Applications of Proof Theory to Core Mathematics: Recent Developments

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Though in this specific form, in general impossible (Gödel), the basic approach is largely correct for existing ordinary mathematics: **proof-theoretic tameness** of ordinary mathematics!

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'What more do we know if we have proved a theorem by restricted means than if we merely know that it is true?' (G. Kreisel)

Proof Mining in core mathematics

- During (mainly) the last 20 years this proof-theoretic approach has resulted in **numerous new quantitative results** as well as **qualitative uniformity results** in particular in: nonlinear analysis, fixed point theory, ergodic theory, topological dynamics, approximation theory, convex optimization, abstract Cauchy problems, pursuit-evasion games ($\geq$ 100 papers mostly in specialized journals in the resp. areas or general mathematics journals).

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- General **logical metatheorems** explain applications as instances of logical phenomena (K. 2005, Gerhardy/K. 2008, TAMS).
- Some of the logical tools used have been rediscovered in 2007 in special cases by Terence Tao prompted by concrete mathematical needs **"finitary analysis"!**

Logical Metatheorems based on Functional Interpretations

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- **compact** metric spaces K (**if separability is used**),
- metrically bounded subsets of suitable **abstract** metric structures X,
- only depend on **majorizing data** in the case if unbounded subsets.

The running theme: convergence statements in analysis

Let (x_n) be a Cauchy sequence in a metric space (X, d), i.e.

$$\forall k \in \mathbb{N}\, \exists n \in \mathbb{N}\, \forall i, j \geq n\, (d(x_i, x_j) \leq 2^{-k}) \in \forall\exists\forall$$

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A bound $\Phi(k, g)$ on '$\exists n$' in the latter formula is a **rate of metastability**.

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$$\forall \varepsilon > 0\, \forall x, y \in X \left(d(x, T(x)), d(y, T(y)) < \Phi(\varepsilon) \to d(x, y) < \varepsilon\right)$$

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- Possible also in the nonunique case for **Fejér monotone algorithms** if one has a **modulus of metric regularity** (see below).

Applications to Pursuit-evasion games

Pursuit-Evasion Games: The Lion and Man Game

The lion and man problem, going back to R. Rado, is one of the most challenging pursuit-evasion games. **Littlewood's Miscellany** it is described as follows:

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This fact, as well as the potential applications in different fields such as robotics, biology and random processes.

We focus on a **discrete-time** equal-speed game and **ε-capture**.

Let (X,d) be a uniquely geodesic space, $D>0$. $L_0, M_0 \in A$ starting points of the lion L and the man M. After n-steps, M moves to any point M_n s.t. $d(M_n, M_{n+1}) \leq D$ and L moves via the geodesic $[L_n, M_n]$ s.t. $d(L_n, L_{n+1}){=}\min\{D, d(L_n, M_n)\}$.

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'$\mathbf{\lim d(L_{n+1}, M_n) = 0}$' $\in \forall\exists$ since the sequence is nonincreasing!

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Details in: K./López-Acedo/Nicolae, Pacific J. Math. 2021.

Applications to the Proximal Point Algorithm

Proximal mappings in Hilbert space

Let H be a real Hilbert space. $f : H \to (-\infty, \infty]$ proper lsc convex.
The **proximal mapping** $\mathrm{prox}_f : H \to H$ is defined (for $\lambda > 0$) by

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Example: Let $C \subseteq H$ be nonempty, closed and convex and

$$\iota_C : H \to [0, \infty],\ x \mapsto \begin{cases} 0, & \text{if } x \in C \\ \infty, & \text{otherwise.} \end{cases}$$

its **indicator function**, then prox_{ι_C} is the metric projection onto C.

Monotone operators

A set-valued mapping $A \subseteq H \to 2^H$ is **monotone** if

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If A is monotone then the **resolvent**

$$J_A : R(I+A) \to D(A),\ x \mapsto (I+A)^{-1}(x)$$

is single-valued and **firmly nonexpansive**, i.e. for
$T := J_A, D := R(I+A)$

$$\forall x,y \in D\ \left(\|Tx-Ty\|^2 + \|(I-T)x-(I-T)y\|^2 \leq \|x-y\|^2\right).$$

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$$\forall x, y \in D \ \left(\| Tx - Ty \|^2 + \| (I - T)x - (I - T)y \|^2 \leq \| x - y \|^2 \right) .$$

A is **maximally monotone** if is has no proper monotone extension. In this case $R(I + A) = H$.

