# Boxing modal logics

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**June 19, 2021**
**Logical Perspectives, Moscow**

# Modal propositional logics

Modal propositional formulas are build from the set $PL$ of proposition letters using the connectives $\to, \bot, \Box$. Other connectives ($\wedge, \vee, \Diamond, \top$ etc.) are abbreviations.

A modal logic is a set of modal formulas containing

- the classical tautologies;
- the axiom of **K**: $\Box(p_1 \to p_2) \to (\Box p_1 \to \Box p_2)$,

and closed under the rules

- (MP) $A, A \to B \,/\, A$;
- (Nec) $A \,/\, \Box A$;
- (Sub) $A \,/\, SA$, where $S$ is a propositional substitution.

The minimal modal logic is **K**.

# Kripke semantics

A Kripke frame is a non-empty set with a binary relation $F = (W, R)$.

A Kripke model over $F$ is a pair $M = (\Phi, \theta)$, where $\theta : PL \longrightarrow 2^W$ is a valuation.

The inductive definition of the truth of a modal formula $A$ at a point $u$ of a model $M$ ($M, u \vDash A$) is standard.

A formula $A$ is valid on a frame $F$ ($F \vDash A$) if
$M, u \vDash A$ for every point $u$ of every model $M$ over $F$.

$\mathbf{L}(F) := \{A \mid F \vDash A\}$ is the modal logic of a frame $F$.

$\mathbf{L}(\mathcal{C}) := \bigcap\{\mathbf{L}(F) \mid F \in \mathcal{C}\}$ is the modal logic of a class of frames $\mathcal{C}$, or the modal logic determined by $\mathcal{C}$.

# Completeness, strong completeness, FMP, canonicity

Logics of the form $\mathbf{L}(\mathcal{C})$ (or, equivalently, $\mathbf{L}(F)$)) are called (Kripke) complete.

A logic $\mathbf{\Lambda}$ is strongly (Kripke) complete if every $\mathbf{\Lambda}$-consistent set of formulas is satisfiable at some point of a Kripke model over a frame validating $\mathbf{\Lambda}$.

Logics of the form $\mathbf{L}(\mathcal{C})$, where $\mathcal{C}$ is a class of finite frames, are said to have the finite model property (FMP).

**FACT 1** Every finitely axiomatizable logic with the FMP is decidable.

**FACT 2** For any propositional logic $\mathbf{\Lambda}$ there exists a canonical model (whose worlds are maximal $\mathbf{\Lambda}$-consistent sets of formulas) $M_{\mathbf{\Lambda}} = (F_{\mathbf{\Lambda}}, \theta_{\mathbf{\Lambda}})$ such that

$$M_{\mathbf{\Lambda}} \vDash A \text{ iff } \mathbf{\Lambda} \vdash A.$$

A logic $\mathbf{\Lambda}$ is canonical if $F_{\mathbf{\Lambda}} \vDash \mathbf{\Lambda}$. So

every canonical logic is strongly Kripke complete.

# Boxing propositional logics

**Definition**

For a set of modal formulas $\Gamma$, put

$$\Box\Gamma := \{\Box A \mid A \in \Gamma\}.$$

For a modal propositional logic $\mathbf{\Lambda}$ put

$$\Box\cdot\mathbf{\Lambda} := \mathbf{K} + \Box\mathbf{\Lambda}.$$

**Lemma 1**

- $\mathbf{K} + \Gamma \vdash A$ iff $\mathbf{K} + \Box\Gamma \vdash \Box A$.
- $\Box\cdot(\mathbf{K} + \Gamma) = \mathbf{K} + \Box\Gamma$.

It turns out that $\Box\cdot\mathbf{\Lambda}$ inherits many properties of $\mathbf{\Lambda}$.

