Comparing Π^1_2-Problems in Computability Theory and Reverse Mathematics

Denis R. Hirschfeldt

Based on joint work with Carl G. Jockusch, Jr. and with Damir D. Dzhafarov and Sarah C. Reitzes

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(HJ) On Notions of Computability-Theoretic Reduction between Π^1_2 Principles,
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From now on, P and Q will denote such Π^1_2-problems.

Reverse Mathematics

In reverse mathematics, we compare the relative strength of principles over a weak base theory.

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Full second-order arithmetic consists of the basic axioms for a discrete ordered commutative semiring,

set induction:

$$\forall X \left[(0 \in X \wedge \forall n [n \in X \rightarrow n+1 \in X]) \rightarrow \forall n [n \in X]\right],$$

and full comprehension:

$$\exists X \forall n [n \in X \leftrightarrow \varphi(n)]$$

for all φ s.t. X is not free in φ.

The usual base theory RCA_0 consists of the basic axioms,

Δ^0_1-comprehension:

$$\forall n\,[\varphi(n) \leftrightarrow \psi(n)] \rightarrow \exists X\,\forall n\,[n \in X \leftrightarrow \varphi(n)]$$

for all φ, ψ s.t. φ is Σ^0_1 and ψ is Π^0_1, and X is not free in φ,

and Σ^0_1-induction:

$$(\varphi(0) \wedge \forall n\,[\varphi(n) \rightarrow \varphi(n+1)]) \rightarrow \forall n\,\varphi(n)$$

for all Σ^0_1 formulas φ.

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Replacing Σ^0_1-induction by set induction (and adding the totality of exponentiation) yields RCA^*_0.

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Theorem (Folklore / Wang). If

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Jockusch gave a proof using the Compactness Theorem.

A model $\mathcal{M}$ in the language of second-order arithmetic consists of a first-order part $\mathcal{N}$ and a second-order part $\mathcal{S} \subseteq 2^{|\mathcal{N}|}$.

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Write $P \leqslant_\omega Q$ if every ω-model of $\mathsf{RCA}_0 + Q$ is a model of P.

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Write $P \leqslant_\omega Q$ if every ω-model of $\mathsf{RCA}_0 + Q$ is a model of P.

If $\mathsf{RCA}_0 \vdash Q \to P$ then $P \leqslant_\omega Q$, but not always vice-versa.

An Example: Versions of Ramsey’s Theorem

$[X]^n$ is the set of n-element subsets of X.

A k-coloring of $[X]^n$ is a map $c : [X]^n \to k$. (We assume $k \geqslant 2$.)

$H \subseteq X$ is homogeneous for c if $|c([H]^n)| = 1$.

RT^n_k: Every k-coloring of $[\mathbb{N}]^n$ has an infinite homogeneous set.

RT^n: $\forall k \, \mathsf{RT}^n_k$.

RT: $\forall n \, \mathsf{RT}^n$.

Theorem (Jockusch). Let $n \geqslant 2$.

Every computable instance of RT^n has a Π^0_n solution.

There is a computable instance of RT^n_2 with no Σ^0_n solution.

There is a computable instance of RT^n_2 s.t. every solution computes $\emptyset^{(n-2)}$, the $(n-2)$nd jump.

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Theorem (Cholak, Jockusch, and Slaman). $\mathsf{RCA}_0 + \mathsf{RT}^2_2 \nvdash \mathsf{RT}^2$.

$\mathsf{RCA}_0 \vdash \mathsf{RT}^1_k$ for each k, so RT^1 is true in every ω-model of RCA_0, but:

Theorem (Hirst). $\mathsf{RCA}_0 \nvdash \mathsf{RT}^1$.

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Recall that if

$$\mathsf{ACA}_0 \vdash \forall X\,[\varphi(X) \ \to\ \exists Y\,\psi(X, Y)]$$

where φ and ψ are arithmetic, then there is an $n \in \omega$ s.t.

$$\mathsf{ACA}_0 \vdash \forall X\,[\varphi(X) \ \to\ \exists Y \in \Sigma^{0,X}_n\ \psi(X, Y)].$$

Corollary. $\mathsf{RT} \leqslant_\omega \mathsf{ACA}_0$ but $\mathsf{ACA}_0 \nvdash \mathsf{RT}$.

