Term Rewriting

Basic Concepts, Tools, and Applications

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Outline

Term Rewriting

- abstract, Turing complete model of computation
- can model programs, simplification systems, other change processes
- rewriting techniques can serve as toolbox to analyze properties

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Crash course: Term rewriting in a nutshell

- analysis of termination, confluence, completion, complexity
- applications: fun/profit examples

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Rewrite Tools Developed @ CL Group Innsbruck

$\mathsf{T_TT_2}$, $\mathsf{T_CT}$, CSI, mkbtt, KBCV, mædmax, FORT, ProTeM, CeTA, ConCon, MiniSmt, AutoStrat, Ctrl, . . .

Teaching Example 1: Virus Genes

A COVID medication research team wants to develop a medicine that transforms the DNA of SARS-CoV-2:

$$\text{TAGCTAGCTAGCT} \tag{1}$$

into the DNA of a known and relatively benign influenza virus:

$$\text{CTGACTGACT} \tag{2}$$

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Techniques exist to perform the following DNA transformations:

$$\text{TCAT} \leftrightarrow \text{T} \quad \text{GAG} \leftrightarrow \text{AG} \quad \text{CTC} \leftrightarrow \text{TC} \quad \text{AGTA} \leftrightarrow \text{A} \quad \text{TAT} \leftrightarrow \text{CT}$$

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- are the known transformations sufficient to transform (1) into (2)?
- is it possible that the mad cow disease DNA is created in this process?

Teaching Example 2: Bean Game

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● ● ○ ● ○ ● ● ○ ○ ● ○ ○ ● ● ○

player 1   ○ ○ ● ○ ● ● ○ ○ ● ○ ○ ● ● ○

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player 1	○ ○ ● ○ ● ● ○ ○ ● ○ ○ ● ● ○
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player 1	$\circ\ \circ\ \bullet\ \circ\ \bullet\ \bullet\ \circ\ \circ\ \bullet\ \circ\ \circ\ \bullet\ \bullet\ \circ$
player 2	$\bullet\ \circ\ \bullet\ \circ\ \bullet\ \bullet\ \circ\ \circ\ \bullet\ \circ\ \circ\ \bullet\ \circ$
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Questions

- does the game terminate for every initial state?

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$$\bullet\,\bullet\,\circ\,\bullet\,\circ\,\bullet\,\bullet\,\circ\,\circ\,\bullet\,\circ\,\circ\,\bullet\,\bullet\,\circ$$

Questions

- does the game terminate for every initial state?
- which strategies are winning strategies for player 2?

Research Example 3: Functional Programs

TRS $\mathcal{R}$ models sieve of Eratosthenes to enumerate prime numbers:

$$
\begin{aligned}
\mathsf{primes} &\to \mathsf{sieve}(\mathsf{from}(\mathsf{s}(\mathsf{s}(0)))) & \mathsf{sieve}(0 : y) &\to \mathsf{sieve}(y)\\
\mathsf{from}(x) &\to x : \mathsf{from}(\mathsf{s}(x)) & \mathsf{sieve}(\mathsf{s}(x) : y) &\to \mathsf{s}(x) : \mathsf{sieve}(\mathsf{filter}(x, y, x))\\
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\end{aligned}
$$

Research Example 3: Functional Programs

TRS $\mathcal{R}$ models sieve of Eratosthenes to enumerate prime numbers:

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Questions About (Functional) Programs

- is the given program terminating?

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Questions About (Functional) Programs

- is the given program terminating?
- if yes, what is its worst-case computational complexity?
- are results unique?

Practice Example 4: LLVM Expression Simplification

```
int foo(int z) {
  int x = 4*(z|101);
  return -256^x;
}
```

$\overset{*}{\dashrightarrow}$

```
01011111010101011011010
11111111101110100001 0
01001110010101100101 0
11111011101110100101 0
```

- LLVM provides widely used compilation toolchain

Practice Example 4: LLVM Expression Simplification

```
int foo(int z) {
  int x = 4*(z|101);
  return -256^x;
}
```

*

```
define i32 @foo(i32) #0 {
  %2 = or i32 %0, 101
  %3 = mul nsw i32 4, %2
  %4 = xor i32 -256, %3
  ret i32 %4
}
```

