First-Order Theorem Proving and Vampire

Laura Kovács

Outline

Inference System

- inference has the form

$$\frac{F_1 \quad \ldots \quad F_n}{G},$$

where $n \geq 0$ and $F_1, \ldots, F_n, G$ are formulas.
- The formula G is called the conclusion of the inference;
- The formulas $F_1, \ldots, F_n$ are called its premises.
- An Inference system $\mathbb{I}$ is a set of inference rules.
- Axiom: inference rule with no premises.

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Derivation, Proof

- Derivation in an inference system $\mathbb{I}$: a tree built from inferences in $\mathbb{I}$.
- If the root of this derivation is E, then we say it is a derivation of E.
- Proof of E: a finite derivation whose leaves are axioms.

Arbitrary First-Order Formulas (recap)

- A first-order signature (vocabulary): function symbols (including constants), predicate symbols. Equality is part of the language.
- A set of variables.
- Terms are buit using variables and function symbols. For example, $f(x) + g(x)$.
- Atoms, or atomic formulas are obtained by applying a predicate symbol to a sequence of terms. For example, $p(a, x)$ or $f(x) + g(x) \geq 2$.
- Formulas: built from atoms using logical connectives $\neg$, $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$ and quantifiers $\forall$, $\exists$. For example, $(\forall x)x = 0 \vee (\exists y)y > x$.

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- Clause: a disjunction $L_1 \vee \ldots \vee L_n$ of literals, where $n \geq 0$.

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- A formula in Clausal Normal Form (CNF): a conjunction of clauses.
- From now on: A clause is ground if it contains no variables.
- If a clause contains variables, we assume that it implicitly universally quantified. That is, we treat $p(x) \vee q(x)$ as $\forall x(p(x) \vee q(x))$.

Binary Resolution Inference System

The binary resolution inference system, denoted by $\mathbb{BR}$ is an inference system on propositional clauses (or ground clauses). It consists of two inference rules:

- Binary resolution, denoted by BR:

$$\frac{p \vee C_1 \quad \neg p \vee C_2}{C_1 \vee C_2} \text{ (BR)}.$$

- Factoring, denoted by Fact:

$$\frac{L \vee L \vee C}{L \vee C} \text{ (Fact)}.$$

Soundness

- An inference is sound if the conclusion of this inference is a logical consequence of its premises.
- An inference system is sound if every inference rule in this system is sound.

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$\mathbb{BR}$ is sound.

Consequence of soundness: let S be a set of clauses. If $\square$ can be derived from S in $\mathbb{BR}$, then S is unsatisfiable.

Example

Consider the following set of clauses

$$\{\neg p \vee \neg q,\ \neg p \vee q,\ p \vee \neg q,\ p \vee q\}.$$

The following derivation derives the empty clause from this set:

$$\cfrac{\cfrac{\cfrac{p \vee q \quad p \vee \neg q}{p \vee p}\ (\mathrm{BR})}{p}\ (\mathrm{Fact}) \qquad \cfrac{\cfrac{\neg p \vee q \quad \neg p \vee \neg q}{\neg p \vee \neg p}\ (\mathrm{BR})}{\neg p}\ (\mathrm{Fact})}{\square}\ (\mathrm{BR})$$

Hence, this set of clauses is unsatisfiable.

Can this be used for checking (un)satisfiability

1. What happens when the empty clause cannot be derived from S?
2. How can one search for possible derivations of the empty clause?

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   Let S be an unsatisfiable set of clauses. Then there exists a derivation of $\square$ from S in $\mathbb{BR}$.

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   Let S be an unsatisfiable set of clauses. Then there exists a derivation of $\square$ from S in $\mathbb{BR}$.
2. We have to formalize search for derivations.

However, before doing this we will introduce a slightly more refined inference system.

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Selection Function

A literal selection function selects literals in a clause.

- If C is non-empty, then at least one literal is selected in C.

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Note: selection function does not have to be a function. It can be any oracle that selects literals.

