On multilattice counterparts of **MNT4**, **S4**, and **S5**

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- Multilattice logic generalizes logics of billatices, trilattices, and tetralattices which were studied, respectively, by Arieli & Avron (1996), Shramko & Wansing (2005), and Zaitsev (2009).
- The logic of bilattices itself generalizes Belnap and Dunn's logic which algebraic semantics is based on De Morgan lattices.

The family of multilattice logics

- Multilattice logic $\mathbf{ML}_n$ itself (Shramko, 2016), which is also refered as classical multilattice logic,
- first-order multilattice logic $\mathbf{FML}_n$ (Kamide and Shramko, 2017),
- modal multilattice logic $\mathbf{MML}_n$ (Kamide and Shramko, 2017),
- linear multilattice logics $\mathbf{LML}_n$ and $\mathbf{EML}_n$ (Kamide, 2017),
- bi-intuitionistic multilattice logic $\mathbf{BML}_n$ as well as connexive multilattice logic $\mathbf{CML}_n$ (Kamide, Shramko, Wansing, 2017).

We are extending this list by the following logics:

- $\mathbf{MML}_n^{\mathbf{S5}}$, $\mathbf{MML}_n^{\mathbf{S4}}$, $\mathbf{MML}_n^{\mathbf{MNT4}}$.

The language of multilattice logic

Let $n > 1$ and $1 \leqslant j \leqslant n$. $\mathcal{P} = \{p_i, q_i, r_i \mid i \in \mathbb{N}\}$ is a set of propositional variables; $\mathcal{P}^j = \{p^j \mid p \in \mathcal{P}\}$ us a set of indexed propositional variables; $\mathcal{P}^* = \bigcup_{i=1}^{i=n} \mathcal{P}^i$; $\mathcal{C} = \{\neg, \Box, \Diamond, \wedge, \vee, \rightarrow, \leftarrow\}$; $\mathcal{C}^* = \bigcup_{i=1}^{i=n} \{\neg_i, \wedge_i, \vee_i, \rightarrow_i, \leftarrow_i\}$; $\mathcal{C}^\star = \bigcup_{i=1}^{i=n} \{\Box_i, \Diamond_i\}$. Let us fix three languages which we are going to use in the sequel.

- $\mathscr{L}$ is the language of the modal logics **MNT4**, **S4**, and **S5**. It has the alphabet $\langle \mathcal{P}, \mathcal{P}^*, \mathcal{C}, (,) \rangle$.
- $\mathscr{L}_{\mathrm{N}}$ is the language of multilattice logic $\mathbf{ML}_n$. It has the alphabet $\langle \mathcal{P}, \mathcal{C}^*, (,) \rangle$.
- $\mathscr{L}_{\mathrm{M}}$ is the language of the modal multilattice logics $\mathbf{MML}_n^{\mathbf{MNT4}}$, $\mathbf{MML}_n^{\mathbf{S4}}$, and $\mathbf{MML}_n^{\mathbf{S5}}$. It has the alphabet $\langle \mathcal{P}, \mathcal{C}^*, \mathcal{C}^\star, (,) \rangle$.

The sets $\mathscr{F}$, $\mathscr{F}_{\mathrm{N}}$, and $\mathscr{F}_{\mathrm{M}}$, respectively, of all $\mathscr{L}$-, $\mathscr{L}_{\mathrm{N}}$-, and $\mathscr{L}_{\mathrm{M}}$-formulas are defined in a standard inductive way.

What is multilattice?

(Kamide and Shramko, 2017, Definitions 2.1 and 2.2)

An n-dimensional *multilattice* (or just multilattice or n-lattice) is a structure $\mathscr{M}_n = \langle \mathscr{S}, \leqslant_1, \ldots, \leqslant_n \rangle$, where $n > 1$, $\mathscr{S} \neq \emptyset$, $\leqslant_1, \ldots, \leqslant_n$ are partial orders such that $\langle \mathscr{S}, \leqslant_1 \rangle, \ldots, \langle \mathscr{S}, \leqslant_n \rangle$ are lattices with the corresponding pairs of meet and join operations $\langle \cap_1, \cup_1 \rangle, \ldots, \langle \cap_n, \cup_n \rangle$ as well as the corresponding j-inversion operations $-_1, \ldots, -_n$ which satisfy the following conditions, for each $j \leqslant n$, $k \leqslant n$, $j \neq k$, and $x, y \in \mathscr{S}$:

