Term Rewriting
Basic Concepts, Tools, and Applications

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Course Content

Day 1

abstract rewriting, properties of abstract rewrite systems, Newman's Lemma, term rewriting

Day 2

termination, polynomial interpretations, lexicographic path order, Knuth-Bendix order, derivational complexity

Day 3

critical pairs, confluence, orthogonality, Knuth-Bendix completion

Outline

- define positions as means to identify certain portion of term t

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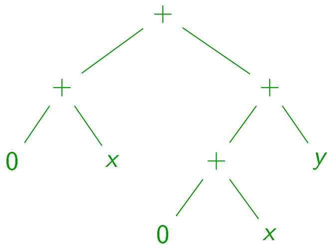

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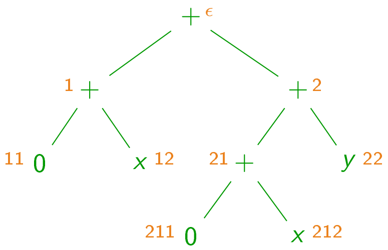

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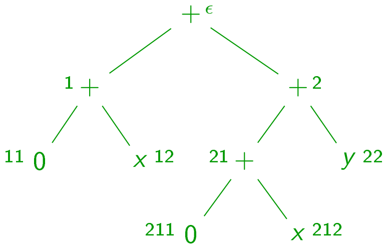

$\mathcal{P}os(t) = \{\, \epsilon, 1, 11, 12, 2, 21, 211, 212, 22 \,\}$

Definitions (Operations on Terms and Positions)

- $t|_p$ — subterm of t at position p

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- $t[\,s\,]_p = (t[\,]_p)[\,s\,]$ replace subterm in t at position p by s

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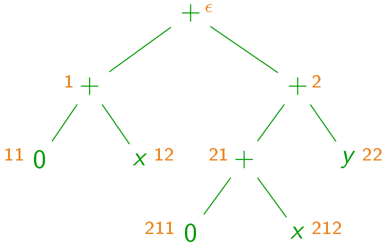

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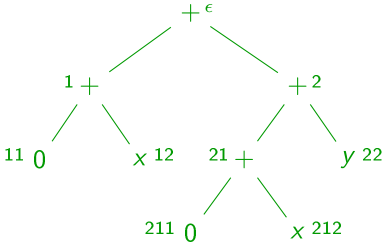

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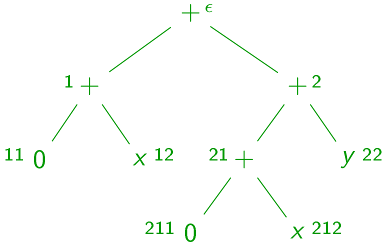

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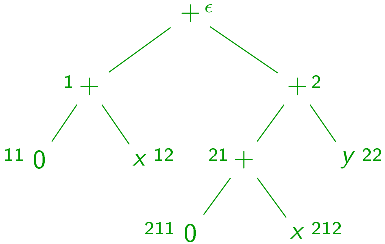

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- $t[z]_{11} = (z + x) + ((0 + x) + y) \qquad t[0]_2 = (0 + x) + 0$

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(d) decomposition

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Definitions (Critical Pair)

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  1. $\ell_1 \to r_1$ and $\ell_2 \to r_2$ are renamings of rewrite rules in $\mathcal{R}$ without common variables
  2. $p \in \mathcal{P}os(\ell_2)$ such that $\ell_2|_p$ is not a variable
  3. ℓ_1 and $\ell_2|_p$ are unifiable with mgu σ
  4. if $p = \epsilon$ then $\ell_1 \to r_1$, $\ell_2 \to r_2$ are not renamings of each other
- $\ell_2\sigma[r_1\sigma]_p \;\leftarrow\; \ell_2\sigma[\ell_1\sigma]_p = \ell_2\sigma \;\to\; r_2\sigma$ critical peak
- $\ell_2\sigma[r_1\sigma]_p \approx r_2\sigma$ critical pair

Definition

TRS $\mathcal{R}$ is locally confluent if ${}_{\mathcal{R}}\!\leftarrow \cdot \to_{\mathcal{R}} \subseteq \to^*_{\mathcal{R}} \cdot {}^{*}_{\mathcal{R}}\!\leftarrow$

Lemma (Critical Pair Lemma)

TRS is locally confluent $\iff$ $s \downarrow t$ *for all critical pairs* $s \approx t$

Example

$$\mathsf{e} \cdot x \to x \qquad x^{-} \cdot x \to \mathsf{e} \qquad (x \cdot y) \cdot z \to x \cdot (y \cdot z)$$

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- overlaps
  1. $\langle \mathsf{e} \cdot u \to u, 1, (x \cdot y) \cdot z \to x \cdot (y \cdot z) \rangle$

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  3. $\langle (u \cdot v) \cdot w \to u \cdot (v \cdot w), 1, (x \cdot y) \cdot z \to x \cdot (y \cdot z) \rangle$

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- critical peaks
  1. $u \cdot z \leftarrow (\mathsf{e} \cdot u) \cdot z \to \mathsf{e} \cdot (u \cdot z)$
  2. $\mathsf{e} \cdot z \leftarrow (u^- \cdot u) \cdot z \to u^- \cdot (u \cdot z)$
  3. $(u \cdot (v \cdot w)) \cdot z \leftarrow ((u \cdot v) \cdot w) \cdot z \to (u \cdot v) \cdot (w \cdot z)$

Example

$$\mathsf{e} \cdot x \to x \qquad x^- \cdot x \to \mathsf{e} \qquad (x \cdot y) \cdot z \to x \cdot (y \cdot z)$$