Binary Resolution with Selection

We introduce a family of inference systems, parametrised by a literal selection function σ.

The binary resolution inference system, denoted by $\mathbb{BR}_\sigma$, consists of two inference rules:

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$$\frac{\underline{p} \lor C_1 \quad \underline{\neg p} \lor C_2}{C_1 \lor C_2} \text{ (BR).}$$

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- Positive factoring, denoted by Fact:

$$\frac{\underline{p} \vee \underline{p} \vee C}{p \vee C} \text{ (Fact)}.$$

Completeness?

Binary resolution with selection may be incomplete, even when factoring is unrestricted (also applied to negative literals).

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Consider this set of clauses:

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(1) & \neg q \vee \underline{r} \\
(2) & \neg p \vee \underline{q} \\
(3) & \neg r \vee \underline{\neg q} \\
(4) & \neg q \vee \underline{\neg p} \\
(5) & \neg p \vee \underline{\neg r} \\
(6) & \neg r \vee \underline{p} \\
(7) & r \vee q \vee \underline{p}
\end{array}$$

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$$

It is unsatisfiable:

$$
\begin{array}{lll}
(8) & q \lor p & (6,7) \\
(9) & q & (2,8) \\
(10) & r & (1,9) \\
(11) & \neg q & (3,10) \\
(12) & \square & (9,11)
\end{array}
$$

However, any inference with selection applied to this set of clauses give either a clause in this set, or a clause containing a clause in this set.

Literal Orderings

Take any well-founded ordering $\succ$ on atoms, that is, an ordering such that there is no infinite decreasing chain of atoms:

$$A_0 \succ A_1 \succ A_2 \succ \cdots$$

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- If $p \succ q$, then $p \succ \neg q$ and $\neg p \succ q$;
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Exercise: prove that the induced ordering on literals is well-founded too.

Orderings and Well-Behaved Selections

Fix an ordering $\succ$. A literal selection function is well-behaved if

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To be well-behaved, we sometimes must select more than one different literal in a clause. Example: $p \vee p$ or $p(x) \vee p(y)$.

Completeness of Binary Resolution with Selection

Binary resolution with selection is complete for every well-behaved selection function.

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$$

A well-behave selection function must satisfy:

1. $r \succ q$, because of (1)
2. $q \succ p$, because of (2)
3. $p \succ r$, because of (6)

There is no ordering that satisfies these conditions.

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How to Establish Unsatisfiability?

Completeness is formulated in terms of derivability of the empty clause $\square$ from a set S_0 of clauses in an inference system $\mathbb{I}$. However, this formulations gives no hint on how to search for such a derivation.

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Idea:

- Take a set of clauses S (the search space), initially $S = S_0$. Repeatedly apply inferences in $\mathbb{I}$ to clauses in S and add their conclusions to S, unless these conclusions are already in S.
- If, at any stage, we obtain $\square$, we terminate and report unsatisfiability of S_0.

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In first-order logic it is often the case that all saturated sets are infinite (due to undecidability), so in practice we can never build a saturated set.

The process of trying to build one is referred to as saturation.

Saturated Set of Clauses

Let $\mathbb{I}$ be an inference system on formulas and S be a set of formulas.

- S is called saturated with respect to $\mathbb{I}$, or simply $\mathbb{I}$-saturated, if for every inference of $\mathbb{I}$ with premises in S, the conclusion of this inference also belongs to S.
- The closure of S with respect to $\mathbb{I}$, or simply $\mathbb{I}$-closure, is the smallest set S' containing S and saturated with respect to $\mathbb{I}$.

Inference Process

Inference process: sequence of sets of formulas $S_0, S_1, \ldots$, denoted by

$$S_0 \Rightarrow S_1 \Rightarrow S_2 \Rightarrow \ldots$$

$(S_i \Rightarrow S_{i+1})$ is a step of this process.

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We say that this step is an $\mathbb{I}$-step if

1. there exists an inference

$$\frac{F_1 \quad \ldots \quad F_n}{F}$$

   in $\mathbb{I}$ such that $\{F_1, \ldots, F_n\} \subseteq S_i$;
2. $S_{i+1} = S_i \cup \{F\}$.