$$x \leqslant_j y \text{ implies } -_j y \leqslant_j -_j x; \tag{anti}$$

$$x \leqslant_k y \text{ implies } -_j x \leqslant_k -_j y; \tag{iso}$$

$$-_j -_j x = x. \tag{per2}$$

Ultramultifilter

Kamide and Shramko, 2017, Definition 2.3

Let $\mathscr{M}_n = \langle \mathscr{S}, \leqslant_1, \ldots, \leqslant_n \rangle$ be an n-lattice. Then $\mathscr{U}_n \subset \mathscr{S}$ is an *n-ultrafilter* (ultramultifilter) on $\mathscr{M}_n$ iff it satisfies the following conditions, for each $j, k \leqslant n$, $j \neq k$, and $x, y \in \mathscr{S}$:

- $x \cap_i y \in \mathscr{U}_n$ iff $x \in \mathscr{U}_n$ and $y \in \mathscr{U}_n$ ($\mathscr{U}_n$ is an n-filter (multifilter) on $\mathscr{M}_n$);
- $x \cup_i y \in \mathscr{U}_n$ iff $x \in \mathscr{U}_n$ or $y \in \mathscr{U}_n$ ($\mathscr{U}_n$ is a prime n-filter on $\mathscr{M}_n$);
- $x \in \mathscr{U}_n$ iff $-_j -_k x \notin \mathscr{U}_n$.

Ultralogical multilattice

Kamide and Shramko, 2017, Definition 2.4

A pair $\langle \mathscr{M}_n, \mathscr{U}_n \rangle$ is called an ultralogical n-lattice (ultralogical multilattice) iff $\mathscr{M}_n$ is a multilattice and $\mathscr{U}_n$ is an ultramultifilter on $\mathscr{M}_n$.

Implications and coimplications

Let $\mathscr{M}_n = \langle \mathscr{S}, \leqslant_1, \ldots, \leqslant_n \rangle$ be a multilattice. Then we can define for all $\langle \mathscr{S}, \leqslant_1 \rangle, \ldots, \langle \mathscr{S}, \leqslant_n \rangle$ the corresponding pseudo-complement operations $\supset_1, \ldots, \supset_n$ as well as pseudo-difference ones $\subset_1, \ldots, \subset_n$ as follows ($x, y \in \mathscr{S}$, $j \leqslant n$, $k \leqslant n$ is fixed and $j \neq k$):

$$\begin{aligned} x \supset_j y &= -_k -_j x \cup_j y; \\ x \subset_j y &= x \cap_j -_k -_j y. \end{aligned}$$

The entailment relation

(Shramko, 2016, Definitions 4.6 and 5.3)

The entailment relation in multilattice logic $\mathbf{ML}_n$ is defined as follows, for each finite sets of formulas Γ, Δ and each formulas α, β:

(1) $\alpha \models_j \beta$ iff for each multilattice $\mathscr{M}_n$ and each valuation v, it holds that $v(\alpha) \leqslant_j v(\beta)$.

(2) $\Gamma \models_{\mathbf{ML}_n} \Delta$ iff for each ultralogical multilattice $\langle \mathscr{M}_n, \mathscr{U}_n \rangle$ and each valuation v, it holds that if $v(\gamma) \in \mathscr{U}_n$ (for each $\gamma \in \Gamma$), then $v(\delta) \in \mathscr{U}_n$ (for some $\delta \in \Delta$).

Modal multilattices

Kamide and Shramko, 2017, Definition 2.5

A multilattice $\mathscr{M}_n = \langle \mathscr{S}, \leqslant_1, \ldots, \leqslant_n \rangle$ is said to be *modal* iff for each $j \leqslant n$ the unary operations of *interior* I_j and *closure* C_j can be defined on $\mathscr{S}$ and satisfy the following conditions ($x, y \in \mathscr{S}$):

$$I_j(x) \leqslant_j x; \qquad \text{(decreasing)}$$
$$I_j(x) = I_j I_j(x); \qquad \text{(idempotent)}$$
$$I_j(x \cap_j y) \leqslant_j I_j(x) \cap_j I_j(y); \qquad \text{(sub-multiplicative)}$$
$$x \leqslant_j C_j(x); \qquad \text{(increasing)}$$
$$C_j(x) = C_j C_j(x); \qquad \text{(idempotent)}$$
$$C_j(x) \cup_j C_j(y) \leqslant_j C_j(x \cup_j y). \qquad \text{(sub-additive)}$$

