Exactly true and non-falsity versions of Deutsch's logic

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Introduction: «Exactly True» and «Non-Falsity»

- Terms «Exactly True» and «Non-Falsity» logic take their roots in recent works on extensions of **FDE**:
  - Pietz, A., Rivieccio, U. (2013). Nothing but the truth. Journal of Philosophical Logic, 42, 125–135.
  - Shramko, Y., Zaitsev, D., Belikov, A. (2017). First-degree entailment and its relatives. Studia Logica, 105, 1291–1317.
  - Shramko, Y., Zaitsev, D., Belikov, A. (2018). The FMLA-FMLA Axiomatizations of the Exactly True and Non-falsity Logics and Some of Their Cousins. Journal of Philosophical Logic, Online First.

Introduction: «Exactly True» and «Non-Falsity»

Recall the definition of the consequence relation in **FDE**.

Definition (Consequence relation in FDE)

$\varphi \vDash_{fde} \psi$ iff for any **FDE**-model, if φ *is true* in this model, then ψ *is true* in the same model.

- Notice that **FDE**-models permit inconsistent and incomplete valuations.
- Technically, it can be reflected by using a logical matrix with four truth-values: T (true and non-false), B (true and false), N (neither true, nor false), F (false and non-true).

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We know that in Classical Logic a consequence relation can be defined by different (but equivalent) ways:

1. Direct (from premise to conclusion) preservation of truth;
2. Backward (from conclusion to premise) preservation of falsity;
3. Direct preservation of truth and non-falsity;
4. Backward preservation of falsity and non-truth;

So, what if we put (3) and (4) in **FDE** setting?

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This way leads us to two consequence relations which give rise to two logics: **Exactly True Logic** (Pietz-Rivieccio) and **Non-Falsity Logic** (Shramko-Zaitsev-Belikov).

Definition (Consequence relation in ETL)

$\varphi \vDash_{etl} \psi$ iff for any **FDE**-model, if φ *is true and non-false* in this model, then ψ *is true and non-false* in the same model.

Definition (Consequence relation in NFL)

$\varphi \vDash_{nfl} \psi$ iff for any **FDE**-model, if ψ *is false and non-true* in this model, then φ *is false and non-true* in the same model.

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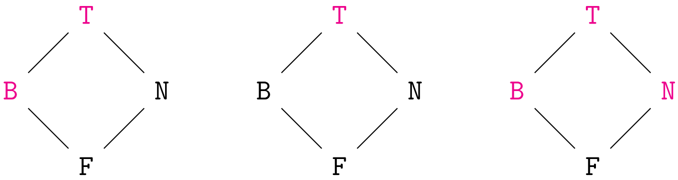

Рис.: Lattice $\mathcal{FOUR}$ and possible sets of designated truth-values: (from left-to-right) FDE setting, ETL setting, NFL setting

- Harry Deutsch introduced a relevant logic S_{fde}:
  - Deutsch, H., Relevant analytic entailment, The Relevance Logic Newsletter 2(1) (1977): 26–44.
  - Deutsch, H., The completeness of S, Studia Logica 38(2) (1979): 137–147.
- S_{fde} has a four-valued semantics in the spirit of Belnap's one.
- S_{fde} is closely related to the family of infectious logics.
- Conjunction, disjunction and negation in S_{fde} are defined by means of the following tables:

$\wedge$	T	B	N	F
T	T	B	N	F
B	B	B	N	F
N	N	N	N	N
F	F	F	N	F

$\vee$	T	B	N	F
T	T	T	N	T
B	T	B	N	B
N	N	N	N	N
F	T	B	N	F

φ	$\sim\varphi$
T	F
B	B
N	N
F	T

- Our main motivation is quite natural. In the original case of ETL and NFL we obtain exactly true and non-falsity versions of the relevant logic FDE, which is obeying the 'Variable Sharing Principle':

  $$\text{if } \varphi \vDash \psi, \text{ then } \mathit{Var}(\varphi) \cap \mathit{Var}(\psi) \neq \varnothing.$$

  Now, we want to extend this approach to the relevant logic S_{fde}, which is obeying different criteria of relevance, namely 'Proscriptive Principle':

  $$\text{if } \varphi \vDash \psi, \text{ then } \mathit{Var}(\psi) \subseteq \mathit{Var}(\varphi).$$

Semantics

- We introduce two new logics – S_{etl} and S_{nfl} – which differ from S_{fde} by the definition of the consequence relation.
- More precisely, the difference lies in the set of the designated truth-values. For S_{etl} we use $\{\mathtt{T}\}$ instead of $\{\mathtt{T}, \mathtt{B}\}$. For S_{nfl} we use $\{\mathtt{T}, \mathtt{B}, \mathtt{F}\}$ instead of $\{\mathtt{T}, \mathtt{B}\}$.

