Pseudomeromorphic currents

Mats Andersson

Chalmers University of Technology and University of Gothenburg

Multidimensional residues and tropical geometry

Sochi, June 2021

Why multivariable residue theory?

Why multivariable residue theory?

Find 'canonical' analytic representation of algebraic objects,
for instance, ideals of holomorphic functions, cohomology classes.

Why multivariable residue theory?

Find ’canonical’ analytic representation of algebraic objects,
for instance, ideals of holomorphic functions, cohomology classes.

Solve problems, for instance, membership problems and $\bar{\partial}$-equation.

Why multivariable residue theory?

Find 'canonical' analytic representation of algebraic objects,
for instance, ideals of holomorphic functions, cohomology classes.

Solve problems, for instance, membership problems and $\bar{\partial}$-equation.

???????

The principal value current $[1/f]$

Assume f non-trivial holomorphic function on complex manifold X.
Can find a distribution ((0, 0)-current) U such that $fU = 1$.

The principal value current $[1/f]$

Assume f non-trivial holomorphic function on complex manifold X.
Can find a distribution ($(0,0)$-current) U such that $fU = 1$.

Then $R := \bar{\partial}U$ has support on $Z = Z(f)$ and

$$\phi R = 0 \iff \phi \in \langle f \rangle.$$

The principal value current $[1/f]$

Assume f non-trivial holomorphic function on complex manifold X.
Can find a distribution ($(0,0)$-current) U such that $fU = 1$.
Then $R := \bar{\partial} U$ has support on $Z = Z(f)$ and

$$\phi R = 0 \iff \phi \in \langle f \rangle.$$

Canonical choice of U (Herrera-Lieberman 1971):

$$\left[\frac{1}{f}\right].\xi = \lim_{\epsilon \to 0} \int_{|f|^2 v > \epsilon} \frac{\xi}{f}, \quad \xi \text{ test form},$$

$v > 0$ smooth. Can write

The principal value current $[1/f]$

Assume f non-trivial holomorphic function on complex manifold X.
Can find a distribution ($(0,0)$-current) U such that $fU = 1$.

Then $R := \bar{\partial} U$ has support on $Z = Z(f)$ and

$$\phi R = 0 \iff \phi \in \langle f \rangle.$$

Canonical choice of U (Herrera-Lieberman 1971):

$$\left[\frac{1}{f}\right].\xi = \lim_{\epsilon \to 0} \int_{|f|^2 v > \epsilon} \frac{\xi}{f}, \quad \xi \text{ test form},$$

$v > 0$ smooth. Can write

$$\left[\frac{1}{f}\right] = \lim_{\epsilon \to 0} \chi(|f|^2 v/\epsilon)\frac{1}{f}, \tag{0.1}$$

$$\chi(t) = \chi_{[1,\infty)}(t),$$

The principal value current $[1/f]$

Assume f non-trivial holomorphic function on complex manifold X.
Can find a distribution ((0,0)-current) U such that $fU = 1$.

Then $R := \bar{\partial} U$ has support on $Z = Z(f)$ and

$$\phi R = 0 \iff \phi \in \langle f \rangle.$$

Canonical choice of U (Herrera-Lieberman 1971):

$$\left[\frac{1}{f}\right].\xi = \lim_{\epsilon \to 0} \int_{|f|^2 v > \epsilon} \frac{\xi}{f}, \quad \xi \text{ test form},$$

$v > 0$ smooth. Can write

$$\left[\frac{1}{f}\right] = \lim_{\epsilon \to 0} \chi(|f|^2 v/\epsilon) \frac{1}{f}, \tag{0.1}$$

$$\chi(t) = \chi_{[1,\infty)}(t),$$

or a slight regularization of it.

Extends to a section of a line bundle

If a holomorphic and $a \neq 0$, then

$$\left[\frac{1}{af}\right] = \frac{1}{a}\left[\frac{1}{f}\right]$$

so (0.1) exists if f holomorphic section of a line bundle $L \to X$.
Then $[1/f]$ is an L^{-1}-valued current.

Sketch of proof.