Algebraic semantics for $\mathbf{MML}_n$

An algebraic completeness of $\mathbf{MML}_n$ with respect to modal multilattices is left as an open problem by Kamide and Shramko (2017). We solved this problem, but the formulation of the notion of modal multilattice has been changed. The conditions (sub-multiplicative) and (sub-additive) have been strengthened as follows:

$$I_j(x \cap_j y) = I_j(x) \cap_j I_j(y); \qquad \text{(multiplicative)}$$
$$C_j(x) \cup_j C_j(y) = C_j(x \cup_j y). \qquad \text{(additive)}$$

Moreover, the following 6 conditions have been added:

$$x \leqslant_j y \text{ implies } I_j(x) \leqslant_j I_j(y); \qquad (I\text{-monotonicity})$$
$$x \leqslant_j y \text{ implies } C_j(x) \leqslant_j C_j(y); \qquad (C\text{-monotonicity})$$
$$-_j I_j(x) = C_j(-_j x); \qquad (-_j I_j\text{-definition})$$
$$-_j C_j(x) = I_j(-_j x); \qquad (-_j C_j\text{-definition})$$
$$-_k I_j(x) = I_j(-_k x); \qquad (-_k I_j\text{-definition})$$
$$-_k C_j(x) = C_j(-_k x). \qquad (-_k C_j\text{-definition})$$

Tarski multilattice

A modal multilattice $\mathscr{M}_n = \langle \mathscr{S}, \leqslant_1, \ldots, \leqslant_n \rangle$ is said to be Tarski multilattice (or a modal multilattice with Tarski operators) iff for each $j \leq n$ operations I_j and C_j satisfy the following conditions ($-_j I_j$-definition), ($-_j C_j$-definition), ($-_k I_j$-definition), ($-_k C_j$-definition), and the following ones:

$$I_j(1) = 1; \qquad \text{(1 is open)}$$

$$C_j(0) = 0; \qquad \text{(0 is closed)}$$

$$I_j(x) = -_j -_k C_j(-_j -_k x); \qquad (I\text{-definition})$$

$$C_j(x) = -_j -_k I_j(-_j -_k x). \qquad (C\text{-definition})$$

Kuratowski multilattice

A Tarski multilattice $\mathscr{M} = \langle \mathscr{S}, \leqslant_1, \ldots, \leqslant_n \rangle$ is said to be *Kuratowski* one (or a modal multilattice with Kuratowski operators) iff for each $j \leqslant n$ the operations I_j and C_j satisfy the conditions (multiplicative) and (additive).

Halmos multilattice

A Kuratowski multilattice $\mathscr{M}_n = \langle \mathscr{S}, \leqslant_1, \ldots, \leqslant_n \rangle$ is said to be *Halmos* one (or a modal multilattice with Halmos operators) iff for each $j \leqslant n$ the operations I_j and C_j satisfy the following condition:

$$I_j(-_j I_j(x)) = -_j I_j(x); \qquad \text{(interior interconnection)}$$

$$C_j(-_j C_j(x)) = -_j C_j(x). \qquad \text{(closure interconnection)}$$

Let $\mathscr{M}_n = \langle \mathscr{S}, \leqslant_1, \ldots, \leqslant_n \rangle$ be S4-modal multilattice. Let v be a valuation as in propositional case. Then we extend it for modal formulas as follows:

(1) $v(\Box_j \alpha) = I_j v(\alpha)$;
(2) $v(\Diamond_j \alpha) = C_j v(\alpha)$.

The definition of an entailment relation for the case of a modal multilattice logic $\mathbf{L} \in \{\mathbf{MML}_n^{\mathbf{S4}}, \mathbf{MML}_n^{\mathbf{MNT4}}, \mathbf{MML}_n^{\mathbf{S5}}\}$ is almost the same as for the $\mathbf{ML}_n$ case. The only difference concerns with a type (Tarski, Kuratowski or Halmos) of a corresponding ultralogical multilattice. Thus we presume that $\Gamma \models_{\mathbf{MML}_n^{\mathbf{MNT4}}} \Delta$ ($\Gamma \models_{\mathbf{MML}_n^{\mathbf{S4}}} \Delta$, $\Gamma \models_{\mathbf{MML}_n^{\mathbf{S5}}} \Delta$) iff for each Tarski (Kuratowski, Halmos) ultralogical multilattice $\langle \mathscr{M}_n, \mathscr{U}_n \rangle$ and each valuation v, it holds that if $v(\gamma) \in \mathscr{U}_n$, for each $\gamma \in \Gamma$, then $v(\delta) \in \mathscr{U}_n$, for some $\delta \in \Delta$.