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- critical pairs
  1. $u \cdot z \approx \mathsf{e} \cdot (u \cdot z)$
  2. $\mathsf{e} \cdot z \approx u^- \cdot (u \cdot z)$
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Outline

Lemma (Critical Pair Lemma)
TRS is locally confluent $\iff$ $s \downarrow t$ *for all critical pairs* $s \approx t$

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Lemma (Newman's Lemma)

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terminating TRS is confluent $\iff$ *all critical pairs are joinable*

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$\bullet\ \bullet \to \circ \qquad \circ\ \circ \to \circ \qquad \bullet\ \circ \to \bullet \qquad \circ\ \bullet \to \bullet$

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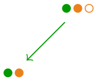

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$$\begin{tikzcd} & \bullet\bullet\circ \arrow[dl] \arrow[dr] & \\ \bullet\bullet & & \circ\circ \end{tikzcd}$$

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$$\begin{tikzcd}
 & \bullet\bullet\circ \arrow[dl] \arrow[dr] & \\
\bullet\bullet \arrow[dr] & & \circ\circ \arrow[dl] \\
 & \circ &
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$$\bullet\ \bullet \to \circ \qquad \circ\ \circ \to \circ \qquad \bullet\ \circ \to \bullet \qquad \circ\ \bullet \to \bullet$$

- critical pairs:

$$\begin{tikzcd}[column sep=small] & \bullet\bullet\circ \arrow[dl] \arrow[dr] & \\ \bullet\bullet \arrow[dr] & & \circ\circ \arrow[dl] \\ & \circ & \end{tikzcd} \quad \begin{tikzcd}[column sep=small] & \circ\circ\circ \arrow[dl] \arrow[dr] & \\ \bullet\circ & & \circ\bullet \end{tikzcd} \quad \begin{tikzcd}[column sep=small] & \bullet\circ\circ \arrow[dl] \arrow[dr] & \\ \bullet\circ & & \bullet\circ \end{tikzcd} \quad \begin{tikzcd}[column sep=small] & \circ\circ\bullet \arrow[dl] \arrow[dr] & \\ \bullet\circ & & \circ\bullet \end{tikzcd}$$

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$$
\begin{tikzcd}[column sep=small]
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\bullet\bullet \arrow[dr] & & \circ\circ \arrow[dl] \\
& \circ &
\end{tikzcd}
\quad
\begin{tikzcd}[column sep=small]
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\bullet\circ \arrow[dr] & & \circ\bullet \arrow[dl] \\
& \bullet &
\end{tikzcd}
\quad
\begin{tikzcd}[column sep=small]
& \bullet\circ\circ \arrow[dl] \arrow[dr] & \\
\bullet\circ & & \bullet\circ
\end{tikzcd}
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$$
\begin{tikzcd}[column sep=small]
 & \bullet\bullet\circ \arrow[dl] \arrow[dr] & \\
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every strategy is winning strategy for player 2!

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$\bullet\ \bullet \to \circ \qquad \circ\ \circ \to \circ \qquad \bullet\ \circ \to \bullet \qquad \circ\ \bullet \to \bullet$

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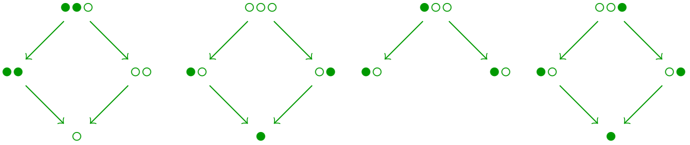

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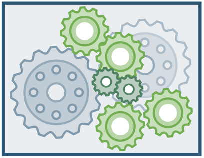
Confluence: Tool Support

CSI

input: term rewrite system $\mathcal{R}$

output: YES + confluence proof, or NO + counterexample

$\mathcal{R}$ ⟶  ⟶ YES / ⋯⋯⟶ ? / ⟶ NO

Nagele, Felgenhauer, and Middeldorp, **CSI: New Evidence — A Progress Report**. 2017.

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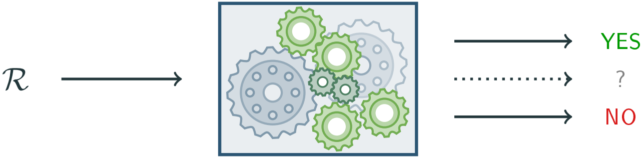

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$\bullet\bullet \to \circ \qquad \circ\circ \to \circ \qquad \bullet\circ \to \bullet \qquad \circ\bullet \to \bullet$

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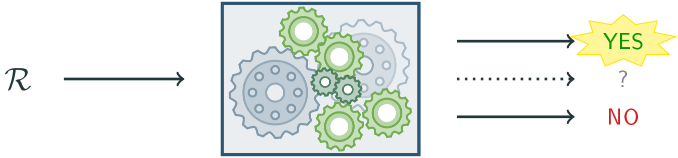

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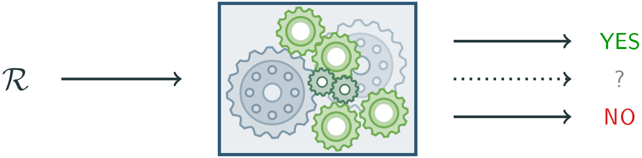

Example (Sieve of Eratosthenes)

$$
\begin{aligned}
\mathsf{primes} &\to \mathsf{sieve}(\mathsf{from}(\mathsf{s}(\mathsf{s}(0)))) & \mathsf{sieve}(0 : y) &\to \mathsf{sieve}(y)\\
\mathsf{from}(x) &\to x : \mathsf{from}(\mathsf{s}(x)) & \mathsf{sieve}(\mathsf{s}(x) : y) &\to \mathsf{s}(x) : \mathsf{sieve}(\mathsf{filter}(x, y, x))\\
\mathsf{hd}(x : y) &\to x & \mathsf{filter}(0, y : z, w) &\to 0 : \mathsf{filter}(w, z, w)\\
\mathsf{tl}(x : y) &\to y & \mathsf{filter}(\mathsf{s}(x), y : z, w) &\to y : \mathsf{filter}(x, z, w)
\end{aligned}
$$