Computability-Theoretic Reductions

P is computably reducible to Q, written $P \leqslant_c Q$, if

for every instance X of P,

there is an X-computable instance $\widehat{X}$ of Q s.t.,

for every solution $\widehat{Y}$ to $\widehat{X}$,

there is an $X \oplus \widehat{Y}$-computable solution to X.

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Problems: P Q

Instances:

Solutions:

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	$\downarrow$		
Solutions:	Y	$\longleftarrow$	$\widehat{Y}$

P is Weihrauch reducible to Q, written $P \leqslant_{\mathrm{W}} Q$, if

there are Turing functionals Φ and Ψ s.t.,

for every instance X of P,

Φ^X is an instance of Q,

and for every solution $\widehat{Y}$ to Φ^X,

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Theorem (Jockusch). Let $n \geqslant 2$.

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Corollary. $\mathsf{RT}^1_k <_{\mathrm{c}} \mathsf{RT}^2_k <_{\mathrm{c}} \mathsf{RT}^3_k <_{\mathrm{c}} \mathsf{RT}^4_k <_{\mathrm{c}} \cdots$ and similarly for $<_{\mathrm{W}}$.

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Theorem (HJ / Brattka and Rakotoniaina / Patey).
$\mathsf{RT}^n_2 <_{\mathrm{W}} \mathsf{RT}^n_3 <_{\mathrm{W}} \mathsf{RT}^n_4 <_{\mathrm{W}} \cdots$.

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Theorem (Patey). $\mathsf{RT}^n_2 <_{\mathrm{c}} \mathsf{RT}^n_3 <_{\mathrm{c}} \mathsf{RT}^n_4 <_{\mathrm{c}} \cdots$ for $n \geqslant 2$.

Reduction Games

We describe two-player reduction games for P and Q defined in (HJ).

Player 1 will play a P-instance X_0.

Player 2 will try to obtain a solution to X_0 by asking Player 1 to solve various Q-instances.

If Player 2 ever plays such a solution, it wins, and the game ends.

If the game never ends then Player 1 wins.

If a player cannot make a move, the opponent wins.

Reduction Games over ω

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First Move:

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Player 2: Either an $(X_0 \oplus X_1)$-computable solution to X_0, or an $(X_0 \oplus X_1)$-computable Q-instance Y_2.

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Third Move:

Player 1: A solution X_2 to Y_2.

Player 2: Either an $(X_0 \oplus X_1 \oplus X_2)$-computable solution to X_0, or an $(X_0 \oplus X_1 \oplus X_2)$-computable Q-instance Y_3.

$\vdots$

Theorem (HJ). If $P \leqslant_\omega Q$ then Player 2 has a winning strategy for $G(Q \to P)$. Otherwise, Player 1 has a winning strategy for $G(Q \to P)$.

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Neumann and Pauly independently gave an equivalent definition using an operator $^\diamond$ on Weihrauch degrees.

Write $P \leqslant^n_\omega Q$ if Player 2 has a winning strategy for $G(Q \to P)$ that wins in at most $n+1$ many moves, and similarly for gW.

Theorem (HJ). Let $n \geqslant 3$ and $j \geqslant 1$, and let m be s.t.

$$n + (j-1)(n-2) < m \leqslant n + j(n-2).$$

Then

$$\mathrm{RT}_k^m \leqslant_{\mathrm{gW}}^{j+1} \mathrm{RT}_k^n \quad \text{but} \quad \mathrm{RT}_k^m \nleqslant_{\omega}^{j} \mathrm{RT}_k^n.$$

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So $\mathsf{RT} \nleqslant^{j}_{\omega} \mathsf{RT}^3_2$ for all j, even though $\mathsf{RT} \leqslant_{\omega} \mathsf{RT}^3_2$.