First prove when f a monomial. Not obvious but quite elementary. Then choose modification $p\colon X' \to X$ such that p^*f locally is a monomial in appropriate coordinates (log resolution). Thus

$$\left[\frac{1}{p^*f}\right] = \lim_{\epsilon\to 0} \chi(|p^*f|^2 p^*v/\epsilon)\frac{1}{p^*f}$$

exists, and we get

$$\left[\frac{1}{f}\right] = \lim_{\epsilon\to 0} \chi(|f|^2 v/\epsilon)\frac{1}{f} = \lim_{\epsilon\to 0} p_*\Big(\chi(|p^*f|^2 p^*v/\epsilon)\frac{1}{p^*f}\Big) = p_*\left[\frac{1}{p^*f}\right].$$

□

Sketch of proof.

First prove when f a monomial. Not obvious but quite elementary. Then choose modification $p\colon X' \to X$ such that p^*f locally is a monomial in appropriate coordinates (log resolution). Thus

$$\left[\frac{1}{p^*f}\right] = \lim_{\epsilon\to 0} \chi(|p^*f|^2 p^*v/\epsilon)\frac{1}{p^*f}$$

exists, and we get

$$\left[\frac{1}{f}\right] = \lim_{\epsilon\to 0} \chi(|f|^2 v/\epsilon)\frac{1}{f} = \lim_{\epsilon\to 0} p_*\Big(\chi(|p^*f|^2 p^*v/\epsilon)\frac{1}{p^*f}\Big) = p_*\left[\frac{1}{p^*f}\right].$$

□

Obs

$$\bar\partial\left[\frac{1}{f}\right] = \lim_{\epsilon\to 0} \bar\partial\chi(|f|^2 v/\epsilon)\,\frac{1}{f}$$

Sketch of proof.

First prove when f a monomial. Not obvious but quite elementary. Then choose modification $p\colon X' \to X$ such that p^*f locally is a monomial in appropriate coordinates (log resolution). Thus

$$\left[\frac{1}{p^*f}\right] = \lim_{\epsilon \to 0} \chi(|p^*f|^2 p^*v/\epsilon)\frac{1}{p^*f}$$

exists, and we get

$$\left[\frac{1}{f}\right] = \lim_{\epsilon \to 0} \chi(|f|^2 v/\epsilon)\frac{1}{f} = \lim_{\epsilon \to 0} p_*\left(\chi(|p^*f|^2 p^*v/\epsilon)\frac{1}{p^*f}\right) = p_*\left[\frac{1}{p^*f}\right].$$

□

Obs

$$\bar\partial\left[\frac{1}{f}\right] = \lim_{\epsilon \to 0} \bar\partial\chi(|f|^2 v/\epsilon)\,\frac{1}{f}$$

We skip brackets [ ] from now on!

Elementary pseudomeromorphic currents

Notice that $1/f$ and $\bar\partial(1/f)$ are the pushforwards of (locally finite sums of) currents of the type

$$\frac{\alpha}{s_1^{a_1} s_2^{a_2} \cdots s_N^{a_N}} \tag{0.2}$$

and

$$\bar\partial \frac{1}{s_1^{a_1}} \wedge \frac{\alpha}{s_2^{a_2} \cdots a_N^{a_N}} \tag{0.3}$$

where α is a smooth form with compact support.

Elementary pseudomeromorphic currents

Notice that $1/f$ and $\bar{\partial}(1/f)$ are the pushforwards of (locally finite sums of) currents of the type

$$\frac{\alpha}{s_1^{a_1} s_2^{a_2} \cdots s_N^{a_N}} \tag{0.2}$$

and

$$\bar{\partial}\frac{1}{s_1^{a_1}} \wedge \frac{\alpha}{s_2^{a_2} \cdots a_N^{a_N}} \tag{0.3}$$

where α is a smooth form with compact support.

We say that (0.2) and (0.3) are *elementary pseudomeromorphic currents*.

Products of principal value currents

If f and g holomorphic on X, then

$$\frac{1}{g} \cdot \frac{1}{f} = \lim_{\epsilon_2 \to 0} \lim_{\epsilon_1 \to 0} \chi(|g|^2/\epsilon_2) \frac{1}{g} \cdot \chi(|f|^2/\epsilon_1) \frac{1}{f},$$

well-defined and commuting in f, g.