The Proximal Point Algorithm I

Example: Let f be as before. Then the **subdifferential** of f

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Let $(\lambda_n) \subset (0, \infty)$ and A maximally monotone, then the **Proximal Point Algorithm (PPA)** is defined by

$$x_{n+1} := J_{\lambda_n A}(x_n), \ x_0 \in H.$$

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- CAT(0) spaces: Leuștean/Sipoș J. Nonlin. Var. Anal. 2018.

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- Hilbert space: K./Leuştean/Nicolae. Comm.Contemp. Math. 2018.
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The proximal point algorithm II

$$x_{n+1} := J_{\lambda_n A} x_n, \quad \textbf{Proximal Point Algorithm (PPA).}$$

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- uniformly convex Banach spaces: K. J. Convex Anal. 2021.

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In general: **strong convergence** (even in infinite dimensional Hilbert spaces) **only for** so-called **Halpern type variant of PPA**:

$$x_{n+1} := \alpha_n u + (1-\alpha_n) J_{\lambda_n A} x_n, \quad u, x_0 \in H \quad \textbf{(HPPA)}$$

(necessary conditions: $\lim \alpha_n = 0, \sum \alpha_n = \infty$).

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The proofs and their resp. minings are very different!

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Definition

A sequence (x_n) in a metric space (X, d) is Fejér monotone w.r.t. a subset $S \subseteq X$ if $\forall n \in \mathbb{N}\, \forall p \in S\ (d(x_{n+1}, p) \leq d(x_n, p))$.

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Why is this important?

If one has metric regularity one not only gets strong convergence but even a **rate of convergence!**

Moduli of regularity for mappings

In continuous optimization notions of **linear** or **Hölder metric regularity**, **error bounds** and **weak sharp minima** etc. play an important role which can be viewed as (often local forms of) special cases of (see also R.M. Anderson: 'Almost' implies 'Near', TAMS 1986):

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Definition (K./Lopéz-Acedo/Nicolae, Israel J. Math 2019)

Let $F : X \to \overline{\mathbb{R}}$ with $zer\, F := \{x \in X : F(x) = 0\} \neq \emptyset$.
F is **regular** w.r.t. $zer\, F$ if

$$\forall n \in \mathbb{N} \exists k \in \mathbb{N}\, \forall x \in X \big(|F(x)| < 2^{-k} \to \exists z' \in zer\, F\, (d(x, z') < 2^{-n})\big).$$

A function $\omega : \mathbb{N} \to \mathbb{N}$ providing $k = \omega(n)$ is a **modulus of regularity**.

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This also covers fixed point and equilibrium problems.

Computational use of moduli of regularity

Proposition (K./Lopéz-Acedo/Nicolae Israel J. Math. 2019)

Let $F : X \to \overline{\mathbb{R}}$ be with $zer\ F \neq \emptyset$ and with modulus of metric regularity ω. Let (x_n) be a sequence in X and $\psi : \mathbb{N} \to \mathbb{N}$ be s.t.

$$\forall k \in \mathbb{N}\, \exists n \leq \psi(k)\ \left(|F(x_n)| < 2^{-k}\right),$$

where (x_n) is **Fejér monotone** w.r.t. $zer\ F$. Then (x_n) is Cauchy:

$$\forall k \in \mathbb{N}\, \forall n, \tilde{n} \geq \Phi(k) := \psi(\omega(k+1))\ \left(d(x_n, x_{\tilde{n}}) < 2^{-k}\right)$$

and $$\forall k \in \mathbb{N}\, \forall n \geq \Phi(k)\ \left(dist(x_n, zer\ F) < 2^{-k}\right).$$

If X is complete and F is continuous, then $\lim x_n \in zer\ F$.

If X is **compact** and T is continuous a modulus of regularity always exists:

Proposition

If F is continuous, X is compact and $zerF \neq \emptyset$, then F has a modulus of regularity.

Noncomputability of moduli of metric regularity

Proposition (K./López-Acedo/Nicolae Israel J. Math. 2019)

There exists a **computable firmly nonexpansive** *mapping* $T : [0,1] \to [0,1]$ *which has* **no computable modulus** *of metric regularity* ϕ *w.r.t.* $Fix(T)$.