Instcombine

```
define i32 @foo(i32) #0 {
  %2 = shl i32 %0, 2
  %3 = or i32 %2, 404
  %4 = xor i32 %3, -256
  ret i32 %4
}
```

*

```
01011111010101011011010
11111111101110100001
0100111001010110010 10
1111101110111010010 10
```

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$\xrightarrow{\text{Instcombine}}$

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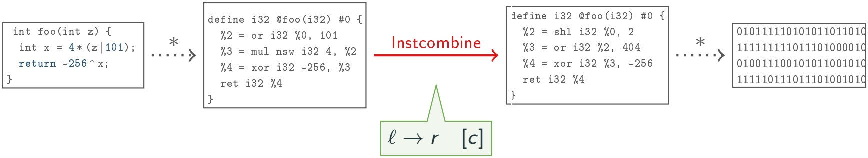
Practice Example 4: LLVM Expression Simplification



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- applies "rewrite rules" with SMT side constraints

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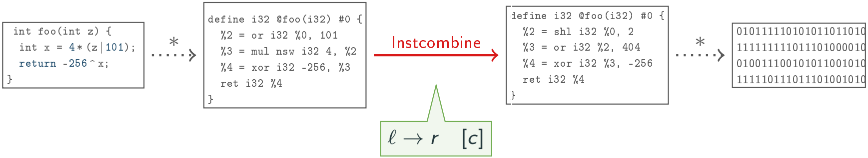

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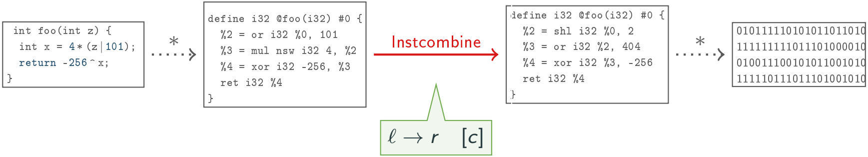

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- applies "rewrite rules" with SMT side constraints
- optimization set is community maintained, interference unclear
- termination is crucial

Course Content

Day 1

abstract rewriting, properties of abstract rewrite systems, Newman's Lemma, term rewriting

Day 2

termination, polynomial interpretations, lexicographic path order, Knuth-Bendix order, derivational complexity

Day 3

critical pairs, confluence, orthogonality, Knuth-Bendix completion

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Outline

Motivating Examples

Abstract Rewriting

Term Rewriting

Definitions (Abstract Rewrite System)

- abstract rewrite system (ARS) consists of
  - carrier set A
  - binary relation $\to$ on A

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$$\text{ARS } \mathcal{A} = (A, \to)$$

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- $\to\; = \left\{ \begin{array}{l} (\mathsf{a},\mathsf{b}),\ (\mathsf{b},\mathsf{d}),\ (\mathsf{b},\mathsf{i}),\ (\mathsf{c},\mathsf{a}),\ (\mathsf{c},\mathsf{e}), \\ (\mathsf{d},\mathsf{c}),\ (\mathsf{d},\mathsf{f}),\ (\mathsf{d},\mathsf{i}),\ (\mathsf{d},\mathsf{j}),\ (\mathsf{e},\mathsf{f}), \\ (\mathsf{e},\mathsf{g}),\ (\mathsf{f},\mathsf{h}),\ (\mathsf{f},\mathsf{j}),\ (\mathsf{f},\mathsf{k}),\ (\mathsf{g},\mathsf{h}), \\ (\mathsf{h},\mathsf{k}),\ (\mathsf{j},\mathsf{j}) \end{array} \right\}$

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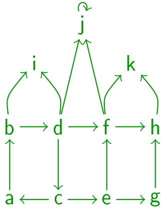

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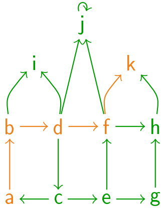

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- rewrite sequence
  - finite $\quad \mathsf{a} \to \mathsf{b} \to \mathsf{d} \to \mathsf{f} \to \mathsf{k}$

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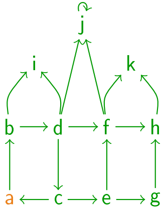

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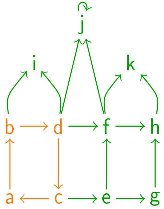