Products of residue currents

$$\bar\partial\frac{1}{g}\wedge\bar\partial\frac{1}{f} = \lim_{\epsilon_2\to 0}\lim_{\epsilon_1\to 0}\bar\partial\chi(|g|^2/\epsilon_2)\frac{1}{g}\wedge\bar\partial\chi(|f|^2/\epsilon_1)\frac{1}{f} \tag{0.4}$$

well-defined (Coleff-Herrera 1978). Coleff and Herrera used limits where $\epsilon_1 << \epsilon_2$ but one can just as well let $\epsilon_1 \to 0$ first.

Products of residue currents

$$\bar{\partial}\frac{1}{g}\wedge\bar{\partial}\frac{1}{f} = \lim_{\epsilon_2\to 0}\lim_{\epsilon_1\to 0}\bar{\partial}\chi(|g|^2/\epsilon_2)\frac{1}{g}\wedge\bar{\partial}\chi(|f|^2/\epsilon_1)\frac{1}{f} \tag{0.4}$$

well-defined (Coleff-Herrera 1978). Coleff and Herrera used limits where $\epsilon_1 << \epsilon_2$ but one can just as well let $\epsilon_1 \to 0$ first.

Example

$$\bar{\partial}\frac{1}{z}\wedge\bar{\partial}\frac{1}{zw} = \bar{\partial}\frac{1}{z^2}\wedge\bar{\partial}\frac{1}{w} \neq 0, \qquad \bar{\partial}\frac{1}{zw}\wedge\bar{\partial}\frac{1}{z} = 0.$$

□

Products of residue currents

$$\bar\partial\frac{1}{g}\wedge\bar\partial\frac{1}{f} = \lim_{\epsilon_2\to 0}\lim_{\epsilon_1\to 0}\bar\partial\chi(|g|^2/\epsilon_2)\frac{1}{g}\wedge\bar\partial\chi(|f|^2/\epsilon_1)\frac{1}{f} \qquad (0.4)$$

well-defined (Coleff-Herrera 1978). Coleff and Herrera used limits where $\epsilon_1 << \epsilon_2$ but one can just as well let $\epsilon_1 \to 0$ first.

Example

$$\bar\partial\frac{1}{z}\wedge\bar\partial\frac{1}{zw} = \bar\partial\frac{1}{z^2}\wedge\bar\partial\frac{1}{w} \neq 0, \qquad \bar\partial\frac{1}{zw}\wedge\bar\partial\frac{1}{z} = 0.$$

□

However, Coleff-Herrera proved that if

$$\mathrm{codim}\,\{f = g = 0\} = 2, \qquad (0.5)$$

then (0.4) anti-commutes, i.e., is independent of the order of the limits.

Limits in the complete intersection case

In case (0.5) holds and, $\chi \equiv \chi_{[1,\infty,)}$, Passare (1988) proved that limit (0.4) exists 'almost' unrestrictedly when $\epsilon_1, \epsilon_2 \to 0$

Limits in the complete intersection case

In case (0.5) holds and, $\chi \equiv \chi_{[1,\infty,)}$, Passare (1988) proved that limit (0.4) exists 'almost' unrestrictedly when $\epsilon_1, \epsilon_2 \to 0$

but Passare-Tsikh (1995) found conterexamples for unrestricted limits.

Limits in the complete intersection case

In case (0.5) holds and, $\chi \equiv \chi_{[1,\infty,)}$, Passare (1988) proved that limit (0.4) exists 'almost' unrestrictedly when $\epsilon_1, \epsilon_2 \to 0$

but Passare-Tsikh (1995) found conterexamples for unrestricted limits.

Samuelsson Kalm (2004) showed that limit exists unrestrictedly if χ is smooth.

Limits in the complete intersection case

In case (0.5) holds and, $\chi \equiv \chi_{[1,\infty,)}$, Passare (1988) proved that limit (0.4) exists 'almost' unrestrictedly when $\epsilon_1, \epsilon_2 \to 0$

but Passare-Tsikh (1995) found conterexamples for unrestricted limits.

Samuelsson Kalm (2004) showed that limit exists unrestrictedly if χ is smooth.

Thus: when (0.5) holds one should think of the Coleff-Herrera product

$$\bar{\partial}\frac{1}{g}\wedge\bar{\partial}\frac{1}{f}$$

as a robust object in itself, and not as some particular limit!

