The Number of Axioms

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- on the number of steps in a minimal cut-free proof of a provable sequent S -

in terms of the number of steps in a minimal proof of S and the number of symbols in S.

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No.

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$$\sqrt[|S|^3]{\frac{1}{2|S|^4+1}\log_2(m)} \leq \alpha$$

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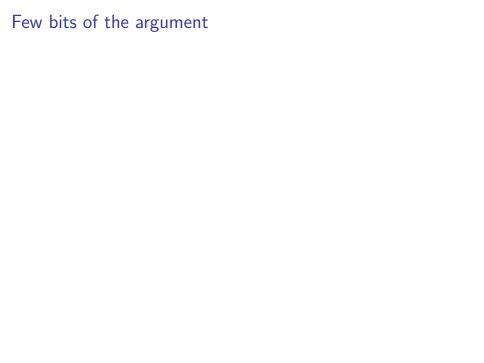
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in terms of the number of symbols in S and the number of steps in a minimal proof of S.

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Main Idea:

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number of positions can be bounded in terms of α

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"any" calculus with contraction and weakening rule

probably does not hold for Linear Logic

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