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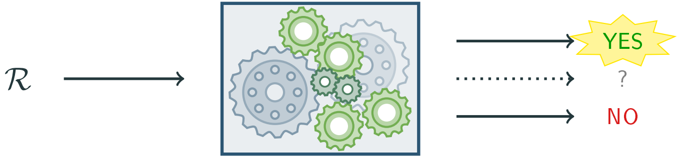

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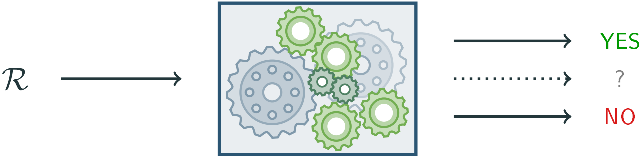

Definition

TRS $\mathcal{R}$ is orthogonal if

- in left-hand sides of $\mathcal{R}$ every variable occurs at most once
- $\mathcal{R}$ has no critical pairs

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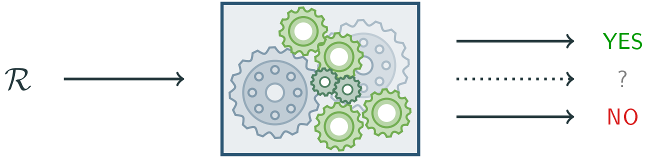
output: YES + confluence proof, or NO + counterexample



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Theorem (Rosen 1973)

orthogonal TRSs are confluent

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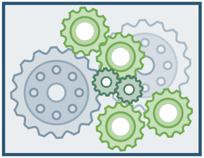
$\mathcal{R}$ ⟶  ⟶ YES ⋯⟶ ? ⟶ NO

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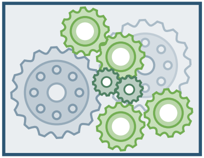
Outline

Knuth-Bendix Completion

input: set of equations $\mathcal{E}$

output: terminating and confluent TRS $\mathcal{R}$ such that $\leftrightarrow^*_{\mathcal{E}} = \leftrightarrow^*_{\mathcal{R}}$

$\mathcal{E} \longrightarrow$



$\longrightarrow \mathcal{R}$

$\dashrightarrow$?

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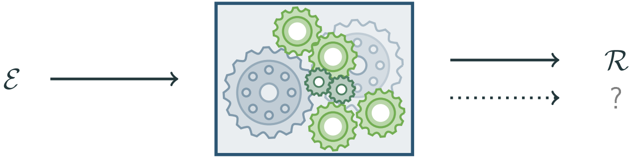

Definition

TRS $\mathcal{R}$ is complete if terminating and confluent

Knuth-Bendix Completion

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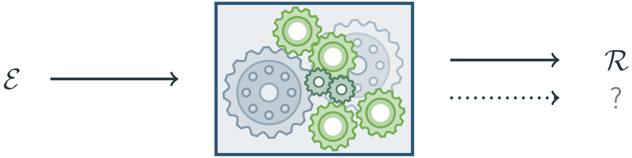

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Aim: Theorem Proving via Completion

- in complete TRS every term has unique normal form

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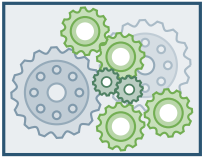

$\longrightarrow \mathcal{R}$

$\dashrightarrow$?

Definition

TRS $\mathcal{R}$ is complete if terminating and confluent

Aim: Theorem Proving via Completion

- in complete TRS every term has unique normal form
- if $\mathcal{R}$ complete and $\leftrightarrow_{\mathcal{E}}^{*} = \leftrightarrow_{\mathcal{R}}^{*}$: $\forall s, t$ can decide whether $s \leftrightarrow_{\mathcal{E}}^{*} t$: check whether $\mathcal{R}$-normal forms of s and t are equal

Knuth-Bendix Completion

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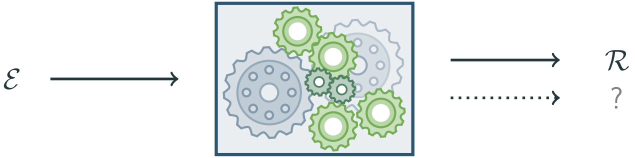

Example (Group Theory)

$$\mathcal{E}: \quad \begin{aligned} 1 \cdot x &\approx x \\ (-x) \cdot x &\approx 1 \\ x \cdot (y \cdot z) &\approx (x \cdot y) \cdot z \end{aligned}$$

$$\mathcal{R}: \quad \begin{aligned} -1 &\to 1 & -(-x) &\to x \\ 1 \cdot x &\to x & (-x) \cdot (x \cdot y) &\to y \\ x \cdot 1 &\to x & y \cdot ((-y) \cdot x) &\to x \\ (-x) \cdot x &\to 1 & x \cdot (y \cdot z) &\to (x \cdot y) \cdot z \\ x \cdot (-x) &\to 1 & -(x \cdot y) &\to -y \cdot (-x) \end{aligned}$$

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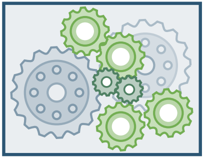