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This can be recasted in terms of reverse mathematics:

Proposition (K. Computability 2019)

Over RCA_0, the statement that every continuous function $T : [0,1] \to [0,1]$ has modulus of metric regularity is equivalent to ACA.

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Example of a modulus of regularity

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However there are important cases where this is true!

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Closed convex $C_1, C_2 \subseteq \mathbb{R}^n$: consider Douglas-Rachford operator

$$T_{C_1,C_2} := \frac{1}{2}(I + R_{C_2} R_{C_1}), \text{ where } R_{C_i} := 2P_{C_i} - I.$$

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Borwein/Li/Tam SIAM 2017: if C_1, C_2 are convex semialgebraic sets in $\mathbb{R}^n$ with $0 \in C_1 \cap C_2$ which can be described by polynomials on $\mathbb{R}^n$ of degree > 1 then for given $r > 0$ T_{C_1,C_2} has modulus of regularity (w.r.t. $Fix(T_{C_1,C_2})$ on $B(0,r)$)

$$\omega(\varepsilon) := (\varepsilon/\mu)^\gamma$$

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for all $x \in B_b(0)$ for suitable $\mu > 0, \gamma > 1$. There interesting connections to **o-minimality** which gave rise to **'tame optimization'** (Bolte, Daniilidis, Lewis, Ioffe,...), see e.g.

A.D. Ioffe: In invitation to tame optimization. Siam J. Optimiz. 2009.

Applications in Nonconvex Optimization

The monotonicity of ∂f is due to the convexity assumption on f.

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Definition (Bauschke/Moursi/Wang 2020; Combettes/Pennanen 2004)

Let $\rho \in \mathbb{R}$. $A : H \to 2^H$ is called ρ-comonotone if

$$\forall (x,u),(y,v) \in gr(A)\,(\langle x-y, u-v\rangle \geq \rho \|u-v\|^2).$$

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Recently (arXiv Oct.2020), Diakonikolas/Daskalakis/Jordan considered this and even more general forms in the context of nonconvex-nonconcave min-max optimization and machine learning!

Uniform strong nonexpansivity of families of functions

Our proof mining of convergence results on the PPA and the HPPA show that these results essentially only need use (though implicitly) that $(J_{\gamma_n}A)$ has a **common modulus of strong nonexpansivity** (SNE-modulus):

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Definition (Bruck/Reich 1977, K. 2016)

$C \subseteq X$ subset of some Banach space X. $T : C \to X$ is **strongly nonexpansive** with **SNE-modulus** $\omega : (0,\infty)^2 \to (0,\infty)$ if

$\forall b, \varepsilon > 0\, \forall x, y \in C$

$\|x-y\| \leq b \wedge \|x-y\| - \|Tx - Ty\| < \omega(b,\varepsilon) \to \|(x-y)-(Tx-Ty)\| < \varepsilon.$

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Proposition (K. Israel J. Math. 2016)

If X is uniformly convex with modulus η and $T : C \to X$ is firmly nonexpansive, then T is SNE with modulus

$$\omega_\eta(b,\varepsilon) = \frac{1}{4}\eta(\varepsilon/b)\cdot\varepsilon.$$

In **Hilbert space** $\omega(b,\varepsilon) := \frac{1}{16b}\varepsilon^2$.

Proposition (K. Optimization Letters 2021)

Let H be a real Hilbert space and $(\gamma_n) \subset (0, \infty), \gamma > 0$ be such that $\gamma_n \geq \gamma > 0$ for all $n \in \mathbb{N}$. Let $\rho \in (-\frac{\gamma}{2}, 0]$ and $A \subseteq H \times H$ be ρ-comonotone.

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Then $J_{\gamma_n A} : R(I + \gamma_n A) \to D(A)$ is strongly nonexpansive with **common SNE-modulus**

$$\omega_\alpha(b, \varepsilon) := \frac{1-\alpha}{4b\alpha} \cdot \varepsilon^2, \text{ where } \alpha := \frac{1}{2((\rho/\gamma)+1)} \in (0,1).$$

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SNE-modulus for averaged maps in Hilbert space: Sipos 2020.

Results on PPA and HPPA in Hilbert space for ρ-comonotone operators

- Rate of metastability for the convergence of the PPA in the boundedly compact case.

