Seminar "Functional analysis and its applications"

# Isomorphisms of commutative regular algebras

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# Outline

1. Intoduction
   - Short history
   - Prelimanaries
2. Main results
   - The group of all band preserving automorphisms
   - Isomorphisms of homogeneous commutative unital regular algebras

# Isomorphisms of regular rings

In 1930's, motivated by the geometry of lattice of the projections of type $II_1$ factors, von Neumann built the theory on the correspondence between complemented orthomodular lattices and regular rings. Let us recall one of his achievements [1, Part II, Theorem 4.2], applied to the case of $*$-regular rings. Let $\mathfrak{R}$, $\mathfrak{R}'$ be $*$-regular rings such that their lattices of projections $L_{\mathfrak{R}}$ and $L_{\mathfrak{R}'}$ are lattice-isomorphic. If $\mathfrak{R}$ has order $n \geq 3$ (which means that it contains a ring of matrices of order $n$), then there exists a ring isomorphism of $\mathfrak{R}$ and $\mathfrak{R}'$ which generates given lattice isomorphism between $L_{\mathfrak{R}}$ and $L_{\mathfrak{R}'}$.

J. von Neumann, Continuous geometry. Foreword by Israel Halperin, Princeton Mathematical Series, No. 25 Princeton University Press, Princeton, N.J. (1960).

# Problems of homomorphisms on semifields

Note that in the case of commutative regular rings the picture is completely different.

Let us recall a problem of isomorphisms for an important class of commutative regular rings with an atomic Boolean algebra of idempotents, namely, so-called Tychonoff semifields.

M. Ya. Antonovskii, V. G. Boltyanskii, Tikhonov semifields and certain problems in general topology, Russian Mathematical Surveys (1970), 25(3):1-43.

# Problems of homomorphisms on semifields

Given an arbitrary set $\Delta$, a Tychonoff semifield $\mathbb{R}^\Delta$ is defined as the product of $|\Delta|$ copies of the real field, equipped with the pointwise algebraic operations, natural partial order and the Tychonoff's topology. These operations make $\mathbb{R}^\Delta$ a topological regular ring. The set of all idempotents of the semifield $\mathbb{R}^\Delta$ with the induced topology and order is topologically isomorphic to $\{0,1\}^\Delta$. For $g \in \Delta$ denote by $\mathbf{1}_g$ an atom from $\{0,1\}^\Delta$ defined as $\mathbf{1}_g(g) = 1$ and $\mathbf{1}_g(g') = 0$ for $g \neq g'$ $(g' \in \Delta)$ and $\mathbf{1}_\Delta$ is identity of $\mathbb{R}^\Delta$.

M. Ya. Antonovskii, V. G. Boltyanskii, Tikhonov semifields and certain problems in general topology, Russian Mathematical Surveys (1970), 25(3):1-43.

# Problems of homomorphisms on semifields

The following two questions are equivalent:[a]

(a) Does there exist an algebraic homomorphism $\psi : \mathbb{R}^{\Delta} \to \mathbb{R}$ satisfying the condition $\psi(\mathbf{1}_g) = 0$ for all $g \in \Delta$, such that $\psi(\mathbf{1}_{\Delta}) = 1$?

(b) Does there exist a non trivial two-valued countably additive measure $\mu : \{0,1\}^{\Delta} \to \mathbb{R}$ satisfying the condition $\mu(\mathbf{1}_g) = 0$ for all $g \in \Delta$?

The second question is the famous Ulam Problem which is connected with the properties of cardinal $|\Delta|$.[b]

[a] Sh. A. Ayupov, *Homomorphisms of a class of rings and two-valued measures on Boolean algebras*, Funct. Anal. Appl., 11:3 (1977), 217–219.

[b] S. U1am, *Zur Mass theory inder allgsmeinen Mengenlehre*, Fund. Math. **16** (1930), 140–150.

# Kusraev's result

- Let $S(\Omega, \Sigma, \mu)$ be the algebra of all measurable complex functions on $(\Omega, \Sigma, \mu)$;
- A.G. Kusraev by means of Boolean-valued analysis establishes necessary and sufficient conditions for existence of band-preserving non trivial algebra automorphisms on $S(\Omega, \Sigma, \mu)$;
- In particular, he has proved that $S[0,1]$ admits discontinuous algebra automorphisms which identically act on the Boolean algebra $P(L_\infty[0,1])$.[a]

[a] A. G. Kusraev, *Automorphisms and derivations in an extended complex $f$-algebra*, Sib. Math. J. **47** (2006) 97–107.