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Example (Group Theory)

$$\mathcal{E}: \quad \begin{aligned} 1 \cdot x &\approx x \\ (-x) \cdot x &\approx 1 \\ x \cdot (y \cdot z) &\approx (x \cdot y) \cdot z \end{aligned}$$

$$\mathcal{R}: \quad \begin{aligned} -1 &\to 1 & -(-x) &\to x \\ 1 \cdot x &\to x & (-x) \cdot (x \cdot y) &\to y \\ x \cdot 1 &\to x & y \cdot ((-y) \cdot x) &\to x \\ (-x) \cdot x &\to 1 & x \cdot (y \cdot z) &\to (x \cdot y) \cdot z \\ x \cdot (-x) &\to 1 & -(x \cdot y) &\to -y \cdot (-x) \end{aligned}$$

- group theory is decidable

Definition (Completion Inference System)

set of equations $\mathcal{E}$ set of rewrite rules $\mathcal{R}$ reduction order $>$

inference system KB consists of eight rules

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delete $\dfrac{\mathcal{E} \uplus \{ s \approx s \}, \mathcal{R}}{\mathcal{E}, \mathcal{R}}$

deduce $\dfrac{\mathcal{E}, \mathcal{R}}{\mathcal{E} \cup \{ s \approx t \}, \mathcal{R}}$

if $s \;{}_{\mathcal{R}}\!\leftarrow \cdot \rightarrow_{\mathcal{R}} t$

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compose $$\frac{\mathcal{E}, \mathcal{R} \uplus \{ s \rightarrow t \}}{}$$

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if $t \rightarrow_{\mathcal{R}} u$

orient $$\frac{\mathcal{E} \uplus \{ s \approx t \}, \mathcal{R}}{} \qquad \frac{\mathcal{E} \uplus \{ t \approx s \}, \mathcal{R}}{}$$

if $s > t$

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$$\frac{\mathcal{E} \uplus \{ t \approx s \}, \mathcal{R}}{\mathcal{E}, \mathcal{R} \cup \{ s \to t \}}$$

if $s > t$

simplify
$$\frac{\mathcal{E} \uplus \{ s \approx t \}, \mathcal{R}}{}$$

$$\frac{\mathcal{E} \uplus \{ t \approx s \}, \mathcal{R}}{}$$

if $t \to_{\mathcal{R}} u$

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Example

$$\mathsf{f}(\mathsf{f}(x)) \approx \mathsf{g}(x)$$
$$\mathsf{g}(\mathsf{a}) \approx \mathsf{b}$$

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- LPO with precedence $\mathsf{f} > \mathsf{g} > \mathsf{b} > \mathsf{a}$

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- LPO with precedence $\mathsf{f} > \mathsf{g} > \mathsf{b} > \mathsf{a}$
- orient $\quad \mathsf{f}(\mathsf{f}(x)) >_{\mathsf{lpo}} \mathsf{g}(x)$

Example

$$f(f(x)) \to g(x)$$

$$g(a) \approx b$$

- LPO with precedence $f > g > b > a$
- orient $f(f(x)) >_{lpo} g(x)$

Example

$$\mathsf{f}(\mathsf{f}(x)) \to \mathsf{g}(x)$$

$$\mathsf{g}(\mathsf{a}) \approx \mathsf{b}$$

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Example

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$$\mathsf{f}(\mathsf{f}(x)) \to \mathsf{g}(x)$$
$$\mathsf{g}(\mathsf{a}) \to \mathsf{b}$$

- LPO with precedence $\mathsf{f} > \mathsf{g} > \mathsf{b} > \mathsf{a}$
- deduce $\quad \mathsf{f}(\mathsf{g}(x)) \leftarrow \mathsf{f}(\mathsf{f}(\mathsf{f}(x))) \to \mathsf{g}(\mathsf{f}(x)) \quad$ critical pair

Example

$$\mathsf{f}(\mathsf{g}(x)) \approx \mathsf{g}(\mathsf{f}(x)) \qquad\qquad \begin{aligned} \mathsf{f}(\mathsf{f}(x)) &\to \mathsf{g}(x) \\ \mathsf{g}(\mathsf{a}) &\to \mathsf{b} \end{aligned}$$

- LPO with precedence $\mathsf{f} > \mathsf{g} > \mathsf{b} > \mathsf{a}$
- deduce $\quad \mathsf{f}(\mathsf{g}(x)) \leftarrow \mathsf{f}(\mathsf{f}(\mathsf{f}(x))) \to \mathsf{g}(\mathsf{f}(x)) \qquad$ critical pair

Example

$$\mathsf{f}(\mathsf{g}(x)) \approx \mathsf{g}(\mathsf{f}(x)) \qquad\qquad \begin{aligned} \mathsf{f}(\mathsf{f}(x)) &\to \mathsf{g}(x) \\ \mathsf{g}(\mathsf{a}) &\to \mathsf{b} \end{aligned}$$

- LPO with precedence $\mathsf{f} > \mathsf{g} > \mathsf{b} > \mathsf{a}$
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Example

$$\begin{aligned} \mathsf{f}(\mathsf{f}(x)) &\to \mathsf{g}(x) \\ \mathsf{g}(\mathsf{a}) &\to \mathsf{b} \\ \mathsf{f}(\mathsf{g}(x)) &\to \mathsf{g}(\mathsf{f}(x)) \end{aligned}$$

- LPO with precedence $\mathsf{f} > \mathsf{g} > \mathsf{b} > \mathsf{a}$
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Example

$$\mathsf{f}(\mathsf{f}(x)) \to \mathsf{g}(x)$$
$$\mathsf{g}(\mathsf{a}) \to \mathsf{b}$$
$$\mathsf{f}(\mathsf{g}(x)) \to \mathsf{g}(\mathsf{f}(x))$$