A sequent is an ordered pair written as follows: $\Gamma \Rightarrow \Delta$, where Γ and Δ are finite sets of $\mathfrak{L}$-formulas ($\mathfrak{L} \in \{\mathscr{L}, \mathscr{L}_{\mathrm{N}}, \mathscr{L}_{\mathrm{M}}\}$). A sequent is called valid for $\mathbf{L} \in \{\mathbf{ML}_n, \mathbf{MML}_n^{\mathbf{S4}}, \mathbf{MML}_n^{\mathbf{S5}}, \mathbf{MML}_n^{\mathbf{MNT4}}, \mathbf{S4}, \mathbf{S5}, \mathbf{MNT4}\}$ if $\Gamma \models_{\mathbf{L}} \Delta$ holds. When $\Gamma \Rightarrow \Delta$ is valid for $\mathbf{L}$, we write $\mathbf{L} \models \Gamma \Rightarrow \Delta$.

Let us introduce Indrzejczak's (2005) cut-free sequent calculus for **MNT4**. The only axiom is as follows (for any $p \in \mathcal{P} \cup \mathcal{P}^*$):

$$\text{(Ax) } p \Rightarrow p$$

The structural rules are as follows:

$$\text{(Cut) } \frac{\Gamma \Rightarrow \Delta, \alpha \qquad \alpha, \Theta \Rightarrow \Lambda}{\Gamma, \Theta \Rightarrow \Delta, \Lambda} \quad (\text{W}{\Rightarrow})\ \frac{\Gamma \Rightarrow \Delta}{\alpha, \Gamma \Rightarrow \Delta} \quad ({\Rightarrow}\text{W})\ \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \alpha}$$

The non-modal logical rules are as follows:

$$(\wedge\Rightarrow)\ \frac{\alpha,\beta,\Gamma\Rightarrow\Delta}{\alpha\wedge\beta,\Gamma\Rightarrow\Delta}\qquad(\Rightarrow\wedge)\ \frac{\Gamma\Rightarrow\Delta,\alpha\qquad\Gamma\Rightarrow\Delta,\beta}{\Gamma\Rightarrow\Delta,\alpha\wedge\beta}$$

$$(\vee\Rightarrow)\ \frac{\alpha,\Gamma\Rightarrow\Delta\qquad\beta,\Gamma\Rightarrow\Delta}{\alpha\vee\beta,\Gamma\Rightarrow\Delta}\qquad(\Rightarrow\vee)\ \frac{\Gamma\Rightarrow\Delta,\alpha,\beta}{\Gamma\Rightarrow\Delta,\alpha\vee\beta}$$

$$(\rightarrow\Rightarrow)\ \frac{\Gamma\Rightarrow\Delta,\alpha\qquad\beta,\Theta\Rightarrow\Lambda}{\alpha\rightarrow\beta,\Gamma,\Theta\Rightarrow\Delta,\Lambda}\qquad(\Rightarrow\rightarrow)\ \frac{\alpha,\Gamma\Rightarrow\Delta,\beta}{\Gamma\Rightarrow\Delta,\alpha\rightarrow\beta}$$

$$(\leftarrow\Rightarrow)\ \frac{\alpha,\Gamma\Rightarrow\Delta,\beta}{\alpha\leftarrow\beta,\Gamma\Rightarrow\Delta}\qquad(\Rightarrow\leftarrow)\ \frac{\Gamma\Rightarrow\Delta,\alpha\qquad\beta,\Theta\Rightarrow\Lambda}{\Gamma,\Theta\Rightarrow\Delta,\Lambda,\alpha\leftarrow\beta}$$

$$(\neg\Rightarrow)\ \frac{\Gamma\Rightarrow\Delta,\alpha}{\neg\alpha,\Gamma\Rightarrow\Delta}\qquad(\Rightarrow\neg)\ \frac{\alpha,\Gamma\Rightarrow\Delta}{\Gamma\Rightarrow\Delta,\neg\alpha}$$