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# Regular algebras

- A ring $\mathcal{A}$ is said to be regular in the sense of von Neumann if, for every $a \in \mathcal{A}$, the equation $axa = a$ has a solution in $\mathcal{A}$.
- An algebra $\mathcal{A}$ is said to be regular, if $\mathcal{A}$ is a regular ring in the sense of von Neumann.
- Let $\mathcal{A}$ be a commutative unital regular algebra with a unity $\mathbf{1}$ over an algebraically closed field $\mathbb{F}$ of characteristic 0 and let $\nabla = \nabla(\mathcal{A})$ denote the set of idempotent elements of $\mathcal{A}$, that is,
$$\nabla(\mathcal{A}) = \left\{e \in \mathcal{A} : e^2 = e\right\}.$$
- For $a \in \mathcal{A}$, let $i(a)$ be the solution of the equations $axa = a$ and $xax = x$.
- The element $s(a) = ai(a) \in \nabla$ is called the support of $a$.

# A measure on Boolean algebras

- A function $\mu : \nabla \to [0, +\infty]$ is said to be a measure on $\nabla$, if $\mu(0) = 0$ and $\mu(e \vee g) = \mu(e) + \mu(g)$ for all $e, g \in \nabla$ such that $e \wedge g = 0$.
- The measure $\mu$ is called finite, if $\mu(e) < +\infty$ for all $e \in \nabla$; a strictly positive, if from the equality $\mu(e) = 0$, it follows that $e = 0$.
- The measure $\mu$ is called countably additive, if for any countable family $\{e_i\}_{i \in I} \subset \nabla$ of nonzero pairwise disjoint elements such that $\sup\limits_{i \in I} e_i \in \nabla$ the equality

$$\mu\left(\sum_{i \in I} e_i\right) = \sum_{i \in I} \mu(e_i)$$

holds.

# A rank metric on regular algebra

- We assume below that $\mu$ is a strictly positive finite measure on $\nabla$ of all idempotents of a commutative unital regular algebra $\mathcal{A}$.
- Define the function $\rho : \mathcal{A} \times \mathcal{A} \to \mathbb{R}$, by setting:

$$\rho(a, b) = \mu(s(a - b)). \tag{1}$$

- We will assume that $\mathcal{A}$ is complete with respect to the metric $\rho$ defined as in (1).
- In this case, $\nabla$ is a complete Boolean algebra of the countable type.

# Main example

**Example 1.1**

*Let $(\Omega, \Sigma, \mu)$ be a measure space with a finite strictly positive countable-additive measure $\mu$. Denote by $S(\Omega) = S(\Omega, \Sigma, \mu)$ the algebra of all (classes of) measurable functions on $(\Omega, \Sigma, \mu)$ with values in the field $\mathbb{F}$. Clearly, $S(\Omega)$ is a commutative regular algebra with unit $\mathbf{1}$ given by $\mathbf{1}(\omega) = 1, \omega \in \Omega$. The Boolean algebra $\nabla$ of all idempotents in $S(\Omega)$ coincides with the Boolean algebra of classes of almost everywhere equal sets from $\Sigma$. For any $a \in S(\Omega)$ the support $s(a)$ is a class from $\nabla$ with the representative $\{\omega \in \Omega :: |a(\omega)| \neq 0\} \in \Sigma$.*

# Example

## Example 1.2

*If* $\Omega = \{1, \ldots, n\}$ *is a n-point set with a counting measure* $\mu$, *that is,* $\mu(A) = |A|$ *for all* $A \subseteq \Omega$, *where* $|A|$ *is the cardinality of* $A$, *then* $S(\Omega) \cong \mathbb{F}^n$. *In this case* $\nabla = \{0,1\}^n$. *Further, for a non zero element* $a = (a_1, \ldots, a_n) \in \mathbb{F}^n$ *its partial inverse and support are the following:* $i(a) = (i(a_1), \ldots, i(a_n))$, *where* $i(a_k) = \frac{1}{a_k}$ *for* $a_k \neq 0$ *and* $i(a_k) = 0$ *otherwise;* $s(a) = (\varepsilon_1, \ldots, \varepsilon_n)$, *where* $\varepsilon_k = |sign a_k|$ *for all* $k = 1, \ldots, n$. *The metric* $\rho$ *on* $\mathbb{F}^n$ *can be defined as*

$$\rho(a,b) = \left|\{k \in \overline{1,n} : a_k \neq b_k\}\right| = \sum_{k=1}^{n} |sign(a_k - b_k)|.$$