- LPO with precedence $\mathsf{f} > \mathsf{g} > \mathsf{b} > \mathsf{a}$
- deduce $\quad \mathsf{f}(\mathsf{g}(\mathsf{f}(x))) \leftarrow \mathsf{f}(\mathsf{f}(\mathsf{g}(x))) \to \mathsf{g}(\mathsf{g}(x)) \quad$ critical pair

Example

$$\mathsf{f}(\mathsf{g}(\mathsf{f}(x))) \approx \mathsf{g}(\mathsf{g}(x)) \qquad\qquad \begin{aligned} \mathsf{f}(\mathsf{f}(x)) &\to \mathsf{g}(x) \\ \mathsf{g}(\mathsf{a}) &\to \mathsf{b} \\ \mathsf{f}(\mathsf{g}(x)) &\to \mathsf{g}(\mathsf{f}(x)) \end{aligned}$$

- LPO with precedence $\mathsf{f} > \mathsf{g} > \mathsf{b} > \mathsf{a}$
- deduce $\quad \mathsf{f}(\mathsf{b}) \leftarrow \mathsf{f}(\mathsf{g}(\mathsf{a})) \to \mathsf{g}(\mathsf{f}(\mathsf{a}))$ critical pair

Example

$$\begin{aligned} f(g(f(x))) &\approx g(g(x)) \\ f(b) &\approx g(f(a)) \end{aligned} \qquad \begin{aligned} f(f(x)) &\to g(x) \\ g(a) &\to b \\ f(g(x)) &\to g(f(x)) \end{aligned}$$

- LPO with precedence $f > g > b > a$

Example

$$\mathsf{f}(\mathsf{g}(\mathsf{f}(x))) \approx \mathsf{g}(\mathsf{g}(x)) \qquad \mathsf{f}(\mathsf{f}(x)) \to \mathsf{g}(x)$$

$$\mathsf{f}(\mathsf{b}) \approx \mathsf{g}(\mathsf{f}(\mathsf{a})) \qquad \mathsf{g}(\mathsf{a}) \to \mathsf{b}$$

$$\mathsf{f}(\mathsf{g}(x)) \to \mathsf{g}(\mathsf{f}(x))$$

- LPO with precedence $\mathsf{f} > \mathsf{g} > \mathsf{b} > \mathsf{a}$
- simplify $\quad \mathsf{f}(\mathsf{g}(\mathsf{f}(x))) \to \mathsf{g}(\mathsf{f}(\mathsf{f}(x)))$

Example

$$g(f(f(x))) \approx g(g(x)) \qquad f(f(x)) \to g(x)$$
$$f(b) \approx g(f(a)) \qquad g(a) \to b$$
$$f(g(x)) \to g(f(x))$$

- LPO with precedence $f > g > b > a$
- simplify $\quad g(f(f(x))) \to g(g(x))$

Example

$$\mathsf{g}(\mathsf{g}(x)) \approx \mathsf{g}(\mathsf{g}(x)) \qquad \mathsf{f}(\mathsf{f}(x)) \to \mathsf{g}(x)$$

$$\mathsf{f}(\mathsf{b}) \approx \mathsf{g}(\mathsf{f}(\mathsf{a})) \qquad \mathsf{g}(\mathsf{a}) \to \mathsf{b}$$

$$\mathsf{f}(\mathsf{g}(x)) \to \mathsf{g}(\mathsf{f}(x))$$

- LPO with precedence $\mathsf{f} > \mathsf{g} > \mathsf{b} > \mathsf{a}$
- delete $\quad \mathsf{g}(\mathsf{g}(x)) = \mathsf{g}(\mathsf{g}(x))$

Example

$$\mathsf{f}(\mathsf{b}) \approx \mathsf{g}(\mathsf{f}(\mathsf{a}))$$

$$\begin{aligned} \mathsf{f}(\mathsf{f}(x)) &\to \mathsf{g}(x) \\ \mathsf{g}(\mathsf{a}) &\to \mathsf{b} \\ \mathsf{f}(\mathsf{g}(x)) &\to \mathsf{g}(\mathsf{f}(x)) \end{aligned}$$

- LPO with precedence $\mathsf{f} > \mathsf{g} > \mathsf{b} > \mathsf{a}$
- orient $\quad \mathsf{f}(\mathsf{b}) >_{\mathsf{lpo}} \mathsf{g}(\mathsf{f}(\mathsf{a}))$

Example

$$\begin{aligned} \mathsf{f}(\mathsf{f}(x)) &\to \mathsf{g}(x) \\ \mathsf{g}(\mathsf{a}) &\to \mathsf{b} \\ \mathsf{f}(\mathsf{g}(x)) &\to \mathsf{g}(\mathsf{f}(x)) \\ \mathsf{f}(\mathsf{b}) &\to \mathsf{g}(\mathsf{f}(\mathsf{a})) \end{aligned}$$

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- LPO with precedence $\mathsf{f} > \mathsf{g} > \mathsf{b} > \mathsf{a}$
- deduce $\mathsf{f}(\mathsf{g}(\mathsf{f}(\mathsf{a}))) \leftarrow \mathsf{f}(\mathsf{f}(\mathsf{b})) \to \mathsf{g}(\mathsf{b})$ critical pair

Example

$$f(g(f(a))) \approx g(b)$$

$$\begin{aligned} f(f(x)) &\to g(x) \\ g(a) &\to b \\ f(g(x)) &\to g(f(x)) \\ f(b) &\to g(f(a)) \end{aligned}$$

- LPO with precedence $f > g > b > a$
- deduce $f(g(f(a))) \leftarrow f(f(b)) \rightarrow g(b)$ critical pair

