Topology of Phase Tropical Varieties

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Summary

- Overview on tropical geometry

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- Phase tropical varieties

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- Main results

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- Main results
- Perspectives and Questions

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Definition

The *amoeba* $\mathscr{A}(V)$ of an algebraic variety $V \subset (\mathbb{C}^*)^n$ is by definition (see M. Gelfand, M.M. Kapranov and A.V. Zelevinsky (1994)) the image of V under the map :

$$\begin{array}{cccc} \mathrm{Log} & : & (\mathbb{C}^*)^n & \longrightarrow & \mathbb{R}^n \\ & & (z_1, \ldots, z_n) & \longmapsto & (\log|z_1|, \ldots, \log|z_n|). \end{array}$$

Let $\mathbb{K}$ be an algebraically closed valued field whose value group G is a non-zero divisible additive subgroup of $\mathbb{R}$. Its valuation is a surjective homomorphism $\mathbb{K}^\times \to G$, which induces a map $\nu\colon (\mathbb{K}^\times)^n \to G^n$. The closure in $\mathbb{R}^n$ of the image of a variety $V \subset (\mathbb{K}^\times)^n$ under the map ν is its nonarchimedean amoeba, $\mathscr{T}(V)$.

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$$f \; = \; \sum_{a \in \mathcal{A}} c_a x^a \qquad \text{where} \qquad c_a \in \mathbb{K}^\times \, .$$

where $\mathcal{A} \subset \mathbb{Z}^n$ is the support of f which we suppose finite.

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where $\mathcal{A} \subset \mathbb{Z}^n$ is the support of f which we suppose finite.

The coordinate ring of $(\mathbb{K}^\times)^n$ is the ring of Laurent polynomials $\mathbb{K}[x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}]$. Given a vector $w \in \mathbb{R}^n$ and a Laurent polynomial f with support $\mathcal{A} \subset \mathbb{Z}^n$, we have a piecewise linear map

$$\mathbb{R}^n \ni z \longmapsto \min\{\nu(c_a) + w \cdot a \,|\, a \in \mathcal{A}\} . \qquad (1)$$

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Let τ be an $(r-1)$-dimensional polyhedron in $\mathscr{T}(V)$. Modulo the affine span $\langle\tau\rangle$ of τ, each r-dimensional polyhedron σ incident on τ ($\sigma \in \text{star}(\tau)$) determines a primitive vector v_σ. The balancing condition is that

$$\sum_{\sigma\in\text{star}(\tau)} \alpha_\sigma v_\sigma \;=\; 0 \qquad \text{mod } \langle\tau\rangle\,.$$

We are primarily concerned with nonarchimedean amoebas when the field $\mathbb{K}$ is the complex Puiseaux field.

Let $\mathcal{V} \subset \mathbb{C}^* \times (\mathbb{C}^*)^n$ be a subvariety whose every component maps dominantly onto the first factor, $\mathbb{C}^*$, with coordinate s. We consider $\mathcal{V}$ to be a family of varieties over an open subset U of $\mathbb{C}^*$, with fiber $\mathcal{V}_s$ over $s \in U$.

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Then $\mathcal{V}$ is a variety in the torus $(\mathbb{C}(s)^*)^n$ over $\mathbb{C}(s)$. Extending scalars to the Puiseaux field gives a variety $V \subset (\mathbb{K}^*)^n$ with tropicalization $\mathscr{T}(V)$. In this context, Jonsson [1] proved the following.

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Theorem (Jonsson)

We have $\lim_{s \to 0} \frac{-1}{\log |s|} \mathscr{A}(\mathcal{V}_s) \;=\; \mathscr{T}(V).$

This means that the nonarchimedean amoeba of V is the limit of (appropriately scaled) amoebas of fibers of the family $\mathcal{V}$. With this point of view, Jonsson's Theorem holds in the large context of tropicalizations of varieties in $(\mathbb{K}^*)^n$.

