Seminar "Functional analysis and its applications"

Stone Algebras and Kaplansky–Hilbert Modules

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N. Edeko, M. Haase and H. Kreidler, A decomposition theorem for unitary group representations on Kaplansky–Hilbert modules and the Furstenberg–Zimmer structure theorem, https://arxiv.org/abs/2104.04865.

In this paper, a decomposition theorem for (covariant) unitary group representations on Kaplansky–Hilbert modules over Stone algebras is established, which generalizes the well-known Hilbert space case (where it coincides with the decomposition of Jacobs, de Leeuw and Glicksberg).
The proof rests heavily on the operator theory on Kaplansky–Hilbert modules, in particular the spectral theorem for Hilbert–Schmidt homomorphisms on such modules.
As an application, a generalization of the celebrated Furstenberg–Zimmer structure theorem to the case of measure–preserving actions of arbitrary groups on arbitrary probability spaces is established.

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Outline

Let A be a unital commutative C^*-algebra.[a] A unital A-module E equipped with a mapping $(\cdot|\cdot) : E \times E \to A$ is called a pre-Hilbert module over A if the following conditions are satisfied.

1. For $x \in E$ we have $(x|x) \geq 0$. Moreover, $(x|x) = 0$ if and only if $x = 0$.
2. The map $(\cdot|y) : E \to A$, $y \to (x|y)$ is A-linear for every $y \in E$.
3. $\overline{(x|y)} = (y|x)$ for all $x, y \in E$.

[a] By Gelfand–Naimark theorem, $A \cong C(\Omega)$ is the $*$-algebra of all complex-valued continuous functions on a compact set Ω. In particular, if Ω is a singleton, then $A = C(\Omega) \cong \mathbb{C}$.

- In a pre-Hilbert module E the Cauchy–Schwarz inequality

$$|(x|y)| \leq \sqrt{(x|x)}\sqrt{(y|y)}$$

  holds for all $x, y \in E$.

- As a consequence, by

$$||x|| = ||(x|x)^{\frac{1}{2}}||_A = ||(x|x)||_A^{\frac{1}{2}}$$

  for $x \in E$ a norm $||\cdot||$ is defined on E.

- The pre-Hilbert module E is called a Hilbert module, if it is complete with respect to this norm.
- Note that a (pre-)Hilbert module over $A = \mathbb{C}$ is nothing but a usual (pre-)Hilbert space.

- One says that $x, y \in E$ are orthogonal if $(x|y) = 0$, and for a subset $M \subseteq E$ we define the orthogonal complement $M^\perp$ as

$$M^\perp = \{x \in E | (x|y) = 0 \text{ for every } y \in E\}.$$

- Any A-linear map $T : E \to F$ between pre-Hilbert modules is called a module homomorphism.[a]
- The space of bounded module-homomorphisms is

$$\mathrm{Hom}(E; F)$$

with $\mathrm{End}(E) = \mathrm{Hom}(E; E)$.
- Obviously, $\mathrm{Hom}(E; F)$ is a closed subspace of $\mathcal{L}(E; F)$ (even with respect to the weak operator topology) and, canonically, an A-module.

[a] $T(a_1x_1 + a_2x_2) = a_1T(x_2) + a_2T(x_2)$ for all $x_1, x_2 \in E$, $a_1, a_2 \in A$.

- A module homomorphism $T : E \to F$ is called A-isometric if
$$(Tx|Ty) = (x|y) \text{ for all } x, y \in E.$$
- By polarization, this is equivalent to
$$(Tx|Tx) = (x|x)$$
for every $x \in E$.
- Clearly, every A-isometric homomorphism is (norm)-isometric, and hence bounded and injective.

Example 1.1

1. *Let Ω be a compact space and H a Hilbert space. The space $C(\Omega; H)$ of continuous maps from Ω to H equipped with the pointwise scalar product defines a Hilbert module over $C(\Omega)$.*
2. *Let Ω be a non-finite compact space and consider $C(\Omega)$ as a Hilbert module over itself. If $\omega \in \Omega$ is an accumulation point of Ω, then*
$$I_\omega = \{f \in C(\Omega) | f(\omega) = 0\}$$
is a closed submodule of $C(\Omega)$ with $I_\omega^\perp = \{0\}$.

Outline

Any Hilbert module H over a unital commutative C^*-algebra A admits a "vector valued norm"

$$|\cdot| : E \to A_+, \;\; x \longmapsto (x|x)^{\frac{1}{2}}$$

where A_+ denotes the cone of the positive elements of A. This turns E into a so-called lattice-normed space.