# A subalgebra of simple elements

An element $a \in \mathcal{A}$ is called

- finitely valued, if
$$a = \sum_{k=1}^{n} \lambda_k e_k,$$
where $\lambda_k \in \mathbb{F}$, $e_k \in \nabla$, $e_k e_j = 0$, $k \neq j$, $k, j = 1, \ldots, n$, $n \in \mathbb{N}$;
- countably valued, if
$$a = \sum_{k=1}^{\infty} \lambda_k e_k,$$
where $\lambda_k \in \mathbb{F}$, $e_k \in \nabla$, $e_k e_j = 0$, $k \neq j$, $k, j \in \mathbb{N}$.

Denote by $\mathbb{F}(\nabla)$ (respectively, $\mathbb{F}_c(\nabla)$) the set of all finitely valued (respectively, countably valued) elements in $\mathcal{A}$. Then,

$$\nabla \subset \mathbb{F}(\nabla) \subset \mathbb{F}_c(\nabla)$$

and $\mathbb{F}(\nabla)$, $\mathbb{F}_c(\nabla)$ are regular subalgebras in $\mathcal{A}$.

# A band preserving mapping

Recall that two elements $x, y \in \mathcal{A}$ are said to be orthogonal (notation $x \perp y$), if $s(x)s(y) = 0$.
A linear mapping $\Phi : \mathcal{B} \to \mathcal{A}$ is said to be band preserving, if

$$x \perp y \Longrightarrow \Phi(x) \perp y.$$

Since $x \perp \mathbf{1} - s(x)$, we see that $\Phi$ is band preserving iff

$$s\left(\Phi(x)\right) \leq s(x) \tag{2}$$

for all $x \in \mathcal{A}$.

**Proposition 1.3**

*Let $\mathcal{B}$ be a regular subalgebra in $\mathcal{A}$ such that $\mathbb{F}_c(\nabla) \subset \mathcal{B}$ and let $\Phi : \mathcal{B} \to \mathcal{A}$ be a homomorphism such that $\Phi(\mathbf{1}) = \mathbf{1}$. Then $\Phi$ is band preserving if and only if $\Phi|_{\mathbb{F}_c(\nabla)} = id_{\mathbb{F}_c(\nabla)}$.*

# An algebraically independent subset

- Let $\mathbb{F}[x_1, \ldots, x_n]$ be the algebra of polynomials in $x_1, \ldots, x_n$ over the field $\mathbb{F}$.
- A subset $\mathcal{M} \subset \mathcal{A}$ is said to be algebraically independent, if for all $a_1, \ldots, a_n \in \mathcal{M}, e \in \nabla$, and $p(x_1, \ldots, x_n) \in \mathbb{F}[x_1, \ldots, x_n]$ from $ep(a_1, \ldots, a_n) = 0$ it follows that $eq(a_1, \ldots, a_n) = 0$ for every monomial $q \in \mathbb{F}[x_1, \ldots, x_n]$ occurring in a representation of $p$.
- An algebraically independent subset $\mathcal{M}$ is said to be maximal if it is not a proper subset of an algebraically independent set.
- For example, $\left\{t^{\sqrt{2}}, t^{\sqrt{3}}\right\}$ is an algebraically independent subset in $S(0;1)$. But $\{\sin t, \cos t\}$ is an algebraically dependent subset in $S(0;1)$, because $\sin^2 t + \cos^2 t = 1$.

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# An algebraically independent subset

Let $\mathcal{M}$ be a maximal algebraically independent subset of $\mathcal{A}$. Set

$$s(\mathcal{M}) = \{s(a) : a \in \mathcal{M}\} \subset \nabla,$$

and for $e \in s(\mathcal{M})$ set

$$\mathcal{M}_e = \{a \in \mathcal{M} : s(a) = e\}.$$

Denote by $G(\mathcal{M}, \mathcal{A})$ the set of all mappings $g : \mathcal{M} \to \mathcal{A}$ such that

- $s(g(a)) = s(a)$ for all $a \in \mathcal{M}$;
- $g(\mathcal{M}) = \{g(x) : x \in \mathcal{M}\}$ is also a maximal algebraically independent subset of $\mathcal{A}$.