The modal logical rules are as follows:

$$(M_\Box)\ \frac{\alpha \Rightarrow \beta}{\Box\alpha \Rightarrow \Box\beta} \quad (N_\Box)\ \frac{\Rightarrow \alpha}{\Rightarrow \Box\alpha} \quad (4)\ \frac{\Box\alpha \Rightarrow \beta}{\Box\alpha \Rightarrow \Box\beta} \quad (\Box \Rightarrow)\ \frac{\alpha, \Gamma \Rightarrow \Delta}{\Box\alpha, \Gamma \Rightarrow \Delta}$$

The rule $(M_\Box)$ is derivable due to (4) and $(\Box \Rightarrow)$.

- Now we present yet another sequent calculus for **MNT4** in the language with both $\Box$ and $\Diamond$.
- For the modal rules below we use the following convention:
  - the letter γ stands for the empty set or a one element set $\{\Box\beta\}$,
  - the letter δ stands for the empty set or a one element set $\{\Diamond\beta\}$.
- This sequent calculus is obtained from Indrzejczak's one by a replacement of the rules $(M_{\Box})$, $(N_{\Box})$, and (4) with the following ones:

$$(\Rightarrow\Box)\ \frac{\gamma\Rightarrow\Diamond\Lambda,\alpha}{\gamma\Rightarrow\Diamond\Lambda,\Box\alpha}\qquad(\Diamond\Rightarrow)\ \frac{\alpha,\Box\Lambda\Rightarrow\delta}{\Diamond\alpha,\Box\Lambda\Rightarrow\delta}\qquad(\Rightarrow\Diamond)\ \frac{\Gamma\Rightarrow\Delta,\alpha}{\Gamma\Rightarrow\Delta,\Diamond\alpha}$$

The rules (4), $(M_\Box)$, $(M_\Diamond)$, $(N_\Box)$, $(N_\Diamond)$ are derivable in this calculus (where $(M_\Diamond)$ and $(N_\Diamond)$ are presented below).

$$(M_\Diamond)\ \frac{\alpha \Rightarrow \beta}{\Diamond\alpha \Rightarrow \Diamond\beta} \qquad (N_\Diamond)\ \frac{\alpha \Rightarrow}{\Diamond\alpha \Rightarrow}$$

- If one replaces γ and δ in the rules $(\Rightarrow \Box)$ and $(\Diamond \Rightarrow)$ with the sets of formulas (possibly, empty, but not necessarily one element ones) $\Box\Gamma$ and $\Diamond\Delta$, respectively, then one obtains a cut-free sequent calculus for **S4** (Kripke, 1963).
- If in the rule $(\Rightarrow \Box)$ one replaces γ with $\Box\Gamma$ and puts $\delta = \emptyset$ as well as in the rule $(\Diamond \Rightarrow)$ puts $\gamma = \emptyset$ and replaces δ with $\Diamond\Delta$, then one gets an *in*complete version of **S4** (Kripke, 1963) (although its restrictions for the $\Box$-free and $\Diamond$-free languages are complete (Ohnishi, Matsumoto, 1957)), since the sequents $\Box\alpha \Rightarrow \neg\Diamond\neg\alpha$, $\neg\Diamond\neg\alpha \Rightarrow \Box\alpha$, $\neg\Box\neg\alpha \Rightarrow \Diamond\alpha$ and $\Diamond\alpha \Rightarrow \neg\Box\neg\alpha$ are unprovable in it (Kripke, 1963).
- We mention the latter fact, because Kamide and Shramko used this incomplete version of **S4** sequent calculus as a basis of their logic $\mathbf{MML}_n$. As a result (which can be checked by the use of Kamide and Shramko's embedding theorems [?]), in the system $\mathbf{MML}_n$ the sequents $\Box_j\alpha \Rightarrow \neg_k\neg_j\Diamond_j\neg_k\neg_j\alpha$, $\neg_k\neg_j\Diamond_j\neg_k\neg_j\alpha \Rightarrow \Box_j\alpha$, $\neg_k\neg_j\Box_j\neg_k\neg_j\alpha \Rightarrow \Diamond_j\alpha$ and $\Diamond_j\alpha \Rightarrow \neg_k\neg_j\Box_j\neg_k\neg_j\alpha$ are no provable.