Definition 1.2

Let A be a unital commutative C^*-algebra. A vector space E equipped with a mapping $|\cdot| : E \to A_+, \;\; x \longmapsto |x|$ is a lattice-normed space (over A) if the following conditions are satisfied for all $x, y \in E$ and $\lambda \in \mathbb{C}$:

1. $|x| = 0$ *if and only if* $x = 0$;
2. $|\lambda x| = |\lambda||x|$;
3. $|x + y| \leq |x| + |y|$.

Example 1.3

1. *Let A be a unital commutative C^*-algebra. As mentioned before, each pre-Hilbert module over A is a lattice-normed module, canonically.*
2. *A is a lattice-normed space, where the lattice-norm is given by the usual modulus map $A \to A$, $f \longmapsto |f|$ defined via functional calculus.*
3. *Let Ω be a compact space and H be a Hilbert space. Consider the Hilbert module $C(\Omega; H)$ of Examples 1.1 part (1). Then the vector-valued norm is given by*

$$C(\Omega; H) \to C(\Omega), \;\; x \longmapsto (\omega \longmapsto ||x(\omega)||).$$

Definition 1.4

Let E be a lattice-normed space over a unital commutative C^*-algebra A. A net $(f_i)_{i\in I}$ in A decreases to 0 if

$$i \geq j \Longrightarrow 0 \leq f_i \leq f_j \text{ and } \inf\{f_i | i \in I\} = 0.$$

A net $(x_\alpha)_\alpha$ in E order-converges (or: is order-convergent) to $x \in E$ (in symbols: $o - \lim_\alpha x_\alpha = x$), if there is a net $(f_i)_{i\in I}$ in A decreasing to zero and satisfying

$$\forall\, i \in I,\ \exists\, \alpha_i : |x_\alpha - x| \leq f_i \ (\alpha > \alpha_i).$$

A net $(x_\alpha)_\alpha$ in E is order-Cauchy if the net $(x_\alpha - x_\beta)_{(\alpha,\beta)}$ order-converges to zero.

1. Recall that in a normed space E a net $(x_\alpha)_\alpha$ converges to an element $x \in E$ if and only if for every $n \in \mathbb{N}$ there is $\alpha(n)$ such that $||x_\alpha - x|| \leq \frac{1}{n}$ for every $\alpha > \alpha(n)$. Hence, Definition 1.4 is obtained by replacing the scalar–valued norm by a vector–valued norm and the sequence $\left(\frac{1}{n}\right)_{n\in\mathbb{N}}$ by a net decreasing to zero.
2. The order–limit of a net is unique (if it exists). Each order–convergent net is order–Cauchy and each order–Cauchy net is eventually order–bounded.
3. In a lattice–normed space, the vector space operations as well as the lattice–norm are order–continuous. In a pre–Hilbert module, the inner product and the module product are order–continuous.
4. One has $o - \lim x_\alpha = x$ in a lattice–normed space E if and only if $o - \lim_\alpha ||x_\alpha - x| = 0$ in A, and a similar statement holds for order–Cauchy nets.

Lemma 1.5

Let E, F be lattice-normed modules over a unital, commutative C^*-algebra A. Then the following assertions hold:

(i) *If $(x_\alpha)_\alpha$ is a net in E and $x \in E$ then $\lim_\alpha x_\alpha = x$ implies $o-\lim_\alpha x_\alpha = x$.*

(ii) *Order-closed subsets of E are norm-closed, norm-dense subsets are order-dense.*

(iii) *A subset of E is order-bounded iff it is norm-bounded.*

(iv) *A module homomorphism $T : E \to F$ is bounded iff it is order-continuous iff it is order-bounded, i.e. there exists $c > 0$ such that $|Tx| \leq c|x|$ for all $x \in E$. In this case, the latter estimate is true with $c = ||T||$.*

(v) *If $A = \mathbb{C}$ then order-convergence is the same as norm-convergence.*

- A lattice-normed space E is (order-)complete if every order-Cauchy net in E is order-convergent in E.
- A commutative unital C^*-algebra A is a Stone algebra if it is order-complete (as a lattice-normed space over itself).
- A compact space Ω is called a Stonean space if it is extremally disconnected, i.e., if the closure of every open subset is open.[a]
- A unital commutative C^*-algebra (or rather: a Banach lattice) is called Dedekind complete if each subset of real elements, bounded from above, has a supremum.

[a] For example, the Stone–Čech compactification $\beta\mathbb{N}$ of the space of natural numbers $\mathbb{N}$ is an example of Stonean space. Further, it is clear that, the unit interval $[0;1]$ is not a Stonean space.

In view of the Gelfand–Naimark representation theorem, the following result is a complete characterization of Stone algebras.

Proposition 1.6

For a compact space Ω the following assertions are equivalent.