A group of all permutations of $\mathcal{M}$, denoted by $Sym(\mathcal{M})$.
Let $\mathbb{F}_c(\nabla)^\bullet$ be the group of all invertible elements from $\mathbb{F}_c(\nabla)$.

# The group of all band preserving automorphisms

For a commutative unital regular algebra $\mathcal{A}$ denote by $Aut_{\nabla}(\mathcal{A})$ the group of all band preserving automorphisms of $\mathcal{A}$.

## Theorem 2.1

*Let $\mathcal{A}$ be a commutative unital regular algebra over an algebraically closed field $\mathbb{F}$ of characteristic zero such that there exists a finite strictly positive countable-additive measure $\mu$ on the Boolean algebra $\nabla$ of all idempotents in $\mathcal{A}$; $\mathcal{A}$ is a complete with respect the metric $\rho(a, b) = \mu(s(a - b)), a, b \in \mathcal{A}$. Then there is an injective mapping*

$$g \in G(\mathcal{M}, \mathcal{A}) \to \Phi_g \in Aut_{\nabla}(\mathcal{A}). \qquad (3)$$

*In particular, the group $Aut_{\nabla}(\mathcal{A})$ contains a subgroup isomorphic to the group $\prod\limits_{e \in s(\mathcal{M})} Sym(\mathcal{M}_e) \ltimes (\mathbb{F}_c(e\nabla)^{\bullet})^{|e\mathcal{M}|}$.*

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For a subset $\mathcal{S}$ consider the following subalgebras:

- $\mathcal{A}_0(\mathcal{S}) = \mathbb{F}(\mathcal{S}, \nabla)$ is the subalgebra of $\mathcal{A}$ generated by $\mathcal{S}$ and $\nabla$;
- $\mathcal{A}_1(\mathcal{S})$ is the least regular subalgebra of $\mathcal{A}$ containing $\mathcal{A}_0(\mathcal{S})$;
- $\mathcal{A}_2(\mathcal{S})$ is the closure of $\mathcal{A}_1(\mathcal{S})$ with respect to the metric $\rho$;
- $\mathcal{A}_3(\mathcal{S})$ is the integral closure of $\mathcal{A}_2(\mathcal{S})$;
- $\mathcal{A}_4(\mathcal{S})$ is the closure of $\mathcal{A}_3(\mathcal{S})$ with respect to the metric $\rho$.

# A homogeneous Boolean algebra

- Recall that a weight of the Boolean algebra $\nabla$ is the least cardinality of sets which completely generate $\nabla$, i.e.,
$$\tau(\nabla) = \inf\{|X| : X \text{ completely generates } \nabla\}.$$
For $0 \neq e \in \nabla$ set $\nabla_e = \{x \in \nabla : x \leq e\}$.
- A Boolean algebra $\nabla$ is homogeneous, if $\tau(\nabla_e) = \tau(\nabla)$ for any non zero $e \in \nabla$.
- We say that an algebraically independent subset $\mathcal{M}$ in $\mathcal{A}$ is faithful, if $s(x) = \mathbf{1}$ for all $x \in \mathcal{M}$.
- Let $\gamma$ be a cardinal number. We say that a commutative unital regular algebra $\mathcal{A}$ is $\gamma$-homogeneous, if there is a faithful algebraically independent subset $\mathcal{M}$ with the cardinality $\gamma$ such that $\mathcal{A}_4(\mathcal{M}) = \mathcal{A}$.
- In this case we call $\gamma$ the transcendence degree of $\mathcal{A}$ (notation $\mathrm{trdeg}(\mathcal{A}) = \gamma$) and say that $\mathcal{A}$ is a $\gamma$-homogeneous commutative regular algebra.

# Maharam homogeneous measure space

- A measure space $(\Omega, \Sigma, \mu)$ is said to be Maharam homogeneous if the Boolean algebra $\nabla$ of the algebra $S(\Omega)$ is homogeneous.
- An automorphism $\phi$ of the Boolean algebra $\nabla$ is said to be measure-preserving, if $\mu(e) = \mu(\phi(e))$ for all $e \in \nabla$.
- Every measure-preserving automorphism of the Boolean algebra $\nabla$ can be extended to an automorphism of $S(\Omega)$.
- Let $G = \{\phi\}$ be a group of automorphisms of $\nabla$. The group $G$ is called ergodic if for every $0 \neq e \in \nabla$ the following equality holds:
$$\bigvee_{\phi \in G} \phi(e) = \mathbf{1}.$$

