zero.

And so, we have proved that the formal series (0.9) has the form

$$f(z) \sim \sum_{i=0}^{\infty} c_{i,0}(z),$$

and it converges at each point $z^0 \in \Omega$. Moreover, it follows from the conditions of the theorem 3, that $f(z) = \sum\limits_{i=0}^{\infty} c_{i,0}(z), \forall z \in V.$

Now, we use the proof of Theorem 2 in order to establish the uniform convergence of the series $\sum\limits_{i=0}^{\infty} c_{i,0}(z)$ inside Ω. According to inequality (0.6)

$$|c_{i,0}(z)|^{1/i} \leq (1+\varepsilon)\exp V\left(z, \bar{E}_{\varepsilon,N,G}\right), \ z \in \mathbb{C}^n, \ i \geq N. \tag{0.11}$$

Here, $\varepsilon > 0, \ G \subset\subset \Omega$ is a fixed domain and $E_{\varepsilon,N,G} = \left\{z \in \bar{G}: \ |c_{i,0}(z)|^{1/i} \leq 1 + \varepsilon, \ i \geq N\right\}$. Since $V\left(z, \bar{E}_{\varepsilon,N,G}\right) = 0, \forall z \in \bar{E}_{\varepsilon,N,G}$, then according (0.11) $|c_{i,0}(z)|^{1/i} \leq (1+\varepsilon), \forall z \in \bar{E}_{\varepsilon,N,G}$ and $|c_{i,0}(\rho z)|^{1/i} \leq \rho(1+\varepsilon), \forall \ 0 < \rho < 1, \ z \in \bar{E}_{\varepsilon,N,G}$. From here, it follows that the series $\sum\limits_{i=0}^{\infty} c_{i,0}(z)$ converges uniformly on the compact $\frac{1}{1+2\varepsilon}\bar{E}_{\varepsilon,N,G}$. Since $E_{\varepsilon,N,G} \subset E_{\varepsilon,N+1,G}, \ N = 1, 2, ...,$ and $G \subset \bar{G} = \bigcup\limits_{N=1}^{\infty} \bar{E}_{\varepsilon,N,G}$, then the series $\sum\limits_{i=0}^{\infty} c_{i,0}(z)$ converges uniformly inside the domain $\frac{1}{1+2\varepsilon}\bar{G}$. Tending first $\varepsilon \downarrow 0$, and then the domain $G \subset\subset \Omega$ to Ω we get uniform convergence of $\sum\limits_{i=0}^{\infty} c_{i,0}(z)$ inside Ω, i.e. the function $f(z)$, which was infinitely smooth in a neighborhood $V \ni 0$ extends holomorphically to Ω, as a sum $f(z) = \sum\limits_{i=0}^{\infty} c_{i,0}(z)$. *The theorem is proved.*

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