General Coleff-Herrera products

The same works for $m > 2$. Let $f = (f_1, \ldots, f_m)$ be tuple of holomorphic functions. If

$$\operatorname{codim}\{f_1 = \cdots = f_m = 0\} = m, \tag{0.6}$$

then the Coleff-Herrera product

$$\mu^f = \bar{\partial}\frac{1}{f_1}\wedge \ldots \wedge\bar{\partial}\frac{1}{f_m}$$

is anti-commuting in f_j.

Moreover (Passare, Dickenstein-Sessa $\sim$ 1985),

$$\phi\mu^f = 0 \iff \phi \in \langle f_1, \ldots, f_m \rangle, \tag{0.7}$$

General Coleff-Herrera products

The same works for $m > 2$. Let $f = (f_1, \ldots, f_m)$ be tuple of holomorphic functions. If

$$\operatorname{codim}\{f_1 = \cdots = f_m = 0\} = m, \tag{0.6}$$

then the Coleff-Herrera product

$$\mu^f = \bar{\partial}\frac{1}{f_1}\wedge \ldots \wedge \bar{\partial}\frac{1}{f_m}$$

is anti-commuting in f_j.

Moreover (Passare, Dickenstein-Sessa $\sim$ 1985),

$$\phi\mu^f = 0 \iff \phi \in \langle f_1, \ldots, f_m\rangle, \tag{0.7}$$

and μ^f only depends on the ideal $\mathcal{J} = \langle f_1, \ldots, f_m\rangle$
so again we have a unique analytic representation of $\mathcal{J}$.

Bochner-Martinelli type currents

When (0.6) does not hold, not anti-commutative, so one had to find something new. A key step was the Bochner-Martinelli type current R^f (Passare-Tsikh-Yger 2000):

Bochner-Martinelli type currents

When (0.6) does not hold, not anti-commutative, so one had to find something new. A key step was the Bochner-Martinelli type current R^f (Passare-Tsikh-Yger 2000):

Think of f as a section $f = f_1 e_1 + \cdots + f_m e_m$ of a trivial bundle E. Then

$$\sigma = \frac{\bar{f}_1 e_1^* + \cdots + \bar{f}_m e_m^*}{|f|^2}$$

is a section of the dual bundle E^* outside $Z = \{f = 0\}$,

$$f\sigma = 1,$$

Bochner-Martinelli type currents

When (0.6) does not hold, not anti-commutative, so one had to find something new. A key step was the Bochner-Martinelli type current R^f (Passare-Tsikh-Yger 2000):

Think of f as a section $f = f_1 e_1 + \cdots + f_m e_m$ of a trivial bundle E. Then

$$\sigma = \frac{\bar{f}_1 e_1^* + \cdots + \bar{f}_m e_m^*}{|f|^2}$$

is a section of the dual bundle E^* outside $Z = \{f = 0\}$,

$$f\sigma = 1,$$

and

$$\sigma \wedge (\bar{\partial}\sigma)^{k-1} = \sum_{|I|=k} u_{k,I} \wedge e_{I_1}^* \wedge \ldots \wedge e_{I_k}^*$$

is a well-defined smooth $(0, k-1)$-form with values in $\Lambda^k E^*$.

Can define the residue currents, $1 \leq k \leq n$,

$$R_k^f = \lim_{\epsilon \to 0} \bar{\partial}\chi(|f|^2/\epsilon)\wedge\sigma\wedge(\bar{\partial}\sigma)^{k-1}.$$

Can define the residue currents, $1 \leq k \leq n$,

$$R_k^f = \lim_{\epsilon \to 0} \bar{\partial}\chi(|f|^2/\epsilon) \wedge \sigma \wedge (\bar{\partial}\sigma)^{k-1}.$$

Obs

$$R_k^f = \sum_{|I|=k}{}' R_{k,I}^f \wedge e_{I_1}^* \wedge \ldots \wedge e_{I_k}^*,$$

where $R_{k,I}^f$ are $(0,k)$-currents with support on Z.

Can define the residue currents, $1 \le k \le n$,

$$R_k^f = \lim_{\epsilon \to 0} \bar\partial \chi(|f|^2/\epsilon) \wedge \sigma \wedge (\bar\partial \sigma)^{k-1}.$$

Obs

$$R_k^f = \sum_{|I|=k}{}' R_{k,I}^f \wedge e_{I_1}^* \wedge \ldots \wedge e_{I_k}^*,$$

where $R_{k,I}^f$ are $(0,k)$-currents with support on Z.