(a) $C(\Omega)$ *is a Stone algebra.*

(b) $C(\Omega)$ *is Dedekind complete.*

(c) Ω *is a Stonean space.*[a]

[a] Since $\beta\mathbb{N}$ is a Stonean space, the space $C(\beta\mathbb{N})$ is a Stone algebra. It is important that the set of all idempotents of $C(\beta\mathbb{N})$ is an uncountable set. At the same time the algebra $C[0;1]$ consists only two trivial idempotents $\mathbf{0}$ and $\mathbf{1}$.

As a consequence of Proposition 1.6 we note the following simplified description of order-convergence when A is a Stone algebra.

Lemma 1.7

Let E be a lattice-normed space over a Stone algebra A, let $(x_\alpha)_\alpha$ be a net in E and $x \in E$. Then the following assertions hold:

(i) $o - \lim_\alpha x_\alpha = x$ *iff there is an index α_0 and a net $(f_\alpha)_{\alpha \geq \alpha_0}$ in A decreasing to* 0 *with*

$$|x_\alpha - x| \leq f_\alpha \quad (\alpha \geq \alpha_0).$$

(ii) $(x_\alpha)_\alpha$ *is order-Cauchy iff there is α_0 and a net $(f_\alpha)_{\alpha \geq \alpha_0}$ in A decreasing to* 0 *with*

$$|x_\beta - x_\gamma| \leq f_\alpha \quad (\beta, \gamma \geq \alpha \geq \alpha_0).$$

Outline

1. Lattice-Normed Spaces and Stone Algebras
   - Hilbert Modules
   - Lattice-Normed Spaces
2. Kaplansky–Hilbert Modules
   - Bounded Module Homomorphisms
   - Hilbert–Schmidt Homomorphisms

- An order-complete lattice normed module over a Stone algebra A is called a Kaplansky–Banach module (over A).
- A Kaplansky–Hilbert module (in short: KH- module) is an order-complete pre-Hilbert module E over a Stone algebra.
- A submodule of a Kaplansky–Hilbert module E is called a Kaplansky–Hilbert submodule (KH-submodule) of E if it is order-closed in E.

- Let E, F be lattice-normed modules over a Stone algebra A.
- Recall from Lemma 1.5 that a module homomorphism $T : E \to F$ is bounded if and only if it is order-continuous. The space $\mathrm{Hom}(E; F)$ of all bounded module homomorphisms is a unital A-module in a canonical way.
- We now turn $\mathrm{Hom}(E; F)$ into a lattice-normed module. To this end, define the operator lattice-norm of $T \in \mathrm{Hom}(E; F)$ by

$$|T| = \sup_{|x| \leq 1} |Tx|.$$

Here the supremum is taken in A_+, and this supremum exists since the set $\{|Tx| | x \in E, |x| \leq 1\}$ is bounded from above, e.g., by $||T||\mathbf{1}$.

The following result is analogous to a well-known result from Banach space theory.

Proposition 2.1

Let E, F be lattice-normed modules over a Stone algebra A. Then

$$|\cdot| : Hom(E;F) \to A_+, \ \ T \longmapsto |T| = \sup_{|x|\leq 1} |Tx|$$

turns $Hom(E;F)$ into a lattice-normed module. Moreover,

$$||T|| = |||T|||_A$$

and

$$|Tx| \leq |T||x| \ \ (T \in Hom(E;F), \ x \in E).$$

If G is another lattice-normed module over A and $S \in Hom(F;G)$, then $|ST| \leq |S||T|$. If F is order-complete, then so is $Hom(E;F)$.

A module homomorphism $T : E \to F$ between lattice-normed spaces is called A-isometric if $|Tx| = |x|$ for all $x \in E$.

Proposition 2.2

Let E, F be lattice-normed modules over the Stone algebra A and let E_0 be an order-dense submodule of E. Furthermore, let F be order-complete. Then each $T \in Hom(E_0; F)$ has a unique extension to an element $T^E \in Hom(E; F)$.
If T is A-isometric, then so is T^E. The mapping

$$Hom(E_0; F) \to Hom(E; F), \ \ T \longmapsto T^E$$

is an A-isometric isomorphism of lattice-normed modules.

- Let E be a pre-Hilbert module over a unital commutative C^*-algebra A.
- A family $(x_i)_{i\in I}$ in E is called an orthogonal system if $(x_i|x_j) = 0$ whenever $i \neq j$. In this case if I is finite, one has

$$\left|\sum_{i\in I} x_i\right|^2 = \sum_{i\in I} |x_i|^2.$$

- An orthogonal system $(x_i)_{i\in I}$ in E is called a suborthonormal system if each x_i is normalized.
- In this case, the system is called homogeneous if $|x_i| = |x_j|$ for all $i, j \in I$. In other words, a suborthonormal system is homogeneous if $(x_i|x_j) = \delta_{ij}p$ for all i, j and some fixed idempotent $p \in \mathbb{B} = P(A)$.
- If $p = \mathbf{1}$ then $(x_i)_{i\in I}$ is called an orthonormal system.