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So RT $\nleqslant^{j}_{\omega}$ RT^3_2 for all j, even though RT $\leqslant_{\omega}$ RT^3_2.

Theorem (HJ). Let $j \geqslant 2$ and $j^m < k \leqslant j^{m+1}$. Then $\mathsf{RT}^1_k \leqslant^{m+1}_{\mathrm{gW}} \mathsf{RT}^1_j$ but $\mathsf{RT}^1_k \nleqslant^{m}_{\mathrm{gW}} \mathsf{RT}^1_j$.

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So $\mathrm{RT} \nleqslant^j_\omega \mathrm{RT}^3_2$ for all j, even though $\mathrm{RT} \leqslant_\omega \mathrm{RT}^3_2$.

Theorem (HJ). Let $j \geqslant 2$ and $j^m < k \leqslant j^{m+1}$. Then $\mathrm{RT}^1_k \leqslant^{m+1}_{\mathrm{gW}} \mathrm{RT}^1_j$ but $\mathrm{RT}^1_k \nleqslant^m_{\mathrm{gW}} \mathrm{RT}^1_j$.

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Theorem (HJ). Let $n \geqslant 3$ and $j \geqslant 1$, and let m be s.t.

$$n + (j-1)(n-2) < m \leqslant n + j(n-2).$$

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Patey characterized the least m s.t. $\mathrm{RT}^n_k \leqslant^m_\omega \mathrm{RT}^n_j$ for $n \geqslant 2$ and $j < k$.

For $n \geqslant 3$, this m is always 2. For $n = 2$ it is more complicated and goes to infinity as k increases.

Reduction Games over Non-ω Models

(HJ) also considered games over models of RCA_0.

More generally, we can work over models of a consistent extension Γ of Δ^0_1-comprehension by Π^1_1 formulas that proves the existence of a universal Σ^0_1 formula.

Let $\mathcal{N}$ be a model in the language of first-order arithmetic.

The notions of instance and solution of a Π^1_2-problem still make sense over $\mathcal{N}$.

For $X_0, \dots, X_n \subseteq |\mathcal{N}|$, let $\mathcal{N}[X_0, \dots, X_n] = (\mathcal{N}, S)$ where S consists of all subsets of $|\mathcal{N}|$ that are Δ^0_1-definable from parameters in $|\mathcal{N}| \cup \{X_0, \dots, X_n\}$.

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First Move:

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Player 1: A solution X_1 to Y_1 in S.

Player 2: Either a solution to X_0 in $\mathcal{N}[X_0, X_1]$, or a Q-instance $Y_2 \in \mathcal{N}[X_0, X_1]$.

Third Move:

Player 1: A solution X_2 to Y_2 in S.

Player 2: Either a solution to X_1 in $\mathcal{N}[X_0, X_1, X_2]$, or a Q-instance $Y_3 \in \mathcal{N}[X_0, X_1, X_2]$.

⋮

Proposition (HJ / DHR). If $\Gamma \vdash Q \to P$ then Player 2 has a winning strategy for $G^\Gamma(Q \to P)$. Otherwise, Player 1 does.

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Using computability-theoretic results of Patey, we have:

Corollary (DHR / Slaman and Yokoyama). Let Γ be RCA_0 together with all Π^1_1 formulas true in the natural numbers. Then $\Gamma + \mathsf{RT}^2_2 \nvdash \mathsf{RT}^2$.

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We can define $\Gamma \vdash^n Q \to P$ if Player 2 has a winning strategy for $G^\Gamma(Q \to P)$ that wins in at most $n + 1$ many moves.

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P is generalized Weihrauch reducible to Q over Γ, written $P \leqslant^\Gamma_{\mathrm{gW}} Q$, if Player 2 has a computable (i.e., Δ^0_1) winning strategy for $G^\Gamma(Q \to P)$.

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Comparing Π^1_2-Problems in Computability Theory and Reverse Mathematics

Denis R. Hirschfeldt

Based on joint work with Carl G. Jockusch, Jr. and with Damir D. Dzhafarov and Sarah C. Reitzes

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