# A homogeneous commutative regular algebra

**Proposition 2.2**

*Let $(\Omega, \Sigma, \mu)$ be a Maharam homogeneous measure space with a finite strictly positive countable-additive measure $\mu$ and let $\mathcal{A}$ be an integrally closed and $\rho$-closed regular subalgebra in $S(\Omega)$. Suppose that $G$ is a measure-preserving ergodic group of automorpisms on $\nabla$ such that $\phi(\mathcal{A}) = \mathcal{A}$ for all $\phi \in G$. Then $\mathcal{A}$ is homogeneous.*

Since for a Maharam homogeneous measure space $(\Omega, \Sigma, \mu)$ , the group of all measure-preserving automorphisms of $\nabla$ is ergodic, by Proposition 2.2, $S(\Omega)$ is a homogeneous commutative regular algebra.

# A criteria of isomorphism of homogeneous commutative regular algebras

**Theorem 2.3**

*Let $(\Omega, \Sigma, \mu)$ be a Maharam homogeneous measure space with a finite strictly positive countable-additive measure $\mu$. Assume that $\mathcal{A}$ and $\mathcal{B}$ are homogeneous, regular, integrally closed and $\rho$-closed unital subalgebras in $S(\Omega)$. Suppose that $\nabla(\mathcal{A})$ and $\nabla(\mathcal{B})$ are respectively their Boolean algebras of idempotents. Then the following assertions are equivalent:*

1. *the algebras $\mathcal{A}$ and $\mathcal{B}$ are isomorphic;*
2. *the Boolean algebras $\nabla(\mathcal{A})$ and $\nabla(\mathcal{B})$ are isomorphic and $trdeg(\mathcal{A}) = trdeg(\mathcal{B})$.*

# The group of band preserving automorphisms of homogeneous algebra

Let $\gamma$ be a cardinal number and let $Sym_\gamma$ be the group of all permutations of a set of the cardinality $\gamma$. Theorem 2.1 implies the following result.

**Corollary 2.4**

*Let $\mathcal{A}$ be an $\gamma$-homogeneous algebra satisfying conditions of Theorem 2.1 . Then the group $Aut_\nabla(\mathcal{A})$ contains a subgroup isomorphic to the group $Sym_\gamma \ltimes (\mathbb{F}_c(\nabla)^\bullet)^\gamma$.*

# An approximate limit

Recall that a measurable set $E \subset \mathbb{R}$ is said to have density $d$ at $t$ if the

$$\lim_{h\to 0} \frac{m\left(E \cap [t-h, t+h]\right)}{2h}$$

exists and is equal to $d$, where $m$ is the Lebesgue's measure on $\mathbb{R}$. By Lebesgue's density theorem the set $E$ has density 1 at almost all points of $E$.

Consider a Lebesgue measurable set $E \subset \mathbb{R}$, a measurable function $f : E \to \mathbb{R}$ and a point $t_0 \in \mathbb{R}$, in which $E$ has density 1. The approximate upper and lower limits of $f$ at $t_0$ are defined, respectively, as

- the infimum of $a \in \mathbb{R} \cup \{\infty\}$ such that the set $\{f \leq a\}$ has density 1 at $t_0$;
- the supremum of $a \in \mathbb{R} \cup \{\infty\}$ such that the set $\{f \geq a\}$ has density 1 at $t_0$.

# An approximate derivative

They are usually denoted by

$$ap - \limsup_{t \to t_0} f(t) \text{ and } ap - \liminf_{t \to t_0} f(t).$$

If the two numbers coincide then the result is called the approximate limit of $f$ at $t_0$ and it is denoted by $ap - \lim\limits_{t \to t_0} f(t)$.

If the approximate limit

$$f'_{ap}(t_0) := \mathrm{ap} - \lim_{t \to t_0} \frac{f(t) - f(t_0)}{t - t_0}$$

exists and it is finite, then it is called approximate derivative of the function $f$ at $t_0$ and the function $f$ is called approximately differentiable at $t_0$.

# The algebra of approximately differentiable functions

1. The $*$-subalgebra $AD(0;1)$ coincides with the set of all functions of the form
$$\sum_{n=1}^{\infty} e_n g_n,$$
where $\{e_n\}_{n\geq 1}$ is a partition of the unit in $\nabla$, $g_n \in C^{(1)}[0,1]$, $n \in \mathbb{N}$, and $C^{(1)}[0;1]$ is the algebra of all continuously differentiable complex-valued functions on $[0;1]$.