Example

$$\mathsf{f}(\mathsf{g}(\mathsf{f}(\mathsf{a}))) \approx \mathsf{g}(\mathsf{b})$$

$$\begin{aligned} \mathsf{f}(\mathsf{f}(x)) &\to \mathsf{g}(x) \\ \mathsf{g}(\mathsf{a}) &\to \mathsf{b} \\ \mathsf{f}(\mathsf{g}(x)) &\to \mathsf{g}(\mathsf{f}(x)) \\ \mathsf{f}(\mathsf{b}) &\to \mathsf{g}(\mathsf{f}(\mathsf{a})) \end{aligned}$$

- LPO with precedence $\mathsf{f} > \mathsf{g} > \mathsf{b} > \mathsf{a}$
- simplify $\quad \mathsf{f}(\mathsf{g}(\mathsf{f}(\mathsf{a}))) \to \mathsf{g}(\mathsf{f}(\mathsf{f}(\mathsf{a})))$

Example

$$g(f(f(a))) \approx g(b)$$

$$\begin{aligned} f(f(x)) &\to g(x) \\ g(a) &\to b \\ f(g(x)) &\to g(f(x)) \\ f(b) &\to g(f(a)) \end{aligned}$$

- LPO with precedence $f > g > b > a$
- simplify $g(f(f(a))) \to g(g(a))$

Example

$$\mathsf{g}(\mathsf{g}(\mathsf{a})) \approx \mathsf{g}(\mathsf{b})$$

$$\mathsf{f}(\mathsf{f}(x)) \to \mathsf{g}(x)$$
$$\mathsf{g}(\mathsf{a}) \to \mathsf{b}$$
$$\mathsf{f}(\mathsf{g}(x)) \to \mathsf{g}(\mathsf{f}(x))$$
$$\mathsf{f}(\mathsf{b}) \to \mathsf{g}(\mathsf{f}(\mathsf{a}))$$

- LPO with precedence $\mathsf{f} > \mathsf{g} > \mathsf{b} > \mathsf{a}$
- simplify $\quad \mathsf{g}(\mathsf{g}(\mathsf{a})) \to \mathsf{g}(\mathsf{b})$

Example

$$\mathsf{g}(\mathsf{b}) \approx \mathsf{g}(\mathsf{b})$$

$$\begin{aligned} \mathsf{f}(\mathsf{f}(x)) &\to \mathsf{g}(x) \\ \mathsf{g}(\mathsf{a}) &\to \mathsf{b} \\ \mathsf{f}(\mathsf{g}(x)) &\to \mathsf{g}(\mathsf{f}(x)) \\ \mathsf{f}(\mathsf{b}) &\to \mathsf{g}(\mathsf{f}(\mathsf{a})) \end{aligned}$$

- LPO with precedence $\mathsf{f} > \mathsf{g} > \mathsf{b} > \mathsf{a}$
- delete $\quad \mathsf{g}(\mathsf{b}) = \mathsf{g}(\mathsf{b})$

Example

$$\begin{aligned} f(f(x)) &\to g(x) \\ g(a) &\to b \\ f(g(x)) &\to g(f(x)) \\ f(b) &\to g(f(a)) \end{aligned}$$

- LPO with precedence $f > g > b > a$
- complete TRS

Definition (Fair Run)

run for given ES $\mathcal{E}$ is finite sequence

$$\mathcal{E}_0, \mathcal{R}_0 \vdash \mathcal{E}_1, \mathcal{R}_1 \vdash \cdots \vdash \mathcal{E}_n, \mathcal{R}_n$$

such that $\mathcal{E}_0 = \mathcal{E}$ and $\mathcal{R}_0 = \varnothing$

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- run fails if $\mathcal{E}_n \neq \varnothing$

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- run is fair if $\mathrm{CP}(\mathcal{R}_n) \subseteq \downarrow_{R_n} \cup \bigcup_{i=0}^{n} \mathcal{E}_i$

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CP($\mathcal{R}_n$) is set of critical pairs of $\mathcal{R}_n$

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Theorem

for every fair non-failing run

$$\mathcal{E}_0, \mathcal{R}_0 \vdash \mathcal{E}_1, \mathcal{R}_1 \vdash \cdots \vdash \mathcal{E}_n, \mathcal{R}_n$$

$\mathcal{R}_n$ *is complete and* $\leftrightarrow^*_{\mathcal{R}_n} = \leftrightarrow^*_{\mathcal{E}_0}$

Example: Virus Genes

A COVID medication research team wants to develop a medicine that transforms the SARS-CoV-2 DNA:

TAGCTAGCTAGCT

into the DNA of a known and relatively benign influenza virus:

CTGACTGACT

Techniques exist to perform the following DNA transformations:

TCAT $\leftrightarrow$ T   GAG $\leftrightarrow$ AG   CTC $\leftrightarrow$ TC   AGTA $\leftrightarrow$ A   TAT $\leftrightarrow$ CT

Recently it has been discovered that the mad cow disease is caused by a retrovirus with the following DNA sequence

CTGCTACTGACT

What now, if accidentally this virus is created? According to the engineers there is little risk because this never happened in their experiments, but medical authorities demand absolute assurance.