Semantics: a family of consequence relations

- We define a family of consequence relations. For each $\Gamma \cup \Delta \subseteq \texttt{Form}$, each $\mathsf{L}_{\mathsf{etl}} \in \{\mathsf{S}_{\mathsf{etl}}, \mathsf{ETL}\}$, each $\mathsf{L}_{\mathsf{fde}} \in \{\mathsf{S}_{\mathsf{fde}}, \mathsf{FDE}\}$, and each $\mathsf{L}_{\mathsf{nfl}} \in \{\mathsf{S}_{\mathsf{nfl}}, \mathsf{NFL}\}$, it holds that:
  - $\Gamma \models_{\mathsf{L}_{\mathsf{etl}}} \Delta$ iff for each valuation v, $v(\gamma) = \texttt{T}$ (for each $\gamma \in \Gamma$) implies $v(\delta) = \texttt{T}$ (for some $\delta \in \Delta$);
  - $\Gamma \models_{\mathsf{L}_{\mathsf{fde}}} \Delta$ iff for each valuation v, $v(\gamma) \in \{\texttt{T}, \texttt{B}\}$ (for each $\gamma \in \Gamma$) implies $v(\delta) \in \{\texttt{T}, \texttt{B}\}$ (for some $\delta \in \Delta$);
  - $\Gamma \models_{\mathsf{L}_{\mathsf{nfl}}} \Delta$ iff for each valuation v, $v(\gamma) \in \{\texttt{T}, \texttt{B}, \texttt{N}\}$ (for each $\gamma \in \Gamma$) implies $v(\delta) \in \{\texttt{T}, \texttt{B}, \texttt{N}\}$ (for some $\delta \in \Delta$).

Semantics

- It should be noted that both S_{etl} and S_{nfl} appeared in Szmuc's paper 'Defining LFIs and LFUs in extensions of infectious logics' (2017) under the names $L_{\mathfrak{ne}}$ and $L_{\mathfrak{bb}'}$, respectively, where they are mentioned as the subsystems of K_3^w and PWK. S_{etl} is also independently appeared in Priest's paper 'Natural Deduction Systems for Logics in the FDE Family' (2020).
- Although S_{etl} and S_{nfl} are indeed subsystems of K_3^w and PWK, we find it reasonable to trace another, as we think, more closer connection of S_{etl} and S_{nfl} with different logics.

Semantics

$\wedge$	T	B	N	F
T	T	B	N	F
B	B	B	N	F
N	N	N	N	N
F	F	F	N	F

$\vee$	T	B	N	F
T	T	T	N	T
B	T	B	N	B
N	N	N	N	N
F	T	B	N	F

φ	$\sim\varphi$
T	F
B	B
N	N
F	T

Thus, If we delete N, then we obtain tables for K_3 and LP!

$\wedge$	T	B	F
T	T	B	F
B	B	B	F
F	F	F	F

$\vee$	T	B	F
T	T	T	T
B	T	B	B
F	T	B	F

φ	$\sim\varphi$
T	F
B	B
F	T

Sequent Calculi

The connection between S_{etl}, S_{nfl} and K_3, LP, respectively, helps to find a formalization for our logics.

We present S_{etl} and S_{nfl} in the form of Gentzen-type sequent calculi. We can use corresponding calculi for K_3 and LP as the basic step toward this goal. These calculi are well-known in the literature, for example, from A. Avron's paper 'Classical Gentzen-type Methods in Propositional Many-valued Logics' (2005).