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2. $S_{i+1} = S_i \cup \{F\}$.

An $\mathbb{I}$-inference process is an inference process whose every step is an $\mathbb{I}$-step.

Property

Let $S_0 \Rightarrow S_1 \Rightarrow S_2 \Rightarrow \ldots$ be an $\mathbb{I}$-inference process and a formula F belongs to some S_i. Then S_i is derivable in $\mathbb{I}$ from S_0. In particular, every S_i is a subset of the $\mathbb{I}$-closure of S_0.

Limit of a Process

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Question: does completeness imply that the limit of the process contains the empty clause?

Fairness

Let $S_0 \Rightarrow S_1 \Rightarrow S_2 \Rightarrow \ldots$ be an inference process with the limit S_ω.
The process is called fair if for every $\mathbb{I}$-inference

$$\frac{F_1 \quad \ldots \quad F_n}{F},$$

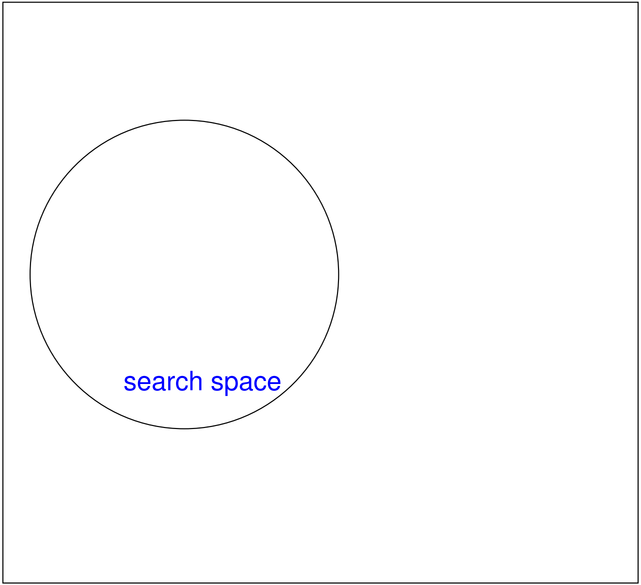
if $\{F_1, \ldots, F_n\} \subseteq S_\omega$, then there exists i such that $F \in S_i$.

Completeness, reformulated

Theorem Let $\mathbb{I}$ be an inference system. The following conditions are equivalent.

1. $\mathbb{I}$ is complete.
2. For every unsatisfiable set of formulas S_0 and any fair $\mathbb{I}$-inference process with the initial set S_0, the limit of this inference process contains $\square$.

Fair Saturation Algorithms: Inference Selection by Clause Selection



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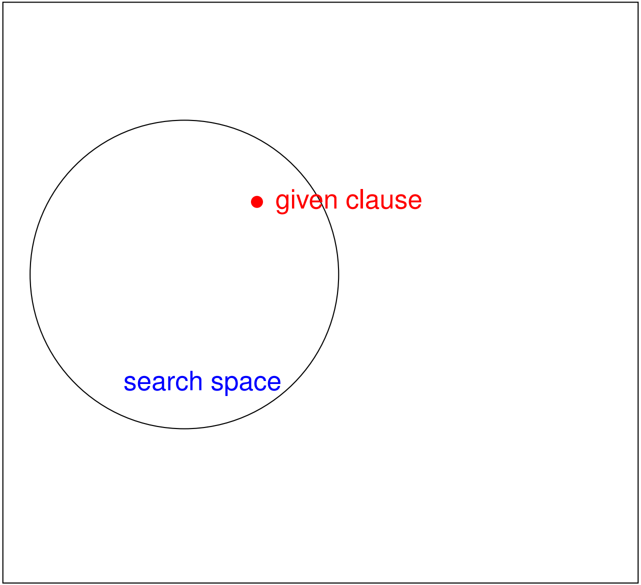