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  - empty $\quad \mathsf{a}$
  - infinite $\quad \mathsf{a} \to \mathsf{b} \to \mathsf{d} \to \mathsf{c} \to \mathsf{a} \to \mathsf{b} \to \mathsf{d} \to \mathsf{c} \to \cdots$

Definitions (Derived Relations)

- $\rightarrow^{+}$ transitive closure of $\rightarrow$

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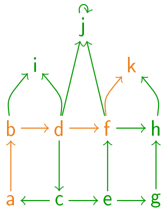
Example



- $a \to^+ k$

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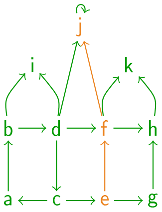

- $a \to^+ k$ $e \to^+ j$

Definitions (Derived Relations)

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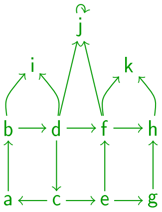
Example



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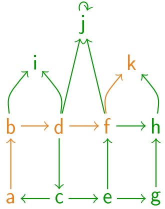

- $a \rightarrow^{+} k$ $e \rightarrow^{+} j$
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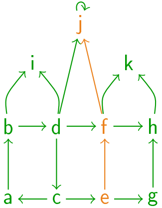

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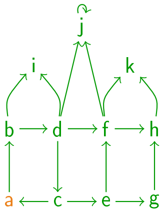

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- $a \to^{*} k$ $e \to^{*} j$ $a \to^{*} a$

Definitions (Derived Relations)

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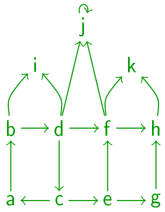
- $\leftrightarrow^*$ conversion (equivalence relation generated by $\to$)

Example



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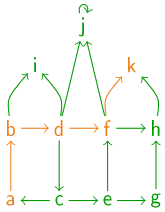

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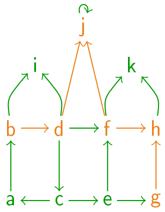

- $a \rightarrow^{+} k$ $e \rightarrow^{+} j$
- $a \rightarrow^{*} k$ $e \rightarrow^{*} j$ $a \rightarrow^{*} a$
- $a \leftrightarrow^{*} k$ $b \leftrightarrow^{*} g$

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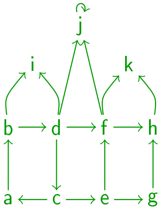
- $\downarrow$ joinability $\downarrow = \to^* \cdot {}^*\!\leftarrow$

Example



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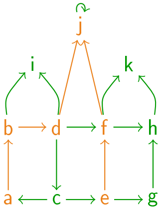

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Example

$$
\begin{tikzcd}
& & & j \arrow[loop above] & & & \\
& i & & & & k & \\
b \arrow[rr] \arrow[ur, bend left] & & d \arrow[rr] \arrow[ul, bend right] \arrow[uur] \arrow[dd] & & f \arrow[rr] \arrow[uul] \arrow[ur, bend left] & & h \arrow[ul, bend right] \\
& & & & & & \\
a \arrow[uu] & & c \arrow[ll] \arrow[rr] & & e \arrow[uu] \arrow[rr] & & g \arrow[uu]
\end{tikzcd}
$$

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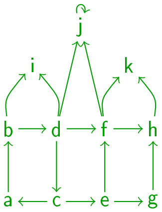
Definitions (Normal Forms)

- normal form is element x such that $x \not\to y$ for all y

Example



- $\mathsf{a} \to^+ \mathsf{k}$ $\mathsf{e} \to^+ \mathsf{j}$
- $\mathsf{a} \to^* \mathsf{k}$ $\mathsf{e} \to^* \mathsf{j}$ $\mathsf{a} \to^* \mathsf{a}$
- $\mathsf{a} \leftrightarrow^* \mathsf{k}$ $\mathsf{b} \leftrightarrow^* \mathsf{g}$
- $\mathsf{a} \downarrow \mathsf{e}$ $\mathsf{e} \downarrow \mathsf{g}$

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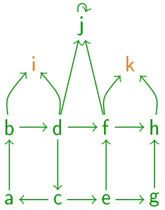

- a $\to^+$ k e $\to^+$ j
- a $\to^*$ k e $\to^*$ j a $\to^*$ a
- a $\leftrightarrow^*$ k b $\leftrightarrow^*$ g
- a $\downarrow$ e e $\downarrow$ g
- normal forms i, k