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Theorem

Let $W \subset (\mathbb{K}^*)^n$ be any variety. Then there is a smooth curve $\mathcal{C}$, a point $o \in \mathcal{C}$, a local parameter u at o, and a family of varieties $\mathcal{V} \subset (\mathcal{C} \setminus \{o\}) \times (\mathbb{C}^*)^n$ over $\mathcal{C} \setminus \{o\}$ with fiber $\mathcal{V}_a$ over $a \in \mathcal{C} \setminus \{o\}$ such that

$$\lim_{a \to o} \frac{-1}{\log|u(a)|} \mathscr{A}(\mathcal{V}_a) \ = \ \mathscr{T}(W) .$$

If W is a complete intersection, then we may choose the family $\mathcal{V}$ so that every fiber $\mathcal{V}_a$ is a complete intersection.

Phase tropical varieties

Now, consider an algebraic hypersurfaces V in the complex algebraic torus $(\mathbb{C}^*)^n$, i.e. the zero locus of a polynomial :

$$f(z) = \sum_{\alpha \in \mathrm{supp}(f)} a_\alpha z^\alpha, \quad z^\alpha = z_1^{\alpha_1} z_2^{\alpha_2} \dots z_n^{\alpha_n},$$

where each a_α is a non-zero complex number and $\mathrm{supp}(f)$ is a finite subset of $\mathbb{Z}^n$.

For a strictly positive real number t we define the self diffeomorphism H_t of $(\mathbb{C}^*)^n$ by :

$$\begin{array}{rccc} H_t & : & (\mathbb{C}^*)^n & \longrightarrow & (\mathbb{C}^*)^n \\ & & (z_1, \ldots, z_n) & \longmapsto & \left(\mid z_1 \mid^{-\frac{1}{\log t}} \dfrac{z_1}{\mid z_1 \mid}, \ldots, \mid z_n \mid^{-\frac{1}{\log t}} \dfrac{z_n}{\mid z_n \mid} \right) \end{array}$$

This defines a new complex structure on $(\mathbb{C}^*)^n$ denoted by $J_t = (dH_t)^{-1} \circ J \circ (dH_t)$ where J is the standard complex structure.

Phase tropical varieties

A J_t-holomorphic hypersurface V_t is a holomorphic hypersurface with respect to the J_t complex structure on $(\mathbb{C}^*)^n$. It is equivalent to say that $V_t = H_t(V)$ where $V \subset (\mathbb{C}^*)^n$ is an holomorphic hypersurface for the standard complex structure J on $(\mathbb{C}^*)^n$.

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Recall that the Hausdorff distance between two closed subsets A, B of a metric space (E, d) is defined by :

$$d_H(A, B) = \max\{\sup_{a \in A} d(a, B), \sup_{b \in B} d(A, b)\}.$$

Here $E = \mathbb{R}^n \times (S^1)^n$ is equipped with the distance defined as the product of the Euclidean metric on $\mathbb{R}^n$ and the flat metric on $(S^1)^n$.

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A phase tropical variety $V_\infty \subset (\mathbb{C}^*)^n$ is the limit (with respect to the Hausdorff metric on compact sets in $(\mathbb{C}^*)^n$) of a sequence of a J_t-holomorphic varieties $V_t \subset (\mathbb{C}^*)^n$ when t tends to ∞.

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Let $a \in \mathbb{K}^*$ be the Puiseux series $a = \sum_{j\in A_a} \xi_j t^j$ with $\xi \in \mathbb{C}^*$ and $A_a \subset \mathbb{R}$ is a well-ordered set with smallest element. Then we have a non-Archimedean valuation on $\mathbb{K}$ defined by $\mathrm{val}(a) = -\min A_a$. We complexify the valuation map as follows :

$$\begin{array}{rcccl} w & : & \mathbb{K}^* & \longrightarrow & \mathbb{C}^* \\ & & a & \longmapsto & w(a) = e^{\mathrm{val}(a) + i \arg(\xi_{-\mathrm{val}(a)})}. \end{array}$$

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Let Arg be the argument map $\mathbb{K}^* \to S^1$ defined by $\mathrm{Arg}(a) = e^{i \arg(\xi_{-\mathrm{val}(a)})}$ for any Puiseux series $a = \sum_{j \in A_a} \xi_j t^j$. Note that this map extends the map $\mathbb{C}^* \to S^1$ defined by $\rho e^{i\theta} \mapsto e^{i\theta}$ which we denote by Arg.