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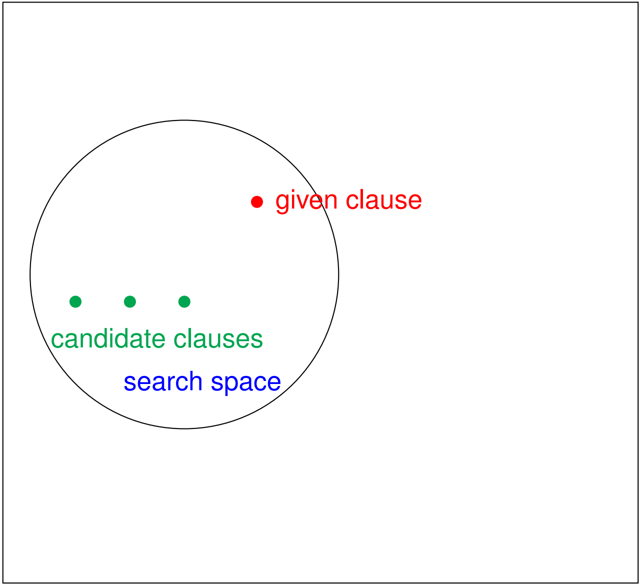

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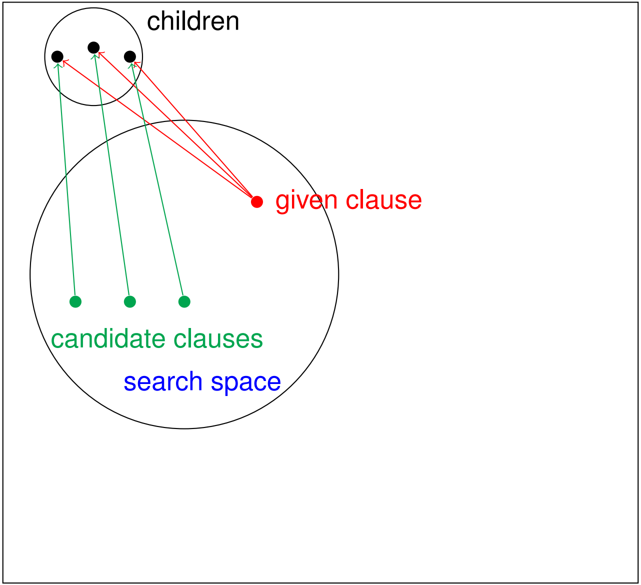

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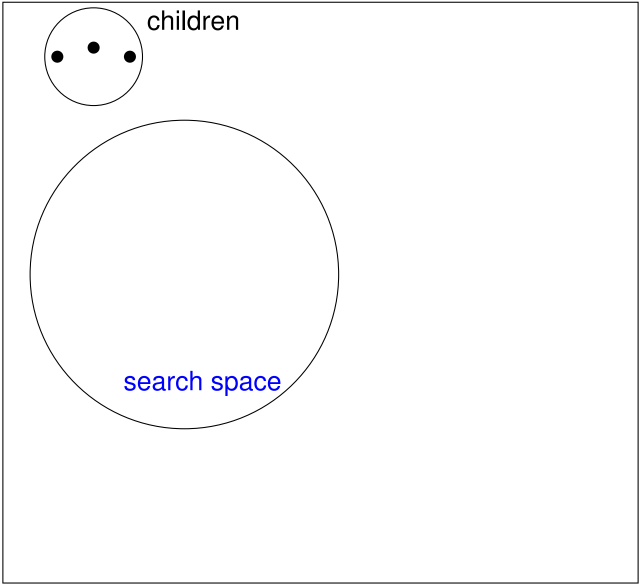