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Definitions (Normal Forms)

- normal form is element x such that $x \not\rightarrow y$ for all y
- $x \rightarrow^! y$ if $x \rightarrow^* y$ for normal form y (x has normal form y)

Example

$$\begin{tikzcd}
 & & & \mathsf{j} \arrow[loop above] & & & \\
 & \mathsf{i} & & & & \mathsf{k} & \\
\mathsf{b} \arrow[ur, bend left] \arrow[rr] & & \mathsf{d} \arrow[ul] \arrow[uur] \arrow[rr] \arrow[d] & & \mathsf{f} \arrow[uul] \arrow[ur, bend left] \arrow[rr] & & \mathsf{h} \arrow[ul] \\
\mathsf{a} \arrow[u] & & \mathsf{c} \arrow[ll] \arrow[rr] & & \mathsf{e} \arrow[u] \arrow[rr] & & \mathsf{g} \arrow[u]
\end{tikzcd}$$

- $\mathsf{a} \rightarrow^+ \mathsf{k}$ $\mathsf{e} \rightarrow^+ \mathsf{j}$
- $\mathsf{a} \rightarrow^* \mathsf{k}$ $\mathsf{e} \rightarrow^* \mathsf{j}$ $\mathsf{a} \rightarrow^* \mathsf{a}$
- $\mathsf{a} \leftrightarrow^* \mathsf{k}$ $\mathsf{b} \leftrightarrow^* \mathsf{g}$
- $\mathsf{a} \downarrow \mathsf{e}$ $\mathsf{e} \downarrow \mathsf{g}$
- normal forms i, k

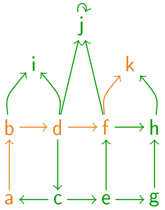
Definitions (Derived Relations)

- $\to^+$ transitive closure of $\to$
- $\to^*$ transitive and reflexive closure of $\to$
- $\leftrightarrow^*$ conversion (equivalence relation generated by $\to$)
- $\downarrow$ joinability $\downarrow = \to^* \cdot {}^*\!\!\leftarrow$

Definitions (Normal Forms)

- normal form is element x such that $x \not\to y$ for all y
- $x \to^! y$ if $x \to^* y$ for normal form y (x has normal form y)

Example



- a $\to^+$ k e $\to^+$ j
- a $\to^*$ k e $\to^*$ j a $\to^*$ a
- a $\leftrightarrow^*$ k b $\leftrightarrow^*$ g
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- if $x \to^* y$ then x rewrites to y and y is reduct of x
- if $x \leftrightarrow^* y$ then x and y are convertible

Definitions

- SN strong normalization
  - no infinite rewrite sequences

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\begin{tikzcd}
 & a \arrow[dl, "*"'] \arrow[dr, "*"] & \\
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\end{tikzcd}
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- WCR local confluence weak Church-Rosser property
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  $\exists\, d$

Definitions

- SN(x)   strong normalization   termination
  - no infinite rewrite sequences starting from $x \in A$
- CR(x)   confluence
  - ${}^*\!\leftarrow x \rightarrow^* \subseteq \downarrow$
  - $\forall$ b, c

$$\begin{tikzcd} & x \arrow[dl, "*"'] \arrow[dr, "*"] & \\ b \arrow[dr, dotted, "*"'] & & c \arrow[dl, dotted, "*"] \\ & d & \end{tikzcd}$$

$\exists\, d$

- WCR(x) local confluence   weak Church-Rosser property
  - $\leftarrow x \rightarrow \subseteq \downarrow$
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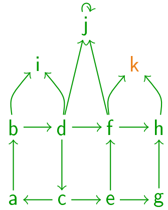
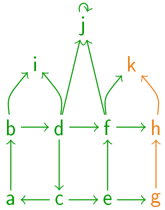
\begin{tikzcd}[column sep=small]
 & & & j \arrow[loop above] & & & \\
 & i & & & & k & \\
b \arrow[ur, bend left] \arrow[rr] & & d \arrow[ul, bend right] \arrow[urr] \arrow[rr] \arrow[dd] & & f \arrow[ul] \arrow[ur, bend left] \arrow[rr] & & h \arrow[ul, bend right] \\
 & & & & & & \\
a \arrow[uu] & & c \arrow[ll] \arrow[rr] & & e \arrow[uu] \arrow[rr] & & g \arrow[uu]
\end{tikzcd}
$$