Example: Virus Genes (2)

equational system of known DNA transformations $\mathcal{E}$:

$\mathsf{TCAT} \approx \mathsf{T} \quad \mathsf{GAG} \approx \mathsf{AG} \quad \mathsf{CTC} \approx \mathsf{TC} \quad \mathsf{AGTA} \approx \mathsf{A} \qquad \mathsf{TAT} \approx \mathsf{CT}$

Example: Virus Genes (2)

equational system of known DNA transformations $\mathcal{E}$:

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has complete presentation $\mathcal{R}$:

$\mathsf{GA} \to \mathsf{A} \quad \mathsf{AGT} \to \mathsf{AT} \quad \mathsf{ATA} \to \mathsf{A} \quad \mathsf{TCA} \to \mathsf{TA} \quad \mathsf{TAT} \to \mathsf{T} \quad \mathsf{CT} \to \mathsf{T}$

Example: Virus Genes (2)

equational system of known DNA transformations $\mathcal{E}$:

$\mathsf{TCAT} \approx \mathsf{T} \quad \mathsf{GAG} \approx \mathsf{AG} \quad \mathsf{CTC} \approx \mathsf{TC} \quad \mathsf{AGTA} \approx \mathsf{A} \quad \mathsf{TAT} \approx \mathsf{CT}$

has complete presentation $\mathcal{R}$:

$\mathsf{GA} \to \mathsf{A} \quad \mathsf{AGT} \to \mathsf{AT} \quad \mathsf{ATA} \to \mathsf{A} \quad \mathsf{TCA} \to \mathsf{TA} \quad \mathsf{TAT} \to \mathsf{T} \quad \mathsf{CT} \to \mathsf{T}$

- (Sars-CoV-2) $\mathsf{TAGCTAGCTAGCT} \overset{*}{\underset{\mathcal{E}}{\longleftrightarrow}} \mathsf{CTGACTGACT}$ (influenza)

  $$\mathsf{TAGCTAGCTAGCT} \overset{!}{\underset{\mathcal{R}}{\longrightarrow}} \mathsf{T} \overset{!}{\underset{\mathcal{R}}{\longleftarrow}} \mathsf{CTGACTGACT}$$

Example: Virus Genes (2)

equational system of known DNA transformations $\mathcal{E}$:

$\mathsf{TCAT} \approx \mathsf{T} \quad \mathsf{GAG} \approx \mathsf{AG} \quad \mathsf{CTC} \approx \mathsf{TC} \quad \mathsf{AGTA} \approx \mathsf{A} \quad \mathsf{TAT} \approx \mathsf{CT}$

has complete presentation $\mathcal{R}$:

$\mathsf{GA} \to \mathsf{A} \quad \mathsf{AGT} \to \mathsf{AT} \quad \mathsf{ATA} \to \mathsf{A} \quad \mathsf{TCA} \to \mathsf{TA} \quad \mathsf{TAT} \to \mathsf{T} \quad \mathsf{CT} \to \mathsf{T}$

- (Sars-CoV-2) $\mathsf{TAGCTAGCTAGCT} \xleftrightarrow[\mathcal{E}]{*} \mathsf{CTGACTGACT}$ (influenza)

  $$\mathsf{TAGCTAGCTAGCT} \xrightarrow[\mathcal{R}]{!} \mathsf{T} \xleftarrow[\mathcal{R}]{!} \mathsf{CTGACTGACT}$$

- (Sars-CoV-2) $\mathsf{TAGCTAGCTAGCT} \not\xleftrightarrow[\mathcal{E}]{*} \mathsf{CTGCTACTGACT}$ (mad cow)

  $$\mathsf{TAGCTAGCTAGCT} \xrightarrow[\mathcal{R}]{!} \mathsf{T} \neq \mathsf{TGT} \xleftarrow[\mathcal{R}]{!} \mathsf{CTGCTACTGACT}$$

Example: Virus Genes (2)

equational system of known DNA transformations $\mathcal{E}$:

$\mathsf{TCAT} \approx \mathsf{T} \quad \mathsf{GAG} \approx \mathsf{AG} \quad \mathsf{CTC} \approx \mathsf{TC} \quad \mathsf{AGTA} \approx \mathsf{A} \quad \mathsf{TAT} \approx \mathsf{CT}$

has complete presentation $\mathcal{R}$:

$\mathsf{GA} \to \mathsf{A} \quad \mathsf{AGT} \to \mathsf{AT} \quad \mathsf{ATA} \to \mathsf{A} \quad \mathsf{TCA} \to \mathsf{TA} \quad \mathsf{TAT} \to \mathsf{T} \quad \mathsf{CT} \to \mathsf{T}$

mkbtt

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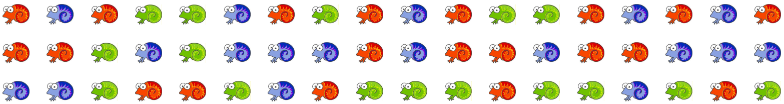
Example: Chameleon Puzzle



A colony of chameleons consists of 20 red, 18 blue, and 16 green animals. Whenever two of different color meet, both change to the third color.

Is it possible that all 54 chameleons become the same color?

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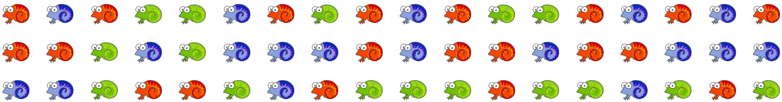

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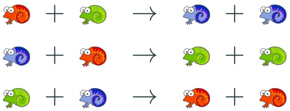

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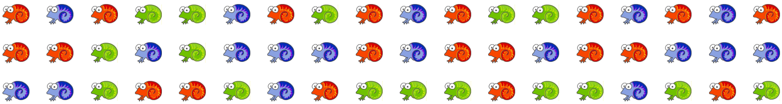

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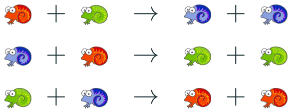

$$x + y \approx y + x$$

$$(x + y) + z \approx x + (y + z)$$

mkbtt

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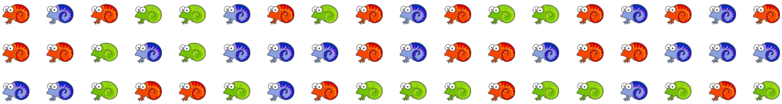