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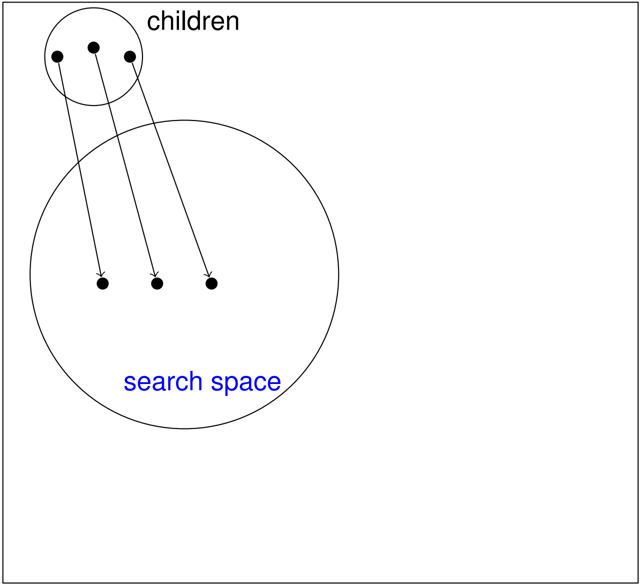

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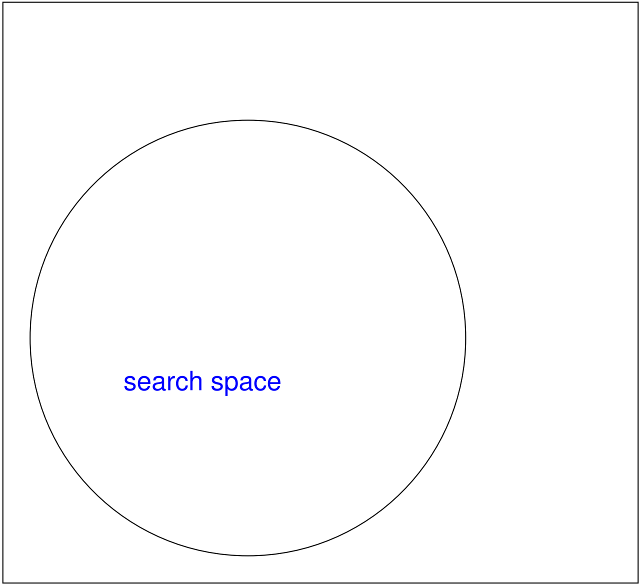

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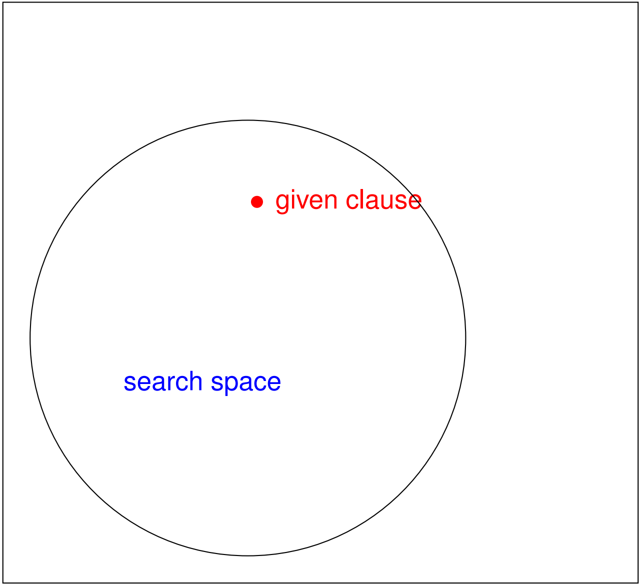

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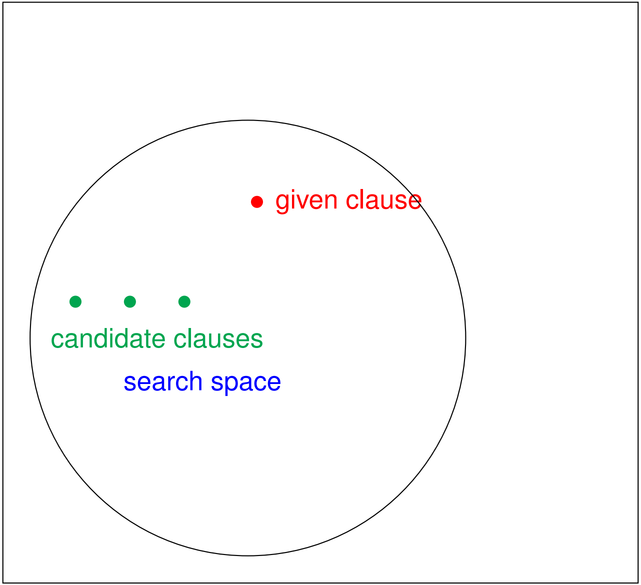