We write $R^f = \sum_k R_k^f$.

Thus R^f is a vector-valued current with lots of components of various bidegrees.

Functoriality of BM-currents

Can define R_k^f for any f holomorphic section of a Hermitian vector bundle $E \to X$.

Functoriality of BM-currents

Can define R_k^f for any f holomorphic section of a Hermitian vector bundle $E \to X$.

If $p\colon X' \to X$ is a modification, then $p^* f$ section of $p^* E$ and

$$R_k^f = p_* R_k^{p^* f}. \tag{0.8}$$

Sketch of proof of existence of R_k^f.

Can find modification $p\colon X' \to X$ such that

$$p^* f = f^0 f'$$

where f^0 section of a line bundle and f' non-vanishing section of a vector bundle.

Sketch of proof of existence of R_k^f.

Can find modification $p\colon X' \to X$ such that

$$p^* f = f^0 f'$$

where f^0 section of a line bundle and f' non-vanishing section of a vector bundle.
Then

$$p^*\left(\sigma \wedge (\bar\partial \sigma)^{k-1}\right) = \frac{1}{(f^0)^k}\,\omega_k$$

where ω_k is smooth,

Sketch of proof of existence of R_k^f.

Can find modification $p\colon X' \to X$ such that

$$p^* f = f^0 f'$$

where f^0 section of a line bundle and f' non-vanishing section of a vector bundle.
Then

$$p^*\left(\sigma \wedge (\bar\partial\sigma)^{k-1}\right) = \frac{1}{(f^0)^k}\,\omega_k$$

where ω_k is smooth, and thus

$$R_k^{p^* f} = \lim_{\epsilon\to 0} \bar\partial\chi(|p^* f|^2/\epsilon)\frac{1}{(f^0)^k}\wedge\omega_k = \bar\partial\frac{1}{(f^0)^k}\wedge\omega_k$$

is well-defined. Thus R_k^f exists and is defined by (0.8). □

General ideals

Proposition

(A 2004) If ϕ holomorphic then

$$\phi R^f = 0 \implies \phi \in \langle f_1, \ldots, f_m \rangle.$$

General ideals

Proposition

(A 2004) If ϕ holomorphic then

$$\phi R^f = 0 \implies \phi \in \langle f_1, \ldots, f_m \rangle.$$

Inspired by *BM*-residues A-Wulcan (2007) introduced a vector-valued (essentially) canonical current $R^{\mathcal{J}}$, only dependent of the ideal $\mathcal{J} = \langle f_1, \ldots, f_m \rangle$, such that

$$\phi R^{\mathcal{J}} = 0 \iff \phi \in \mathcal{J}$$

Thus $R^{\mathcal{J}}$ *represents* the ideal $\mathcal{J}$.

Pseudomeromorphic currents

A-Wulcan introduced (2010) the sheaf $\mathcal{PM}_X$ of pseudomeromorphic currents on X. Defined as locally finite sums of push-forwards of elementary currents as in (0.2) and (0.3), under suitable holomorphic mappings.

All currents mentioned in this talk are in $\mathcal{PM}_X$.

Pseudomeromorphic currents

A-Wulcan introduced (2010) the sheaf $\mathcal{PM}_X$ of pseudomeromorphic currents on X. Defined as locally finite sums of push-forwards of elementary currents as in (0.2) and (0.3), under suitable holomorphic mappings.

All currents mentioned in this talk are in $\mathcal{PM}_X$.

Properties of $\mathcal{PM}_X$:

(0) $\mathcal{PM}_X$ closed under $\bar\partial$ (and ∂) and multiplication by smooth forms

Pseudomeromorphic currents

A-Wulcan introduced (2010) the sheaf $\mathcal{PM}_X$ of pseudomeromorphic currents on X. Defined as locally finite sums of push-forwards of elementary currents as in (0.2) and (0.3), under suitable holomorphic mappings.

All currents mentioned in this talk are in $\mathcal{PM}_X$.

Properties of $\mathcal{PM}_X$:

(0) $\mathcal{PM}_X$ closed under $\bar\partial$ (and ∂) and multiplication by smooth forms

(i) If $\mu \in \mathcal{PM}_X$, h holomorphic and $h = 0$ on $\operatorname{supp}\mu$, then $\bar{h}\mu = 0$.