Recall the sequent calculus for K_3:

- Axioms: $\varphi \Rightarrow \varphi \qquad \varphi, \sim\varphi \Rightarrow$
- Structural rules: (Cut), (L-Weakening), (R-Weakening).
- Logical rules:

$$(\wedge \Rightarrow)\ \frac{\varphi, \psi, \Gamma \Rightarrow \Delta}{\varphi \wedge \psi, \Gamma \Rightarrow \Delta} \qquad (\Rightarrow \wedge)\ \frac{\Gamma \Rightarrow \Delta, \varphi \quad \Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi \wedge \psi}$$

$$(\vee \Rightarrow)\ \frac{\varphi, \Gamma \Rightarrow \Delta \quad \psi, \Gamma \Rightarrow \Delta}{\varphi \vee \psi, \Gamma \Rightarrow \Delta} \qquad (\Rightarrow \vee)\ \frac{\Gamma \Rightarrow \Delta, \varphi, \psi}{\Gamma \Rightarrow \Delta, \varphi \vee \psi}$$

$$(\sim\sim \Rightarrow)\ \frac{\varphi, \Gamma \Rightarrow \Delta}{\sim\sim\varphi, \Gamma \Rightarrow \Delta} \qquad (\Rightarrow \sim\sim)\ \frac{\Gamma \Rightarrow \Delta, \varphi}{\Gamma \Rightarrow \Delta, \sim\sim\varphi}$$

$$(\sim\vee \Rightarrow)\ \frac{\sim\varphi, \sim\psi, \Gamma \Rightarrow \Delta}{\sim(\varphi \vee \psi), \Gamma \Rightarrow \Delta} \qquad (\Rightarrow \sim\vee)\ \frac{\Gamma \Rightarrow \Delta, \sim\varphi \qquad \Gamma \Rightarrow \Delta, \sim\psi}{\Gamma \Rightarrow \Delta, \sim(\varphi \vee \psi)}$$

$$(\sim\wedge \Rightarrow)\ \frac{\sim\varphi, \Gamma \Rightarrow \Delta \qquad \sim\psi, \Gamma \Rightarrow \Delta}{\sim(\varphi \wedge \psi), \Gamma \Rightarrow \Delta} \qquad (\Rightarrow \sim\wedge)\ \frac{\Gamma \Rightarrow \Delta, \sim\varphi, \sim\psi}{\Gamma \Rightarrow \Delta, \sim(\varphi \wedge \psi)}$$

Sequent Calculi

Recall the sequent calculus for LP:

- Axioms: $\varphi \Rightarrow \varphi \qquad \Rightarrow \varphi, \sim\varphi$
- Structural rules: (Cut), (L-Weakening), (R-Weakening).
- Logical rules:

$$(\wedge \Rightarrow)\ \frac{\varphi, \psi, \Gamma \Rightarrow \Delta}{\varphi \wedge \psi, \Gamma \Rightarrow \Delta} \qquad (\Rightarrow \wedge)\ \frac{\Gamma \Rightarrow \Delta, \varphi \quad \Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi \wedge \psi}$$

$$(\vee \Rightarrow)\ \frac{\varphi, \Gamma \Rightarrow \Delta \quad \psi, \Gamma \Rightarrow \Delta}{\varphi \vee \psi, \Gamma \Rightarrow \Delta} \qquad (\Rightarrow \vee)\ \frac{\Gamma \Rightarrow \Delta, \varphi, \psi}{\Gamma \Rightarrow \Delta, \varphi \vee \psi}$$

$$(\sim\sim \Rightarrow)\ \frac{\varphi, \Gamma \Rightarrow \Delta}{\sim\sim\varphi, \Gamma \Rightarrow \Delta} \qquad (\Rightarrow \sim\sim)\ \frac{\Gamma \Rightarrow \Delta, \varphi}{\Gamma \Rightarrow \Delta, \sim\sim\varphi}$$

$$(\sim\vee \Rightarrow)\ \frac{\sim\varphi, \sim\psi, \Gamma \Rightarrow \Delta}{\sim(\varphi \vee \psi), \Gamma \Rightarrow \Delta} \qquad (\Rightarrow \sim\vee)\ \frac{\Gamma \Rightarrow \Delta, \sim\varphi \qquad \Gamma \Rightarrow \Delta, \sim\psi}{\Gamma \Rightarrow \Delta, \sim(\varphi \vee \psi)}$$

$$(\sim\wedge \Rightarrow)\ \frac{\sim\varphi, \Gamma \Rightarrow \Delta \qquad \sim\psi, \Gamma \Rightarrow \Delta}{\sim(\varphi \wedge \psi), \Gamma \Rightarrow \Delta} \qquad (\Rightarrow \sim\wedge)\ \frac{\Gamma \Rightarrow \Delta, \sim\varphi, \sim\psi}{\Gamma \Rightarrow \Delta, \sim(\varphi \wedge \psi)}$$