2. The algebra $AD(0;1)$ is the smallest regular, integrally closed and $\rho$-closed proper subalgebra of $S(0;1)$, which contains $D[0,1]$ and all projections from $S(0;1)$, where $D[0;1]$ is the algebra of (classes of) differentiable functions which have almost everywhere finite derivation on $[0;1]$.[1]

[1] A. F. Ber, K. K. Kudaybergenov and F. A. Sukochev, Notes on derivations of Murray–von Neumann algebras, J. Funct. Anal. 279 (5) (2020) 108589.

# Isomorphism between the algebra of all measurable functions and the algebra of all approximately differentiable functions

Applying Proposition 2.2 for the case where $\mathcal{A} = AD(0;1)$ and $G$ is the group of all rational translations (modulo 1) of $[0,1]$, we see that $AD(0;1)$ is a homogeneous commutative regular algebra.

**Theorem 2.5**

*The regular algebras $S(0;1)$ and $AD(0;1)$ are isomorphic.*

As we have mentioned above $AD(0;1)$ is a proper subalgebra in $S(0;1)$. Recall that the first example of a nowhere approximately differentiable function belonging to the Banach space $C[0,1]$ is due to V. Jarnik.[2]

[2] V. Jarnik, Sur les nombres derivés approximatifs, Fund. Math. 22 (1934) 4-16.

## Remark 2.6

*It should be noted that the algebras* $C[0;1]$ *and* $C^{(1)}[0;1]$ *are not isomorphic. Indeed, assume that* $\Phi : C[0;1] \to C^{(1)}[0;1]$ *is an isomorphism. There is a continuous mapping*[a] $\iota : [0;1] \to [0;1]$ *such that* $\Phi(x) = x \circ \iota$ *for all* $x \in C[0;1]$*. If* $\iota$ *is a bijection, then the image* $Im\Phi = C[0;1]$*. Otherwise there are different points* $t_1, t_2 \in [0;1]$ *such that* $\iota(t_1) = \iota(t_2)$*. Therefore* $\Phi(x)(t_1) = \Phi(x)(t_2)$ *for all* $x \in C[0;1]$*. So, in both cases, we have that* $Im\Phi \neq C^{(1)}[0,1]$*. Thus there is no isomorphism from* $C[0;1]$ *onto* $C^{(1)}[0;1]$*.*

[a] L. Gillman, M. Jerison, Rings of continuous functions, The University Series in Higher Mathematics D. Van Nostrand Co., Inc., Princeton, N.J.-Toronto-London-New York 1960.

# Example

Note that there is an uncountable algebraically independent subset in $S(0;1)$.[3] Therefore we have the following result.

## Corollary 2.7

*The group $Aut_{\nabla}(S(0;1))$ of all band preserving automorphms of the algebra $S(0;1)$ contains a subgroup isomorphic to the group $Sym_{\gamma} \ltimes (\mathbb{F}_c(\nabla)^{\bullet})^{\gamma}$, where $\gamma$ is the continuum.*

[3] A. F. Ber, Derivations on commutative regular algebras, Siberian Advances in Mathematics, 21 (2011) 161–169.

# Conjecture

In connection with the above Corollary the following Conjecture arises.

## Conjecture 2.8

*Let $\mathcal{A}$ be a homogeneous commutative regular algebra with a homogeneous Boolean algebra of idempotents $\nabla$. Then*

$$trdeg(\mathcal{A}) \leq 2^{\max\{\tau(\nabla), \aleph_0\}}.$$

*In particular,*

$$trdeg(S(0;1)) = c.$$

# Example

## Remark 2.9

*We note that any band preserving automorphism of the algebra $S_{\mathbb{R}}(0;1)$ of all classes of real-valued functions on $(0;1)$ is trivial. Indeed, any band preserving automorphism $\Phi$ on $S_{\mathbb{R}}(0;1)$ is positive, because for $0 \leq x \in S_{\mathbb{R}}(0;1)$ we have that $\Phi(x) = \Phi\left((\sqrt{x})^2\right) = \Phi\left(\sqrt{x}\right)^2 \geq 0$. Now for any $x \in S_{\mathbb{R}}(0;1)$ take a sequence $\{x_n\}$ of simple functions on $S_{\mathbb{R}}(0;1)$ such that $x_n \uparrow x$. Then $x_n = \Phi(x_n) \uparrow \Phi(x)$, and therefore $\Phi(x) = x$.*