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$$\begin{array}{ccc} (\mathbb{K}^*)^n & \xrightarrow{\quad W \quad} & (\mathbb{C}^*)^n \\ & {\scriptstyle \mathrm{Log}_{\mathbb{K}}} \searrow \quad \swarrow {\scriptstyle \mathrm{Log}} & \\ & \mathbb{R}^n & \end{array}$$

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Theorem (Mikhalkin, 2002)

The set $V_\infty \subset (\mathbb{C}^*)^n$ is a phase tropical hypersurface if and only if there exists an algebraic hypersurface $V_\mathbb{K} \subset (\mathbb{K}^*)^n$ over $\mathbb{K}$ such that $\overline{W(V_\mathbb{K})} = V_\infty$, where $\overline{W(V_\mathbb{K})}$ is the closure of $W(V_\mathbb{K})$ in $(\mathbb{C}^*)^n \approx \mathbb{R}^n \times (S^1)^n$ as a Riemannian manifold with metric defined by the standard Euclidean metric of $\mathbb{R}^n$ and the standard flat metric of the real torus.

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Indeed, we can consider the Veronese embedding $\rho : (\mathbb{C}^*)^n \to \mathbb{CP}^{\#(\Delta\cap\mathbb{Z}^n)-1}$ defined by the monomial map associated to $\Delta \cap \mathbb{Z}^n$: $(z_1, \cdots, z_n) \mapsto z_1^{\alpha_1} z_2^{\alpha_2} \cdots z_n^{\alpha_n}$, for each $\alpha := (\alpha_1, \cdots, \alpha_n) \in \Delta \cap \mathbb{Z}^n$; and X_Δ is defined as the closure of the image of $(\mathbb{C}^*)^n$

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Theorem (Kim-Nisse & Kerr-Zharkov (2016))

Let $V_t \subset (\mathbb{C}^*)^n$ be a family of smooth complex algebraic hypersurfaces with a fixed degree Δ, and denote by V_∞ the phase tropical hypersurface associated to the family $\{V_t\}_t$ (i.e., the limit of $H_t(V_t)$ when t goes to ∞). Then for a sufficiently large $t \gg 0$ the following statements hold :

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(i) *the hypersurface V_t is homeomorphic to V_∞ ;*

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Definition

Let $\mathscr{H} \subset \mathbb{CP}^k$ be an arrangement of $k+2$ generic hyperplanes in $\mathbb{CP}^k$. Let $\mathscr{U} \subset \mathbb{CP}^k$ be the union of their tubular ε-neighborhood for a small $0 < \varepsilon \ll 1$. The complement $\overline{\mathscr{P}}_k = \mathbb{CP}^k \backslash \mathscr{U}$ is called the k-dimensional pair-of-pants, and $\mathscr{P}_k = \mathbb{CP}^k \backslash \mathscr{H}$ is called the k-dimensional open pair-of-pants.

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Phase tropical varieties

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A phase tropical hyperplane is defined as follows :

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Theorem

A phase tropical hyperplane $H_\infty \subset (\mathbb{C}^*)^{k+1}$ is homeomorphic to a hyperplane in the projective space $\mathbb{CP}^{k+1}$ minus $k+2$ generic hyperplanes.