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Results on PPA and HPPA in Hilbert space for ρ-comonotone operators

- Rate of metastability for the convergence of the PPA in the boundedly compact case.
- Rates of convergence of the PPA in the general case if one has a modulus of regularity.
- Rate of metastability for the convergence of HPPA in the general case together with quantitative information of the limit being a zero of A.

Theorem (K. Optimization Letters 2021)

Let $A \subseteq H \times H$ be ρ-comonotone, $(\gamma_n), \gamma, \rho$ as before. Assume that $\overline{D(A)} \subseteq \bigcap_{n=0}^{\infty} R(I + \gamma_n A)$ is boundedly compact and $x_0 \in \overline{D(A)}$.

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$$(*) \left\{ \begin{array}{l} \forall k \in \mathbb{N}\, \forall g \in \mathbb{N}^{\mathbb{N}}\, \exists n \leq \Psi(k, g, \beta)\, \forall i, j \in [n, n + g(n)] \\ \left(\|x_i - x_j\| \leq \frac{1}{k+1} \text{ and } x_i \in \tilde{F}_k \right), \end{array} \right.$$

where

$$\tilde{F}_k := \bigcap_{i \leq k} \left\{ x \in \overline{D(A)} \, : \, \|x - J_{\gamma_i A} x\| \leq \frac{1}{k+1} \right\}$$

and β is a modulus of total boundedness for $\overline{D(A)} \cap \overline{B}(0, M)$, where $\overline{B}(0, M) := \{x \in H : \|x\| \leq M\}$, with $M \geq b + \|p\|$ and $b \geq \|x_0 - p\|$ for some $p \in zer\, A$.

Here $\Psi(k,g,\beta) := \Psi_0(P,k_0,g)$, with

$$\begin{cases} \Psi_0(0,k_0,g) := 0 \\ \Psi_0(n+1,k_0,g) := \Phi\left(\chi^M_{k,g}\left(\Psi_0(n,k_0,g), 4k_0+3\right)\right), \end{cases}$$

and

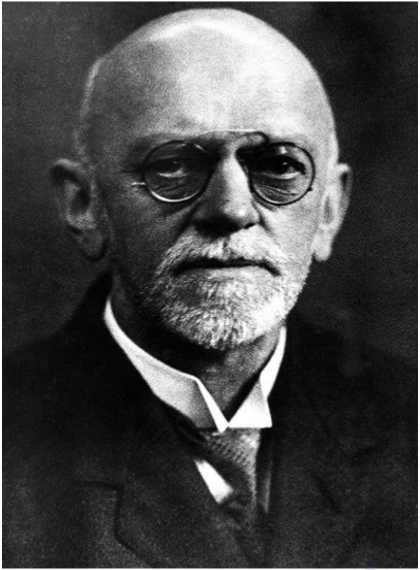
$$\chi_{k,g}(n,r) := \max\{2k+1, \chi(n,g(n),r)\},\ \chi^M_{k,g}(n,r) := \max_{i\le n}\{\chi_{k,g}(i,r)\},$$
$$P := \beta\left(4k_0+3\right),\ k_0 = 2k+1\ \ \chi(n,m,r) := \max\{n+m-1, m(r+1)\}$$
$$\Phi(k) := \left\lceil \frac{b}{\omega_\alpha(b,((k+1)C_k)^{-1})} \right\rceil + 1,\ \ C_k \ge 2 + \frac{\gamma_i}{\gamma} \text{ for all } i \le k.$$

Theorem (K. Optimization Letters 2021)

Let A and $(\gamma_n), \gamma, \rho, b$ be as above and assume that $\overline{D(A)} \subseteq \bigcap_{n=0}^{\infty} R(I + \gamma_n A)$. If A has a modulus ϕ of regularity (suitable adapted for the set-valued case) w.r.t *zer* A and $\overline{B}(p, b)$, then **without compactness assumption** (x_n) converges to a zero $z := \lim x_n$ of A with rate of convergence

$$\boldsymbol{\xi(\varepsilon, \gamma, b) := \left\lceil \frac{b}{\omega_\alpha\left(b, \phi(\varepsilon/2) \cdot \gamma\right)} \right\rceil + 2.}$$

David Hilbert (1862-1943)



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