Basic properties of $\mathcal{PM}_X$ (continued):

(ii) Dimension principle: If $\mu \in \mathcal{PM}_X$ has bidegree $(*, \kappa)$ and support on subvariety V of codimension $> \kappa$, then $\mu = 0$

Basic properties of $\mathcal{PM}_X$ (continued):

(ii) Dimension principle: If $\mu \in \mathcal{PM}_X$ has bidegree $(*, \kappa)$ and support on subvariety V of codimension $> \kappa$, then $\mu = 0$

Example

If U is in $\mathcal{PM}_X$ and $fU = 1$, then U is the pv current $1/f$. □

Basic properties of $\mathcal{PM}_X$ (continued):

(ii) Dimension principle: If $\mu \in \mathcal{PM}_X$ has bidegree $(*, \kappa)$ and support on subvariety V of codimension $> \kappa$, then $\mu = 0$

Example

If U is in $\mathcal{PM}_X$ and $fU = 1$, then U is the pv current $1/f$. □

(iii) if $\mu \in \mathcal{PM}_X$ and V subvariety, then $\mathbf{1}_V\mu$ is in $\mathcal{PM}_X$ and has support on V, and

$$\mathbf{1}_V \mathbf{1}_W \mu = \mathbf{1}_{V\cap W}\mu. \tag{0.9}$$

Basic properties of $\mathcal{PM}_X$ (continued):

(ii) Dimension principle: If $\mu \in \mathcal{PM}_X$ has bidegree $(*, \kappa)$ and support on subvariety V of codimension $> \kappa$, then $\mu = 0$

Example

If U is in $\mathcal{PM}_X$ and $fU = 1$, then U is the pv current $1/f$. □

(iii) if $\mu \in \mathcal{PM}_X$ and V subvariety, then $\mathbf{1}_V\mu$ is in $\mathcal{PM}_X$ and has support on V, and

$$\mathbf{1}_V \mathbf{1}_W \mu = \mathbf{1}_{V \cap W} \mu. \quad (0.9)$$

If h tuple such that $Z(h) = V$, then

$$\mathbf{1}_{X \setminus V} \mu := \mu - \mathbf{1}_V = \lim_{\epsilon \to 0} \chi(|h|^2/\epsilon)\mu$$

Basic properties of $\mathcal{PM}_X$ (continued):

(iv) If h holo function, and $\mu \in \mathcal{PM}_X$, then there is a unique $T \in \mathcal{PM}_X$ that coincides with $(1/h)\mu$ outside $Z(h)$, and such that $\mathbf{1}_{Z(h)} T = 0$.

Basic properties of $\mathcal{PM}_X$ (continued):

(iv) If h holo function, and $\mu \in \mathcal{PM}_X$, then there is a unique $T \in \mathcal{PM}_X$ that coincides with $(1/h)\mu$ outside $Z(h)$, and such that $\mathbf{1}_{Z(h)} T = 0$.

We write T as

$$\frac{1}{h}\mu$$

Basic properties of $\mathcal{PM}_X$ (continued):

(iv) If h holo function, and $\mu \in \mathcal{PM}_X$, then there is a unique $T \in \mathcal{PM}_X$ that coincides with $(1/h)\mu$ outside $Z(h)$, and such that $\mathbf{1}_{Z(h)} T = 0$.

We write T as

$$\frac{1}{h}\mu$$

Notice that

$$\frac{1}{h}\mu = \lim_{\epsilon \to 0} \chi(|h|^2/\epsilon)\, \frac{1}{h}\mu.$$

Basic properties of $\mathcal{PM}_X$ (continued):

(iv) If h holo function, and $\mu \in \mathcal{PM}_X$, then there is a unique $T \in \mathcal{PM}_X$ that coincides with $(1/h)\mu$ outside $Z(h)$, and such that $\mathbf{1}_{Z(h)} T = 0$.

We write T as

$$\frac{1}{h}\mu$$

Notice that

$$\frac{1}{h}\mu = \lim_{\epsilon \to 0} \chi(|h|^2/\epsilon)\,\frac{1}{h}\mu.$$

We have Leibniz' rule (middle term is defined by the equality)

$$\bar\partial\Big(\frac{1}{h}\mu\Big) = \bar\partial\frac{1}{h}\wedge\mu + \frac{1}{h}\wedge\bar\partial\mu.$$

Basic properties of $\mathcal{PM}_X$ (continued):

(iv) If h holo function, and $\mu \in \mathcal{PM}_X$, then there is a unique $T \in \mathcal{PM}_X$ that coincides with $(1/h)\mu$ outside $Z(h)$, and such that $\mathbf{1}_{Z(h)}T = 0$.