- To obtain a sequent calculus for $\mathsf{S}_{\mathsf{etl}}$ it is sufficient to replace $(\Rightarrow \vee)$ and $(\Rightarrow \sim\wedge)$ in K_3 with their restriced versions:

$$(\Rightarrow \vee^*)\ \frac{\Gamma \Rightarrow \Delta, \varphi, \psi}{\Gamma \Rightarrow \Delta, \varphi \vee \psi} \quad \text{provided that } \mathit{var}(\{\varphi, \psi\}) \subseteq \mathit{var}(\Gamma)$$

$$(\Rightarrow \sim\wedge^*)\ \frac{\Gamma \Rightarrow \Delta, \sim\varphi, \sim\psi}{\Gamma \Rightarrow \Delta, \sim(\varphi \wedge \psi)} \quad \text{provided that } \mathit{var}(\{\varphi, \psi\}) \subseteq \mathit{var}(\Gamma)$$

- To obtain a sequent calculus for $\mathsf{S}_{\mathsf{nfl}}$ it is sufficient to replace $(\wedge \Rightarrow)$ and $({\sim}\vee \Rightarrow)$ in LP with their restricted versions:

$$(\wedge^* \Rightarrow)\ \frac{\varphi, \psi, \Gamma \Rightarrow \Delta}{\varphi \wedge \psi, \Gamma \Rightarrow \Delta} \quad \text{provided that } var(\{\varphi, \psi\}) \subseteq var(\Delta)$$

$$({\sim}\vee^* \Rightarrow)\ \frac{{\sim}\varphi, {\sim}\psi, \Gamma \Rightarrow \Delta}{{\sim}(\varphi \vee \psi), \Gamma \Rightarrow \Delta} \quad \text{provided that } var(\{\varphi, \psi\}) \subseteq var(\Delta)$$

The same sequent calculi for S_{etl} and S_{nfl} can be obtained from other source. Recall M. Coniglio and M. I. Corbalan formalization of K_3^w and PWK.

- The sequent calculus for K_3^w is the restriction of the one for classical logic. It has $(\Rightarrow \vee^*)$ instead of the usual $(\Rightarrow \vee)$ and the following rules for negation:

$$(\sim\Rightarrow)\ \frac{\varphi, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \sim\varphi} \qquad (\Rightarrow\sim^*)\ \frac{\Gamma \Rightarrow \Delta, \varphi}{\sim\varphi, \Gamma \Rightarrow \Delta} \quad \text{provided that } \mathit{var}(\varphi) \subseteq \mathit{var}(\Gamma)$$

- The sequent calculus for PWK restricts the calculus for classical logic in a dual way. It has $(\wedge^* \Rightarrow)$ instead of the usual $(\wedge \Rightarrow)$ and the following rules for negation:

$$(\sim^* \Rightarrow)\ \frac{\varphi, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \sim\varphi} \quad \text{provided that } \mathit{var}(\varphi) \subseteq \mathit{var}(\Delta) \qquad (\Rightarrow \sim)\ \frac{\Gamma \Rightarrow \Delta, \varphi}{\sim\varphi, \Gamma \Rightarrow \Delta}$$

Thus, we can naturally obtain the sequent calculi for $\mathsf{S_{etl}}$ and $\mathsf{S_{nfl}}$ from the systems of Coniglio and Corbalan.

- To obtain a sequent calculus for $\mathsf{S_{etl}}$ we add the axiom $\varphi, {\sim}\varphi \Rightarrow$ to $\mathsf{K}_3^{\mathsf{w}}$ and replace the negation rules with the following ones:

$$({\sim}{\sim} \Rightarrow)\ \frac{\varphi, \Gamma \Rightarrow \Delta}{{\sim}{\sim}\varphi, \Gamma \Rightarrow \Delta} \qquad (\Rightarrow {\sim}{\sim})\ \frac{\Gamma \Rightarrow \Delta, \varphi}{\Gamma \Rightarrow \Delta, {\sim}{\sim}\varphi}$$

$$(\Rightarrow {\sim}\wedge^*)\ \frac{\Gamma \Rightarrow \Delta, {\sim}\varphi, {\sim}\psi}{\Gamma \Rightarrow \Delta, {\sim}(\varphi \wedge \psi)} \quad \text{provided that } \mathit{var}(\{\varphi, \psi\}) \subseteq \mathit{var}(\Gamma)$$