A (sub)orthonormal system $(x_i)_{i\in I}$ is called a (sub)orthonormal basis if $\{x_i|i \in I\}^\perp = \{0\}$. A subset $\mathcal{B} \subset E$ is called a (sub)orthonormal subset (basis) if the family $(x)_{x\in\mathcal{B}}$ is a (sub)orthonormal system (basis).

Lemma 2.3

Let E be a pre-Hilbert module over a Stone algebra A, and let $\mathcal{B} \subset E$ be a suborthonormal set in E. Then for each $x \in E$ the formal series (=net of finite partial sums)

$$\sum_{y\in\mathcal{B}} (x|y)y = \left(\sum_{y\in F} (x|y)y\right)_{F\subset\mathcal{B}\ finite}$$

is an order-Cauchy net in E. If it converges, its order-limit z satisfies $x - z \in \mathcal{B}^\perp$, and $|z|^2 = \sum_{y\in\mathcal{B}} |(x|y)|^2 \le |x|^2$.

- If E is a pre-Hilbert module over a unital commutative C^*-algebra A and $y \in E$, then

$$\overline{y} : E \to A, \ \ x \longmapsto (x|y)$$

is an element of the dual module $E^* = \mathrm{Hom}(E; A)$.
- If A is a Stone algebra, one has the following strengthening, comprising a version of the Riesz–Frěchet theorem for Kaplansky-Hilbert modules.

Theorem 2.4

Let E be a pre-Hilbert module over a Stone algebra A. Then $|\overline{y}| = |y|$ for all $y \in E$. If E is a Kaplansky–Hilbert module, then the mapping

$$\Theta : E \to Hom(E; A), \ \ y \longmapsto \overline{y}$$

is bijective.

- If E is a KH-module, then via the bijection Θ the dual module E^* is – canonically – equipped with the structure of a Kaplansky–Hilbert module over A.
- The mapping Θ is then A-antilinear and A-isometric.
- The conjugate homomorphism of $T \in \mathrm{End}(E)$ is $\overline{T} \in \mathrm{End}(E^*)$ defined by $\overline{T} \circ \Theta = \Theta \circ T$ or, equivalently, $\overline{T}\overline{x} = \overline{Tx}$ for all $x \in E$.

Corollary 2.5

Let E, F be Kaplansky–Hilbert modules. For every $T \in Hom(E;F)$ there is a unique module homomorphism $T^* \in Hom(F;E)$ with

$$(Tx|y) = (x|T^*y) \text{ for all } x \in E,\ y \in F$$

Moreover, $(T^*)^* = T$, $|T| = |T^*|$ and $ran(T)^\perp = \ker(T^*)$.

Outline

- Let E and F be Kaplansky–Hilbert modules over a Stone algebra A. Moreover, let $\mathcal{F}$ be the family of all finite sub-orthonormal subsets of E.
- A homomorphism $T \in \mathrm{Hom}(E;F)$ is called a Hilbert–Schmidt homomorphism if

$$|T|_{HS} = \sup\left\{\left(\sum_{x\in\mathcal{B}} |Tx|^2\right)^2 : \mathcal{B}\in\mathcal{F}\right\}$$

  exists in A_+.
- We write $\mathrm{HS}(E;F)$ for the A-module of all A-Hilbert–Schmidt homomorphisms from E to F and $\mathrm{HS}(E)$ if $E = F$.

Proposition 2.6

Let $\mathcal{B}$ be a fixed suborthonormal basis of E. Then for $T \in End(E)$ the following assertions are equivalent:

(a) $T \in HS(E)$;

(b) $T^* \in HS(E)$;

(c) $\sum_{x \in \mathcal{B}} |Tx|^2$ *order-converges in* A;

(d) $\sum_{x \in \mathcal{B}} |T^*x|^2$ *order-converges in* A.

In this case, $|T|_{HS}^2 = \sum_{x \in \mathcal{B}} |Tx|^2$ *and for each* $S \in Hom(E)$ *one has* $ST, TS \in HS(E)$ *with* $|ST|_{HS}, |TS|_{HS} \leq |S||T|_{HS}$.

Proposition 2.7

Let E be a Kaplansky–Hilbert module over a Stone algebra A. Then the mapping

$$HS(E) \times HS(E) \to A, \ \ (T, S) \longmapsto (T|S) = o - \lim_{\mathcal{B} \in \mathcal{F}} \sum_{x \in \mathcal{B}} (Tx|Sx)$$

turns $HS(E)$ into a Kaplansky–Hilbert module over A, and

$$(T|S)_{HS} = \sum_{x \in \mathcal{B}} (Tx|Sx) \ \ (T, S \in HS(E))$$

as an order-convergent series for each suborthonormal basis $\mathcal{B}$ of E.