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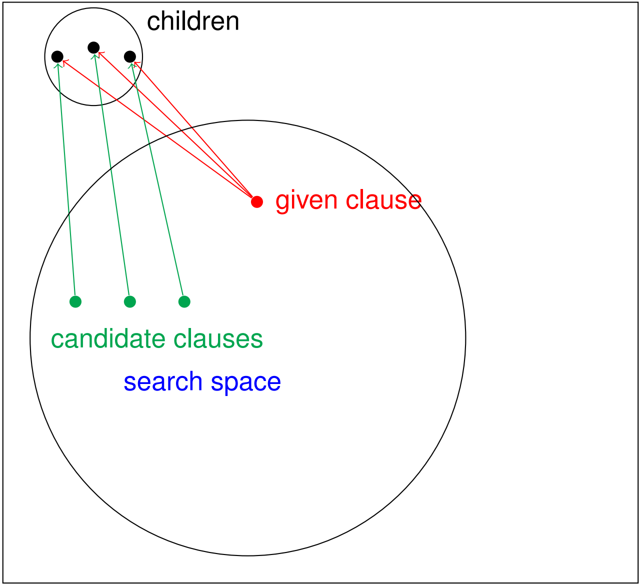

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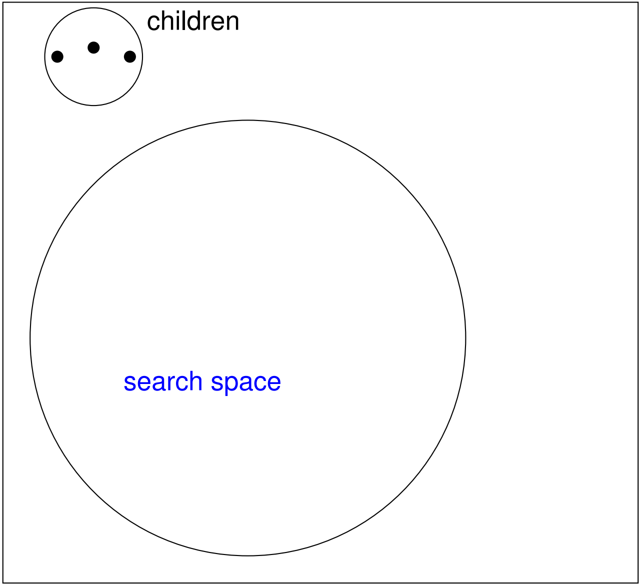

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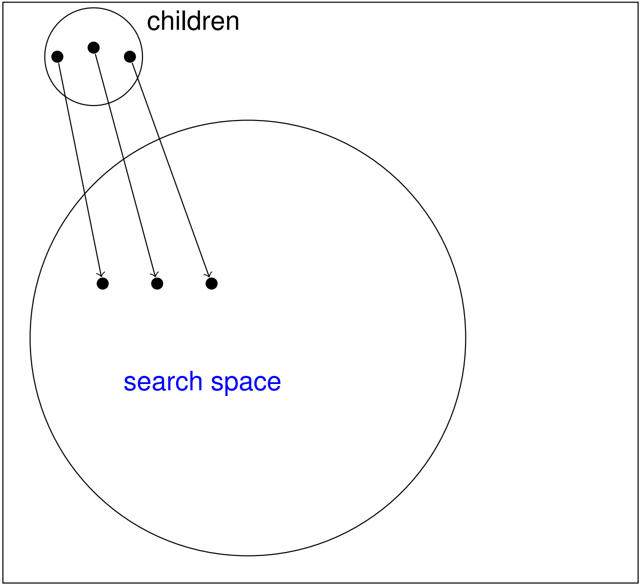