# Boxing propositional logics

**Theorem 1**

- If $\mathbf{\Lambda}$ is Kripke complete, then $\Box{\cdot}\mathbf{\Lambda}$ is Kripke complete.
- If $\mathbf{\Lambda}$ has the FMP, then $\Box{\cdot}\mathbf{\Lambda}$ has the FMP.
- If $\mathbf{\Lambda}$ is canonical, then $\Box{\cdot}\mathbf{\Lambda}$ is canonical.
- If $\mathbf{\Lambda}$ is strongly Kripke complete, then $\Box{\cdot}\mathbf{\Lambda}$ is strongly Kripke complete.
- If $\mathbf{\Lambda}$ is locally tabular, then $\Box{\cdot}\mathbf{\Lambda}$ is locally tabular.
- If $\mathbf{\Lambda}$ is has a finite modal depth, then $\Box{\cdot}\mathbf{\Lambda}$ has a finite modal depth:
$$md(\Box{\cdot}\mathbf{\Lambda}) \leq md(\mathbf{\Lambda}) + 1.$$

# Modal predicate logics

Modal predicate formulas are build from the countable set of individual variables $Var$, predicate letters $P_k^n$ $(n, k \geq 0)$ using the connectives $\rightarrow, \bot, \Box$, and the quantifier $\forall$.

A modal predicate logic is a set of modal predicate formulas containing

- the classical predicate tautologies;
- the axiom of **K**: $\Box(P_1^0 \rightarrow P_2^0) \rightarrow (\Box P_1^0 \rightarrow \Box P_2^0)$,

and closed under the rules

- (MP) $A, A \rightarrow B \,/\, A$;
- (Nec) $A \,/\, \Box A$;
- (Gen) $A \,/\, \forall x A$;
- (Sub) $A \,/\, SA$, where $S$ is a predicate substitution.

**QK** is the minimal modal predicate logic.
$\mathbf{Q\Lambda}$ is the minimal predicate extension of a propositional logic $\mathbf{\Lambda}$.

# Kripke semantics

A *predicate Kripke frame* over a propositional frame $F = (W, R)$ is a pair $\mathbf{F} = (F, D)$, where $D = (D_u)_{u \in W}$, $D_u \neq \varnothing$ is an expanding system of domains:

$$uRv \Rightarrow D_u \subseteq D_v.$$

A *valuation* $\xi$ in $\mathbf{F}$ is a function sending every $n$-ary predicate letter $P_k^n$ to a family of $n$-ary relations on the domains:

$$\xi(P_k^n) = (\xi_u(P_k^n))_{u \in W},$$

where $\xi_u(P_k^n) \subseteq D_u^n$ for $n \neq 0$ and $\xi_u(P_k^0) \in \{0, 1\}$.
The pair $M = (\mathbf{F}, \xi)$ is a *Kripke model* over $\mathbf{F}$.

# Kripke semantics

Given $M$, at every point $u \in W$ we can evaluate *modal* $D_u$*-sentences*, i.e. modal sentences with constants from $D_u$:

$$M, u \vDash P_k^n(a_1, \ldots, a_n) \text{ iff } (a_1, \ldots, a_n) \in \xi_u(P_k^n),$$

$$M, u \vDash P_k^0 \text{ iff } \xi_u(P_k^0) = 1,$$

$$M, u \vDash A \to B \text{ iff } (M, u \nvDash A \text{ or } M, u \vDash B),$$

$$M, u \nvDash \bot,$$

$$M, u \vDash \forall x A(x) \text{ iff } \forall a \in D_u \; M, u \vDash A(a),$$

$$M, u \vDash \Box A \text{ iff } \forall v \in R(u) \; M, v \vDash A.$$

# Kripke semantics

A modal formula $A(x_1, \ldots, x_n)$ is called *true in* $M$ ($M \vDash A(x_1, \ldots, x_n)$) if $M, u \vDash A(\mathbf{a})$ for every $u \in W$ and $\mathbf{a} \in D_u^n$.

A modal formula $A$ is *valid* on a frame $\mathbf{F}$ (in symbols, $\mathbf{F} \vDash A$) if it is true in every Kripke model over $\mathbf{F}$.

$\mathbf{ML}(\mathbf{F}) := \{A \mid \mathbf{F} \vDash A\}$ is the *modal logic of* $\mathbf{F}$.
The *modal logic of a class of frames* $\mathcal{C}$ is
$\mathbf{ML}(\mathcal{C}) := \bigcap\{\mathbf{ML}(\mathbf{F}) \mid \mathbf{F} \in \mathcal{C}\}$.
Logics of this form are called *Kripke complete.*

A frame validating a modal predicate logic $L$ is called an *$L$-frame.*
A formula $A$ is a *logical consequence* of a logic $L$ in Kripke semantics ($L \vDash_{\mathcal{K}} A$) if $A$ is valid on all $L$-frames.