We write T as

$$\frac{1}{h}\mu$$

Notice that

$$\frac{1}{h}\mu = \lim_{\epsilon\to 0} \chi(|h|^2/\epsilon)\,\frac{1}{h}\mu.$$

We have Leibniz' rule (middle term is defined by the equality)

$$\bar\partial\Big(\frac{1}{h}\mu\Big) = \bar\partial\frac{1}{h}\wedge\mu + \frac{1}{h}\wedge\bar\partial\mu.$$

In a similar way one can define

$$R^f\wedge\mu, \quad R^{\mathcal{J}}\wedge\mu.$$

Why introduce $\mathcal{PM}_X$?

Why introduce $\mathcal{PM}_X$?

The sheaf $\mathcal{PM}_X$ is a useful tool!

The Coleff-Herrera product (again)

Let f, g be holomorphic such that $\operatorname{codim}\{f = g = 0\} \geq 2$. Sketch of proof of

$$\bar{\partial}\frac{1}{f}\wedge\bar{\partial}\frac{1}{g} + \bar{\partial}\frac{1}{g}\wedge\bar{\partial}\frac{1}{f} = 0. \tag{0.10}$$

The Coleff-Herrera product (again)

Let f, g be holomorphic such that $\operatorname{codim}\{f = g = 0\} \geq 2$. Sketch of proof of

$$\bar{\partial}\frac{1}{f}\wedge\bar{\partial}\frac{1}{g} + \bar{\partial}\frac{1}{g}\wedge\bar{\partial}\frac{1}{f} = 0. \tag{0.10}$$

Notice that

$$\frac{1}{f}\cdot\bar{\partial}\frac{1}{g} - \bar{\partial}\frac{1}{g}\cdot\frac{1}{f} = 0 \tag{0.11}$$

outside $\{f = g = 0\}$ since $1/f$ is smooth there.

The Coleff-Herrera product (again)

Let f, g be holomorphic such that $\mathrm{codim}\,\{f = g = 0\} \geq 2$. Sketch of proof of

$$\bar{\partial}\frac{1}{f}\wedge\bar{\partial}\frac{1}{g} + \bar{\partial}\frac{1}{g}\wedge\bar{\partial}\frac{1}{f} = 0. \tag{0.10}$$

Notice that

$$\frac{1}{f}\cdot\bar{\partial}\frac{1}{g} - \bar{\partial}\frac{1}{g}\cdot\frac{1}{f} = 0 \tag{0.11}$$

outside $\{f = g = 0\}$ since $1/f$ is smooth there.

By the dimension principle (0.11) holds everywhere.

The Coleff-Herrera product (again)

Let f, g be holomorphic such that $\mathrm{codim}\,\{f = g = 0\} \geq 2$. Sketch of proof of

$$\bar\partial\frac{1}{f}\wedge\bar\partial\frac{1}{g} + \bar\partial\frac{1}{g}\wedge\bar\partial\frac{1}{f} = 0. \tag{0.10}$$

Notice that

$$\frac{1}{f}\cdot\bar\partial\frac{1}{g} - \bar\partial\frac{1}{g}\cdot\frac{1}{f} = 0 \tag{0.11}$$

outside $\{f = g = 0\}$ since $1/f$ is smooth there.

By the dimension principle (0.11) holds everywhere.

By Leibniz rule (0.10) follows.

Non-(anti-)commutativity

In general *not* anti-commutative. This fact is fundamental for various applications, for instance interpolation-division problems, and the $\bar{\partial}$-equation on a singular space.

Non-(anti-)commutativity

In general *not* anti-commutative. This fact is fundamental for various applications, for instance interpolation-division problems, and the $\bar{\partial}$-equation on a singular space.

If $i\colon X \to \Omega \subset \mathbb{C}^N$, then we use residues like

$$R \wedge R^X, \tag{0.12}$$

where R^X localizes to X and R has suitable properties on X.

Non-(anti-)commutativity

In general *not* anti-commutative. This fact is fundamental for various applications, for instance interpolation-division problems, and the $\bar{\partial}$-equation on a singular space.