Example



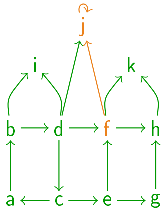
- SN(k)

Example



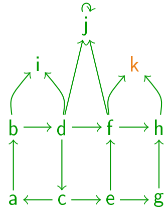
- SN(k) SN(g)

Example



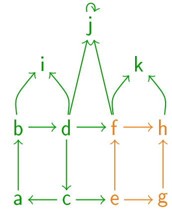
- $\mathrm{SN}(k)$ $\qquad$ $\mathrm{SN}(g)$ $\qquad$ $\neg\mathrm{SN}(f)$

Example



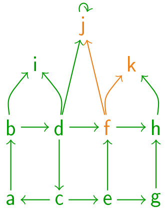
- $\mathsf{SN}(k)$ $\mathsf{SN}(g)$ $\neg\mathsf{SN}(f)$
- $\mathsf{WCR}(k)$

Example



- $\mathsf{SN}(k)$ $\quad$ $\mathsf{SN}(g)$ $\quad$ $\neg\mathsf{SN}(f)$
- $\mathsf{WCR}(k)$ $\quad$ $\mathsf{WCR}(e)$

Example



- $SN(k)$ $\quad SN(g)$ $\quad \neg SN(f)$
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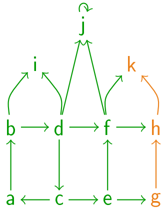

- SN(k) SN(g) ¬SN(f)
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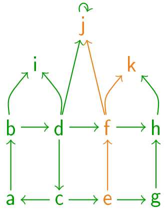
- CR(g)

Example



- SN(k)   SN(g) $\neg$SN(f)
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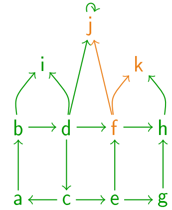

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Example

$$\begin{tikzcd}
 & & & \mathsf{j} \arrow[loop above] & & & \\
 & \mathsf{i} & & & & \mathsf{k} & \\
\mathsf{b} \arrow[ur] \arrow[rr] & & \mathsf{d} \arrow[ul] \arrow[uur] \arrow[rr] \arrow[d] & & \mathsf{f} \arrow[uul] \arrow[ur] \arrow[rr] & & \mathsf{h} \arrow[ul] \\
\mathsf{a} \arrow[u] & & \mathsf{c} \arrow[ll] \arrow[rr] & & \mathsf{e} \arrow[u] \arrow[rr] & & \mathsf{g} \arrow[u]
\end{tikzcd}$$

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 & \mathsf{i} & & \mathsf{j} \arrow[loop above] & & \mathsf{k} & \\
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- $CR(g)$ $\quad$ $\neg CR(e)$ $\quad$ $\neg CR(f)$
- $\neg SN(\mathcal{A})$ $\quad$ $\neg WCR(\mathcal{A})$ $\quad$ $\neg CR(\mathcal{A})$

Relationships between properties

- $CR \implies WCR$

Example

$$
\begin{tikzcd}
 & & & \mathsf{j} \arrow[loop above] & & & \\
 & \mathsf{i} & & & & \mathsf{k} & \\
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Relationships between properties

- CR $\Longrightarrow$ WCR
- WCR $\not\Longrightarrow$ CR

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\begin{tikzcd}
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- $\mathsf{CR} \implies \mathsf{WCR}$
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$$
\begin{tikzcd}
a & b \arrow[l] \arrow[r, bend left] & c \arrow[l, bend left] \arrow[r] & d
\end{tikzcd}
$$

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 & & & j \arrow[loop above] & & & \\
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- SN & $\mathrm{WCR} \implies \mathrm{CR}$

$$
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a & b \arrow[l] \arrow[r, bend left] & c \arrow[l, bend left] \arrow[r] & d
\end{tikzcd}
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Newman's Lemma

Definitions (Terms)

- signature $\mathcal{F}$ function symbols with arities

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Example

- $\mathcal{F} = \{\,0, \mathsf{s}, +\,\}$ for constant 0 (arity 0), unary s, and binary $+$