$\widehat{L} := \{A \mid L \vDash_{\mathcal{K}} A\}$ is the smallest Kripke complete extension of $L$, the *Kripke completion* of $L$.

# Strong completeness

**Definition**

A (modal predicate) theory is a set $\Gamma$ of formulas with constants. $\Gamma$ is $L$-consistent if $\Gamma \not\vdash_L \bot$ (i.e. $\bot$ is not derivable from $L \cup \Gamma$ using MP).

**Definition**

A theory $\Gamma$ with a set of constants $E$ is satisfiable in a Kripke model $M$ at point $u$ if there exists a map $\delta : E \longrightarrow D_u$ such that $M, u \vDash \delta \cdot \Gamma$ (where $\delta \cdot \Gamma$ is obtained from $\Gamma$ by replacing each $c$ with $\delta(c)$).

**Definition**

A modal predicate logic $L$ is strongly Kripke complete if every $L$-consistent theory is satisfiable in some Kripke model over an $L$-frame.

So strong completeness implies completeness.

# Canonical models

**Canonical model theorem**

For any predicate logic $L$ there exists a canonical model (its worlds are maximal $L$-consistent theories with extra constants taken from a fixed set) $VM_L = (VF_L, \theta_L)$ such that for any formula $A$

$$M_L \vDash A \text{ iff } L \vdash A.$$

**Definition**

A logic $L$ is canonical if $VF_L \vDash L$.

**Corollary**

Every canonical logic is strongly Kripke complete.

# Boxing predicate logics

**Definition**

For a modal predicate logic $L$ put $\Box{\cdot}L := \mathbf{QK} + \Box L$.

**Lemma 2**

$L \vdash A$ iff $\Box{\cdot}L \vdash \Box A$.

Thus

$$\mathbf{QK} + \Box\Gamma \subseteq \Box{\cdot}(\mathbf{QK} + \Gamma).$$

**Problem.** Axiomatize $\Box{\cdot}(\mathbf{QK} + \Gamma)$.

**Lemma 3**

If $\mathbf{QT} \subseteq \mathbf{QK} + \Gamma$, then $\Box{\cdot}(\mathbf{QK} + \Gamma) = \mathbf{QK} + \Box\Gamma + \Box\forall ref$, where

$$\Box\forall ref := \Box\forall x(\Box P(x) \to P(x)).$$

# Preservation theorem for boxing

**Theorem 2**

1. Predicate boxing preserves canonicity.
2. If $\mathbf{Q\Lambda}$ is strongly Kripke complete, then $\Box\cdot(\mathbf{Q\Lambda})$ is strongly Kripke complete.

# Some counterexamples

In general predicate boxing does not preserve Kripke completeness (neither weak, nor strong). Consider the logics

$$\mathbf{Q\Lambda U_1} := \mathbf{Q\Lambda} + AU_1,$$

where $\mathbf{\Lambda}$ is a propositional modal logic,

$$AU_1 := \exists x P(x) \to \forall x P(x)$$

is the axiom of singleton domains.

**Proposition 3**

Let $\mathbf{\Lambda}$ be a strongly complete consistent modal propositional logic. Then

- $\mathbf{Q\Lambda U_1}$ is strongly Kripke complete.
- $\Box \cdot \mathbf{Q\Lambda U_1} \vDash_{\mathcal{K}} AU_1$, but $\Box \cdot \mathbf{Q\Lambda U_1} \nvdash AU_1$.

  Thus $\Box \cdot \mathbf{Q\Lambda U_1}$ is Kripke incomplete.

# Kripke sheaves

**Definition**

A *Kripke sheaf* over a propositional Kripke frame $F = (W, R)$ is a triple $\Phi = (F, D, \rho)$ where $(F, D)$ is a system of expanding domains, $\rho = (\rho_{uv})_{(u,v)\in R^*}$ is a family[a] of *transition functions* $\rho_{uv} : D_u \longrightarrow D_v$ such that

- for every $u \in W$, $\rho_{uu} = id_{D_u}$ (the identity function on $D_u$);
- $uR^*vR^*w$ implies $\rho_{vw}\rho_{uv} = \rho_{uw}$.