$$({\sim}\vee \Rightarrow)\ \frac{{\sim}\varphi, {\sim}\psi, \Gamma \Rightarrow \Delta}{{\sim}(\varphi \vee \psi), \Gamma \Rightarrow \Delta} \qquad (\Rightarrow {\sim}\vee)\ \frac{\Gamma \Rightarrow \Delta, {\sim}\varphi \qquad \Gamma \Rightarrow \Delta, {\sim}\psi}{\Gamma \Rightarrow \Delta, {\sim}(\varphi \vee \psi)}$$

$$({\sim}\wedge \Rightarrow)\ \frac{{\sim}\varphi, \Gamma \Rightarrow \Delta \qquad {\sim}\psi, \Gamma \Rightarrow \Delta}{{\sim}(\varphi \wedge \psi), \Gamma \Rightarrow \Delta}$$

Sequent Calculi

- To obtain a sequent calculus for S_{nfl} we add the axiom $\Rightarrow \varphi, {\sim}\varphi$ to PWK and replace the negation rules with the following ones:

$$({\sim}{\sim} \Rightarrow)\ \frac{\varphi, \Gamma \Rightarrow \Delta}{{\sim}{\sim}\varphi, \Gamma \Rightarrow \Delta} \qquad (\Rightarrow {\sim}{\sim})\ \frac{\Gamma \Rightarrow \Delta, \varphi}{\Gamma \Rightarrow \Delta, {\sim}{\sim}\varphi}$$

$$({\sim}\vee^* \Rightarrow)\ \frac{{\sim}\varphi, {\sim}\psi, \Gamma \Rightarrow \Delta}{{\sim}(\varphi \vee \psi), \Gamma \Rightarrow \Delta} \quad \text{provided that } \mathit{var}(\{\varphi, \psi\}) \subseteq \mathit{var}(\Delta)$$

$$(\Rightarrow {\sim}\vee)\ \frac{\Gamma \Rightarrow \Delta, {\sim}\varphi \qquad \Gamma \Rightarrow \Delta, {\sim}\psi}{\Gamma \Rightarrow \Delta, {\sim}(\varphi \vee \psi)}$$

$$({\sim}\wedge \Rightarrow)\ \frac{{\sim}\varphi, \Gamma \Rightarrow \Delta \qquad {\sim}\psi, \Gamma \Rightarrow \Delta}{{\sim}(\varphi \wedge \psi), \Gamma \Rightarrow \Delta} \qquad (\Rightarrow {\sim}\wedge)\ \frac{\Gamma \Rightarrow \Delta, {\sim}\varphi, {\sim}\psi}{\Gamma \Rightarrow \Delta, {\sim}(\varphi \wedge \psi)}$$

Sequent Calculi: Summary

Observation:

- We can treat S_{etl} twofold: either as the weakening of the disjunction in K_3, or as the weakening of the negation in K_3^w.
- We can treat S_{nfl} twofold: either as the weakening of the conjunction in LP, or as the weakening of the negation in PWK.

Results:

- Completeness, soundness and cut-elimination are proved for S_{etl} and S_{nfl}.

Some interesting result concerning the consequence relations

Consequence relations in $\mathsf{S}_{\mathsf{etl}}$ and $\mathsf{S}_{\mathsf{nfl}}$ can be characterized with respect to the consequence relations in K_3 and LP:

- If $Var(\Delta) \subseteq Var(\Gamma)$ and $\Gamma \models_{\mathsf{K}_3} \Delta$, then $\Gamma \models_{\mathsf{S}_{\mathsf{etl}}} \Delta$.
- If $Var(\Gamma) \subseteq Var(\Delta)$ and $\Gamma \models_{\mathsf{LP}} \Delta$, then $\Gamma \models_{\mathsf{S}_{\mathsf{nfl}}} \Delta$.

Conclusion

- Exactly true and non-falsity versions of FDE are extensions of FDE, whereas exactly true and non-falsity versions of S_{fde} are not extensions of S_{fde}.
- S_{etl} and S_{nfl} obey distinct criteria of relevance.
- S_{etl} and S_{nfl} can be seen either as the weakenings of disjunction and conjunction in K_3 and LP, or as the weakenings of negation in K_w^3 and PWK.
- S_{etl} and S_{nfl} can be formalized in terms of Gentzen-type sequent calculi, enjoying cut-elimination.

Thank you!