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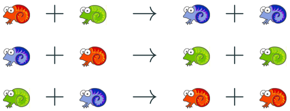

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▸ mkbtt

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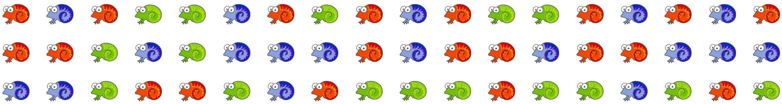

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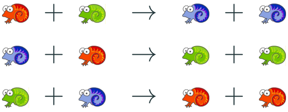

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- initial colony: 20  + 18  + 16 

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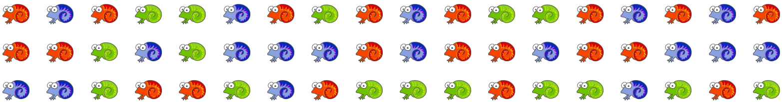

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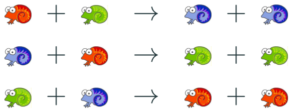

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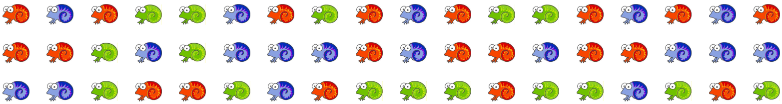
mkbtt

- initial colony: 20  + 18  + 16  $\to^!$ 52  + 2 

Example: Chameleon Puzzle



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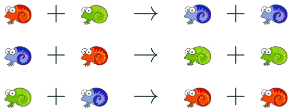

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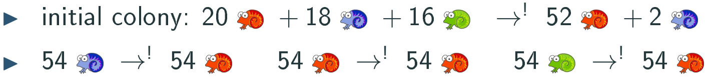


Practice Example: Data Migration

- commercial data migration tool AQL by ConexusAI is based on category theory and uses (ordered) completion
- many symbols and possible orders, ground complete system required
- AQL uses reimplementation of mædmax

$$\mathsf{yIsW}(a_1) \approx w_{29} \quad \mathsf{yIsW}(a_{78}) \approx w_{16} \quad \mathsf{yIsW}(a_{61}) \approx w_{30}$$

$$\mathsf{yIsL}(a_{37}) \approx l_{80} \quad \mathsf{yIsL}(a_{84}) \approx l_{6} \quad \mathsf{yIsL}(a_{29}) \approx l_{47}$$

$$\dots$$

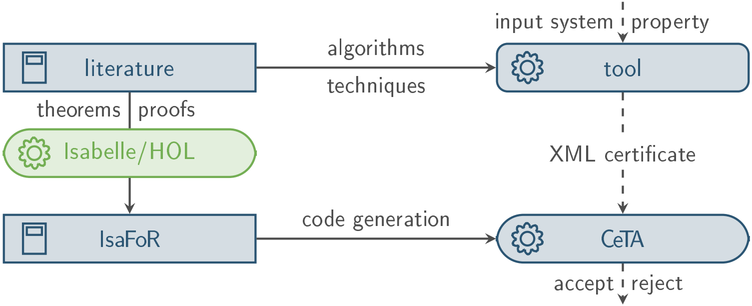
$$\mathsf{yIsL}(a_{51}) \approx l_{77} \quad \mathsf{yIsL}(a_{88}) \approx l_{9} \quad \mathsf{yIsL}(X) \approx \mathsf{yIsWW}(\mathsf{yIsW}(X))$$

`http://conexus.com`

Schultz and Wisnesky, **Algebraic data integration**. JFP 2017.

Why Trust These Tools?

The IsaFoR/CeTA framework



- formalize tool methods in proof assistant Isabelle/HOL
- obtain code-generated proof checker CeTA to verify tool output

Thiemann and Sternagel, **Certification of Termination Proofs Using CeTA**. TPHOL 2009.

Certification of Tool Output: Overview

- TTT2

standard TRS:	829 YES	(750)	200 NO	(193)
standard SRS:	733 YES	(670)	43 NO	(24)

- CSI

standard TRS:	42 YES	(28)	33 NO	(23)

- TCT

TPDB:	203 YES	(165)

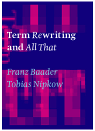
- mædmax

TPTP:	112 SAT	(69)	621 UNSAT	(612)

Data taken from Termination and Confluence Competitions 2019, [AST15], [WM18], [SWZ15].

Outline

Further Reading



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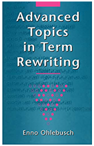

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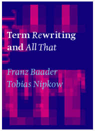

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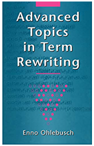

Enno Ohlebusch. *Advanced Topics in Term Rewriting*. Cambridge University Press, 2008.

International Summer School on Rewriting

- general website of IFIP working group
- last and upcoming editions:
  - 2019 in Paris, France
  - 2021 in Madrid, Spain
  - 2022 in Tbilisi, Georgia

Conclusion

Term Rewriting*

- Turing complete model of computation
- rewriting techniques can serve as toolbox to analyze properties

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Conclusion

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Applications

program analysis, theorem proving, simplification systems, ...

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Conclusion

Term Rewriting*

- Turing complete model of computation
- rewriting techniques can serve as toolbox to analyze properties

Applications

program analysis, theorem proving, simplification systems, . . .

Automatic Tools

- termination analysis
- Knuth-Bendix completion
- confluence analysis
- complexity analysis

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Exercises

1.

2.

3.

4.

5.