[a] $R^*$ denotes the reflexive transitive closure of $R$.

**Definition**

A *valuation* on a Kripke sheaf $\Phi$ is a function $\xi$ on predicate letters such that for every $n$-ary predicate letter $P^n_k$

$$\xi(P^n_k) = (\xi_u(P^n_k))_{u\in W},$$

where $\xi_u(P^n_k) \subseteq D^n_u$ for $n \neq 0$ and $\xi_u(P^0_k) \in \{0, 1\}$.
The pair $M = (\mathbf{F}, \xi)$ is a *Kripke sheaf model* over $\Phi$.

# Kripke sheaves

The definition of $M, u \models A$ for $u \in W$ and a $D_u$-sentence $A$ is recursive:

$$
\begin{array}{l}
M, u \models P^n_k(a_1, \dots, a_n) \text{ iff } (a_1, \dots, a_n) \in \xi_u(P^n_k), \\
M, u \models P^0_k \text{ iff } \xi_u(P^0_k) = 1, \\
M, u \models A \to B \text{ iff } (M, u \not\models A \text{ or } M, u \models B), \\
M, u \not\models \bot, \\
M, u \models \forall x A(x) \text{ iff } \forall a \in D_u \; M, u \models A(a), \\
M, u \models \Box A \text{ iff } \forall v \in R(u) \; M, v \models A|v,
\end{array}
$$

where $A|v$ denotes the $D_v$-sentence obtained from $A$ by replacing every individual $a \in D_u$ with $\rho_{uv}(a)$.
A modal formula $A(x_1, \dots, x_n)$ is called *true in* $M$ (in symbols, $M \models A(x_1, \dots, x_n)$) if $M, u \models A(\mathbf{a})$ for every $u \in W$ and $\mathbf{a} \in D^n_u$.
A modal formula $A$ is *valid* on a Kripke sheaf $\Phi$ (in symbols, $\Phi \models A$) if it is true in every Kripke sheaf model over $\Phi$.

# Kripke sheaves

By Soundness theorem, $\mathbf{ML}(\Phi) := \{A \mid \Phi \vDash A\}$ is a modal predicate logic.

The *modal logic of a class of Kripke sheaves* $\mathcal{C}$ is $\mathbf{ML}(\mathcal{C}) := \bigcap\{\mathbf{ML}(\Phi) \mid \Phi \in \mathcal{C}\}$.

Logics $\mathbf{ML}(\mathcal{C})$ are called *Kripke sheaf complete*.

**Remark.** Kripke completeness implies Kripke sheaf completeness.

**Definition**

A modal predicate logic $L$ is called *strongly Kripke sheaf complete* if every $L$-consistent theory is satisfiable in a Kripke sheaf model over a Kripke sheaf validating $L$.

**Theorem 3**

Boxing preserves strong Kripke sheaf completeness.

# Incompleteness and completions

**Theorem 4**

Let $\mathbf{\Lambda}$ be a consistent propositional logic containing $\mathbf{T} = \mathbf{K} + \Box p \to p$. Then

1. $\mathbf{Q}(\Box \cdot \mathbf{\Lambda})$ is Kripke sheaf incomplete.
2. If $\mathbf{Q\Lambda}$ is strongly Kripke complete, then

$$\widehat{\mathbf{Q}(\Box \cdot \mathbf{\Lambda})} = \Box \cdot \mathbf{Q\Lambda} = \mathbf{Q}(\Box \cdot \mathbf{\Lambda}) + \Box \forall ref.$$

# Incompleteness and completions

Examples of the logics $\mathbf{\Lambda}$, for which $\mathbf{Q\Lambda}$ is strongly Kripke complete.

- One-way PTC logics:

$$\mathbf{K} + \Box p \to \Box^n p + \text{ variable-free } \text{ formulas.}$$

  E.g. $\mathbf{T}$, $\mathbf{S4}$, $\mathbf{K4}$,...
- Logics with confluence and density axioms: $\mathbf{S4.2}$, $\mathbf{K4} + \Box^2 p \to \Box p$, $\mathbf{S4.2} + \Box^2 p \to \Box p$,...
- Logics with non-branching axioms: $\mathbf{S4.3}$, $\mathbf{K4.3}$.
- $\mathbf{S5}$ and its extensions.

# THANK YOU!