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- $\mathcal{V} = \{ x, y, z, \ldots \}$

Definitions (Terms)

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Example

- $\mathcal{T}(\mathcal{F}, \mathcal{V}) = \{\, x,\ y,\ z,\ 0,\ \mathsf{s}(0),\ \mathsf{s}(\mathsf{s}(0)),\ \mathsf{s}(0) + x,\ \mathsf{s}(x) + y,\ \ldots \,\}$

Definitions (Terms)

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Example (Examples)

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Example (Examples)

- $\square \qquad \mathsf{s}(0) + \mathsf{s}(\mathsf{s}(\square)) \qquad \square + x$
- $\square[\,\mathsf{s}(0)\,] = \mathsf{s}(0) \qquad (\square + x)[\,0 + x\,] = (0 + x) + x$

Definitions

- substitution is mapping $\sigma \colon \mathcal{V} \to \mathcal{T}(\mathcal{F}, \mathcal{V})$ such that its domain

$$\mathcal{D}om(\sigma) = \{\, x \in \mathcal{V} \mid \sigma(x) \neq x \,\}$$

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- application of substitution σ to term t

$$t\sigma = \begin{cases} \sigma(t) & \text{if } t \in \mathcal{V} \\ f(t_1\sigma, \ldots, t_n\sigma) & \text{if } t = f(t_1, \ldots, t_n) \end{cases}$$

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Example

$t = x + \mathsf{s}(y + z)$ $\qquad$ $\sigma = \{\, x \mapsto \mathsf{s}(y),\ y \mapsto x + \mathsf{s}(0) \,\}$

Definitions

- substitution is mapping $\sigma\colon \mathcal{V} \to \mathcal{T}(\mathcal{F}, \mathcal{V})$ such that its domain

$$\mathcal{D}om(\sigma) = \{\, x \in \mathcal{V} \mid \sigma(x) \neq x \,\}$$

  is finite

- application of substitution σ to term t

$$t\sigma = \begin{cases} \sigma(t) & \text{if } t \in \mathcal{V} \\ f(t_1\sigma, \dots, t_n\sigma) & \text{if } t = f(t_1, \dots, t_n) \end{cases}$$

Example

$t = x + \mathsf{s}(y + z)$ $\qquad \sigma = \{\, x \mapsto \mathsf{s}(y),\ y \mapsto x + \mathsf{s}(0) \,\}$

$t\sigma = \mathsf{s}(y) + \mathsf{s}((x + \mathsf{s}(0)) + z)$

Definition (Matching Problem)

t is instance of s

instance: terms s, t

question: $\exists$ substitution σ such that $s\sigma = t$?

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Definitions (Term Rewrite System)

- rewrite rule $\ell \to r$ is pair of terms (ℓ, r) such that
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normal form

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- *termination is undecidable for confluent TRSs*

Exercises

1 Complete the following table:

	a	b	c	d	e	f	i	j
SN	✗							
WCR								
CR							✓	

2 Show that Newman's Lemma does not hold element-wise: find an ARS $(A, \to)$ such that $a \in A$, $SN(a)$ and $WCR(a)$ hold, but $\neg CR(a)$.

3 Which of the following matching problems have a solution?
$x + \mathsf{s}(y) \mapsto \mathsf{s}(x) + \mathsf{s}(\mathsf{s}(x))$ $\qquad$ $\mathsf{f}(x, \mathsf{g}(\mathsf{x}, \mathsf{y})) \mapsto \mathsf{f}(\mathsf{f}(z, z), \mathsf{g}(\mathsf{a}, z))$

4 Rewrite the term $\mathsf{s}(\mathsf{s}(0)) \times \mathsf{s}(\mathsf{s}(0))$ to normal form wrt the TRS
$0 + y \to y$ $\quad$ $\mathsf{s}(x) + y \to \mathsf{s}(x + y)$ $\quad$ $0 \times y \to 0$ $\quad$ $\mathsf{s}(x) \times y \to (x \times y) + y$

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1

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 & & & j \arrow[loop above] & & & \\
 & i & & & & k & \\
b \arrow[ur] \arrow[rr] & & d \arrow[ul] \arrow[uur] \arrow[rr] \arrow[d] & & f \arrow[uul] \arrow[ur] \arrow[rr] & & h \arrow[ul] \\
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