Let $V \subset (\mathbb{C}^*)^n$ be a smooth k-dimensional algebraic variety with defining ideal $\mathcal{I} = \langle f_1, \ldots, f_l \rangle$, where the polynomials $f_r(z) = \sum_{\alpha \in A_r} a_{r,\alpha} z^\alpha$ with $a_{r,\alpha} \in \mathbb{C}^*$, A_r a finite subset of $\mathbb{Z}^n$ and $z^\alpha = z_1^{\alpha_1} z_2^{\alpha_2} \ldots z_n^{\alpha_n}$. We denote by Δ_r the convex hull of A_r in $\mathbb{R}^n$ which is the Newton polytope of f_r. We can consider the family of hypersurfaces $V_{f_{(t;\, r)}} \subset (\mathbb{C}^*)^n$ defined by the following family of Viro's polynomials :

$$f_{(t;\, r)}(z) = \sum_{\alpha \in A_r} \xi_{r,\alpha} t^{\nu_{PR}(\alpha)} z^\alpha, \tag{2}$$

with $\xi_{r,\alpha} = a_{r,\alpha} e^{\nu_{PR}(\alpha)}$, and we view this family as a deformation of f_r.

So, the ideal $\mathcal{J} = \langle f_{(t;\,1)}, \dots, f_{(t;\,l)} \rangle$ is a defining ideal of an algebraic variety V_t which is a deformation of the original algebraic variety V. We can view $\mathcal{J}$ as an ideal in $\mathbb{K}[z_1^{\pm 1}, \dots, z_n^{\pm 1}]$. Let Γ be the tropical variety image of $W(V(\mathcal{J}))$ under the logarithmic map. Assume that all vertices of Γ are $k+2$ valent. Moreover, assume that if σ is a star composed by a vertex v of Γ and all 1-cells adjacent to v, then σ is dual to a $(k+1)-$dimensional simplex of area $\frac{1}{(k+1)!}$. Then we have the following :

Phase tropical varieties

Definition (Forsberg-Passare-Tsikh (2000))

For a hypersurface $V_{f_t} \subset \mathbb{CP}^n$, let f_t be its defining polynomial. Let $x \in \mathbb{R}^n \subset \mathbb{TP}^n$ be a point outside the amoeba $\mathscr{A}_{f_t}$. We define the functional $\mathrm{ind}_{f_t}(x)$ on the space of loops in T_x the pre-image of x in $\mathbb{C}^n \setminus \{0\}$ under the ma composition of the two maps

$$\mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{CP}^n \to \mathbb{TP}^n$$

where the first map is the usual quotient by $\mathbb{C}^*$, and the second is the Log_t, as follows :

$$\mathrm{ind}_{f_t}(x) : \gamma \mapsto \frac{1}{2\pi i} \int_\gamma d \log f_t.$$

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Each component of $\mathbb{R}^n \setminus \mathscr{A}_{f_t}$ *is a convex domain and there exists a locally constant function :*

$$ind_{f_t} : \mathbb{R}^n \setminus \mathscr{A}_{f_t} \longrightarrow \mathbb{Z}^n \cap \Delta_{f_t}$$

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$$
\begin{array}{ccc}
V_t & \xrightarrow{\;i\;} & (\mathbb{C}^*)^n \\
\Big\downarrow {\scriptstyle \mathrm{Log}_t} & & \Big\downarrow {\scriptstyle \mathrm{Log}_t} \\
\mathscr{A}_t & \xrightarrow{\;i\;} & \mathbb{R}^n
\end{array}
\quad \xrightarrow[t\to\infty]{\lim} \quad
\begin{array}{ccc}
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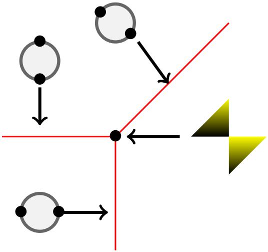
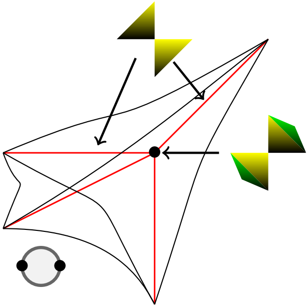
$$





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Let V be a k-dimensional algebraic variety in $(\mathbb{C}^*)^{2k}$ and V_∞ its corresponding phase tropical variety. Is V_∞ the gluing of Lagrangian submanifolds of $(\mathbb{C}^*)^{2k}$ with boundary ? (in other words V_∞ admits a decomposition into Lagrangian submanifolds of $(\mathbb{C}^*)^{2k}$)

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