If $i\colon X \to \Omega \subset \mathbb{C}^N$, then we use residues like

$$R \wedge R^X, \tag{0.12}$$

where R^X localizes to X and R has suitable properties on X.

For instance (0.12) is the basic building block in integral formulas for solving division (membership) problems on X or to solve $\bar{\partial}$ on X.

Analytic representation of sheaf cohomology

By A-Samuelsson Kalm (2012), A-Lärkäng (2019), Samuelsson Kalm (2021), A-Lennartsson, Lärkäng, Samuelsson Kalm (arXiv 2020):

Theorem

Let X be a, possibly non-reduced, singular space of pure dimension. There are fine sheaves $\mathcal{A}_X^{p,q}$ of (p,q)-currents on X, that coincide with sheaves of smooth forms generically in X, so that

$$0 \to \Omega_X^p \to \mathcal{A}_X^{p,0} \xrightarrow{\bar\partial} \mathcal{A}_X^{p,1} \xrightarrow{\bar\partial} \cdots$$

is exact.

Analytic representation of sheaf cohomology

By A-Samuelsson Kalm (2012), A-Lärkäng (2019), Samuelsson Kalm (2021), A-Lennartsson, Lärkäng, Samuelsson Kalm (arXiv 2020):

Theorem

Let X be a, possibly non-reduced, singular space of pure dimension. There are fine sheaves $\mathcal{A}_X^{p,q}$ of (p,q)-currents on X, that coincide with sheaves of smooth forms generically in X, so that

$$0 \to \Omega_X^p \to \mathcal{A}_X^{p,0} \xrightarrow{\bar\partial} \mathcal{A}_X^{p,1} \xrightarrow{\bar\partial} \cdots$$

is exact.

In particular, we get a representation of the cohomology groups (Dolbeault isomorphisms)

$$H^q(X, \Omega_X^p) = \frac{\operatorname{Ker}_{\bar\partial} \mathcal{A}(X)^{p,q}}{\operatorname{Im} \mathcal{A}(X)^{p,q-1}}.$$

Analytic representation of Serre duality

We can also find current representations of the dualizing sheaves and an analytic representation of the abstract Serre duality.

Analytic representation of Serre duality

We can also find current representations of the dualizing sheaves and an analytic representation of the abstract Serre duality.

The case when X is reducedand $p = 0$, was proved by Ruppenthal, Samuelsson Kalm, Wulcan (2017) and for general p by Samuelsson Kalm (2020).

Thanks for your attention!

Meromorphic functions

Example

Let $i\colon Y \to \Omega \subset \mathbb{C}^N$ be a subvariety, and α meromorphic p-form on Y. Then, by (iv), α is in $\mathcal{PM}_Y$. □

Meromorphic functions

Example

Let $i\colon\ Y \to \Omega \subset \mathbb{C}^N$ be a subvariety, and α meromorphic p-form on Y. Then, by (iv), α is in $\mathcal{PM}_Y$. □

By the dimension principle $\mathbf{1}_V\alpha = 0$ if $V \subset Y$ has positive codimension.

If h is any holomorphic tuple such that $Z(h) \subset Y$ has positive codimension, then $\lim_{\epsilon\to 0} \chi(|h|^2/\epsilon)\alpha = \alpha$

Meromorphic functions

Example

Let $i\colon Y \to \Omega \subset \mathbb{C}^N$ be a subvariety, and α meromorphic p-form on Y. Then, by (iv), α is in $\mathcal{PM}_Y$. □

By the dimension principle $\mathbf{1}_V \alpha = 0$ if $V \subset Y$ has positive codimension.

If h is any holomorphic tuple such that $Z(h) \subset Y$ has positive codimension, then $\lim_{\epsilon\to 0} \chi(|h|^2/\epsilon)\alpha = \alpha$

Example

If $Y = \{f = 0\}$, $df \neq 0$ generically, and α meromorphic, e.g., the Poincaré residue, then

$$\int_Y \alpha\wedge\xi = \lim_{\epsilon\to 0} \int_Y \chi(|df|^2/\epsilon)\alpha\wedge\xi = \lim_{\epsilon\to 0} \int_Y \chi(|h|^2/\epsilon)\alpha\wedge\xi$$

if h holo and not identically 0 on Y. □