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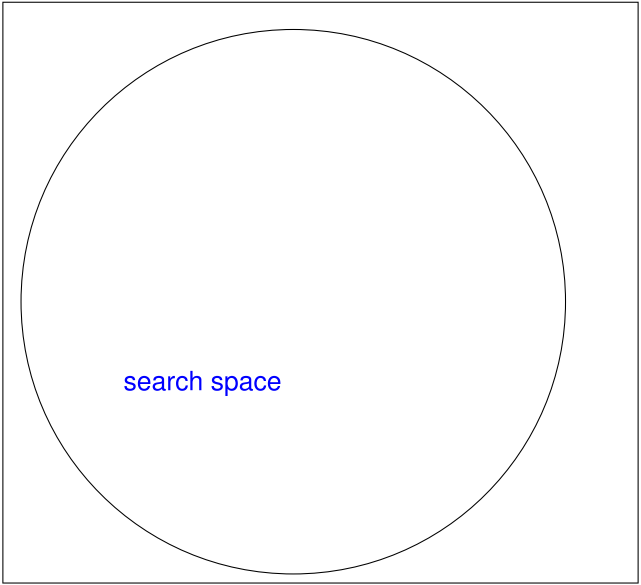

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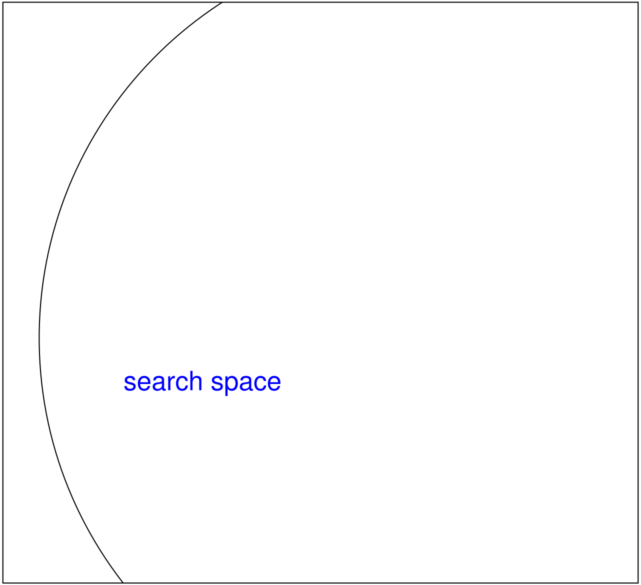

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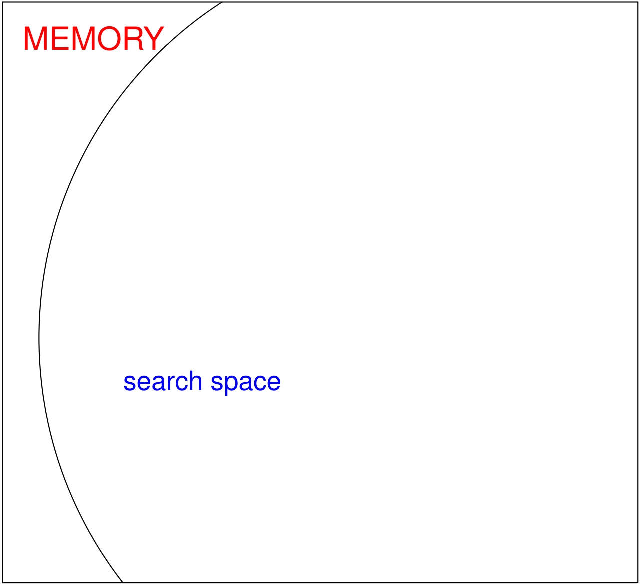

Saturation Algorithm

A saturation algorithm tries to saturate a set of clauses with respect to a given inference system.
In theory there are three possible scenarios:

1. At some moment the empty clause $\square$ is generated, in this case the input set of clauses is unsatisfiable.
2. Saturation will terminate without ever generating $\square$, in this case the input set of clauses in satisfiable.
3. Saturation will run forever, but without generating $\square$. In this case the input set of clauses is satisfiable.

Saturation Algorithm in Practice

In practice there are three possible scenarios:

1. At some moment the empty clause $\square$ is generated, in this case the input set of clauses is unsatisfiable.
2. Saturation will terminate without ever generating $\square$, in this case the input set of clauses in satisfiable.
3. Saturation will run until we run out of resources, but without generating $\square$. In this case it is unknown whether the input set is unsatisfiable.

Outline

Subsumption and Tautology Deletion

A clause is a propositional tautology if it is of the form $p \vee \neg p \vee C$, that is, it contains a pair of complementary literals.
There are also equational tautologies, for example
$a \neq b \vee b \neq c \vee f(c, c) = f(a, a)$.

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A clause C subsumes any clause $C \vee D$, where D is non-empty.

It was known since 1965 that subsumed clauses and propositional tautologies can be removed from the search space.

Problem

How can we prove that completeness is preserved if we remove subsumed clauses and tautologies from the search space?

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Solution: general theory of redundancy.

Bag Extension of an Ordering

Bag = finite multiset.

Let $>$ be any (strict) ordering on a set X. The bag extension of $>$ is a binary relation $>^{bag}$, on bags over X, defined as the smallest transitive relation on bags such that

$$\{x, y_1, \ldots, y_n\} >^{bag} \{x_1, \ldots, x_m, y_1, \ldots, y_n\}$$
$$\text{if } x > x_i \text{ for all } i \in \{1 \ldots m\},$$

where $m \geq 0$.

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where $m \geq 0$.
Idea: a bag becomes smaller if we replace an element by any finite number of smaller elements.
The following results are known about the bag extensions of orderings:

1. $>^{bag}$ is an ordering;
2. If $>$ is total, then so is $>^{bag}$;
3. If $>$ is well-founded, then so is $>^{bag}$.

Clause Orderings

From now on consider clauses also as bags of literals. Note:

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For simpicity we denote the multiset ordering also by $\succ$.

Redundancy

A clause $C \in S$ is called redundant in S if it is a logical consequence of clauses in S strictly smaller than C.

Examples

A tautology $p \vee \neg p \vee C$ is a logical consequence of the empty set of formulas:

$$\models p \vee \neg p \vee C,$$

therefore it is redundant.

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$$\models p \lor \neg p \lor C,$$

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We know that C subsumes $C \lor D$. Note

$$C \lor D \succ C$$
$$C \models C \lor D$$

therefore subsumed clauses are redundant.

Examples

A tautology $p \vee \neg p \vee C$ is a logical consequence of the empty set of formulas:

$$\models p \vee \neg p \vee C,$$

therefore it is redundant.
We know that C subsumes $C \vee D$. Note

$$C \vee D \succ C$$
$$C \models C \vee D$$

therefore subsumed clauses are redundant.

If $\square \in S$, then all non-empty other clauses in S are redundant.

Redundant Clauses Can be Removed

In $\mathbb{BR}_\sigma$ (and in the superposition calculus considered later) redundant clauses can be removed from the search space.

Inference Process with Redundancy

Let $\mathbb{I}$ be an inference system. Consider an inference process with two kinds of step $S_i \Rightarrow S_{i+1}$:

1. Adding the conclusion of an $\mathbb{I}$-inference with premises in S_i.

2. Deletion of a clause redundant in S_i, that is

$$S_{i+1} = S_i - \{C\},$$

   where C is redundant in S_i.

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1. Adding the conclusion of an $\mathbb{I}$-inference with premises in S_i.
   [generating inference]
2. Deletion of a clause redundant in S_i, that is

$$S_{i+1} = S_i - \{C\},$$

   where C is redundant in S_i. [simplifying inference]