- We present the logic $\mathbf{MML}_n^{\mathbf{S4}}$ which extends $\mathbf{MML}_n$ by these sequents. Similarly to the case of **S4**, in **MNT4** we need $\Box\Lambda$ and $\Diamond\Lambda$ in the rules $(\Rightarrow \Box)$ and $(\Diamond \Rightarrow)$, respectively, to make the sequents $\Box\alpha \Rightarrow \neg\Diamond\neg\alpha$, $\neg\Diamond\neg\alpha \Rightarrow \Box\alpha$, $\neg\Box\neg\alpha \Rightarrow \Diamond\alpha$ and $\Diamond\alpha \Rightarrow \neg\Box\neg\alpha$ provable.
- Although Indrzejczak's sequent calculus for **MNT4** in the $\Box$-language is cut-free, our calculus for **MNT4** in the language with both $\Box$ and $\Diamond$, unfortunately, seems to be non-cut-free.

- A standard **S5** sequent calculus is known to be not-cut-free.
- There are various attempts to present cut-free non-standard versions of sequent calculi (in particular, hypersequent calculi).
- Various hypersequent calculi for **S5** were developed by Mints, Pottinger, Avron, Lahav, Restall, Poggiolesi, Kurokawa, Bednarska and Indrzejczak, and Indrzejczak himself.
- We consider Restall's hypersequent calculus, since we think that it is the simplest one.
- A hypersequent is a finite multiset of sequents written as follows: $H := \Gamma_1 \Rightarrow \Delta_1 \mid \ldots \mid \Gamma_m \Rightarrow \Delta_m$. A hypersequent is called valid iff at least one of its sequents is valid.

Restall's hypersequent calculus for **S5**

In Restall's calculus, the (internal) structural and logical rules of classical logic are in the hypersequent form. For example, the rules for conjunction are as follows (here and below H and G are hypersequents):

$$(\wedge \Rightarrow)\ \frac{\alpha, \beta, \Gamma \Rightarrow \Delta \mid H}{\alpha \wedge \beta, \Gamma \Rightarrow \Delta \mid H} \qquad (\Rightarrow \wedge)\ \frac{\Gamma \Rightarrow \Delta, \alpha \mid H \qquad \Gamma \Rightarrow \Delta, \beta \mid G}{\Gamma \Rightarrow \Delta, \alpha \wedge \beta \mid H \mid G}$$

Restall's hypersequent calculus for **S5**

Restall's calculus has (Cut) and several external structural rules presented below:

$$(\mathrm{EW}\Rightarrow)\ \frac{H}{\alpha \Rightarrow\ \mid H} \qquad (\Rightarrow\mathrm{EW})\ \frac{H}{\Rightarrow \alpha \mid H}$$

$$(\mathrm{Merge})\ \frac{\Gamma \Rightarrow \Delta \mid \Theta \Rightarrow \Lambda \mid H}{\Gamma, \Theta \Rightarrow \Delta, \Lambda \mid H}$$

Restall's hypersequent calculus for **S5**

The modal logical rules:

$$(\Box \Rightarrow)\ \frac{\alpha, \Gamma \Rightarrow \Delta \mid H}{\Box\alpha \Rightarrow \mid \Gamma \Rightarrow \Delta \mid H} \qquad (\Rightarrow \Box)\ \frac{\Rightarrow \alpha \mid H}{\Rightarrow \Box\alpha \mid H}$$

$$(\Diamond \Rightarrow)\ \frac{\alpha \Rightarrow \mid H}{\Diamond\alpha \Rightarrow \mid H} \qquad (\Rightarrow \Diamond)\ \frac{\Gamma \Rightarrow \Delta, \alpha \mid H}{\Gamma \Rightarrow \Delta \mid \Rightarrow \Diamond\alpha \mid H}$$

The sequent calculus for $\mathbf{MML}_n^{\mathbf{MNT4}}$ is as follows.

Consider the axioms ($p \in \mathcal{P}$):

$$(\mathrm{Ax})\ p \Rightarrow p \qquad (\mathrm{Ax}_\neg)\ \neg_j p \Rightarrow \neg_j p$$

The structural rules are as follows: (Cut), (W⇒), and (⇒ W).

The non-negated logical rules are presented below:

$$(\wedge_j \Rightarrow)\ \frac{\alpha, \beta, \Gamma \Rightarrow \Delta}{\alpha \wedge_j \beta, \Gamma \Rightarrow \Delta} \qquad (\Rightarrow \wedge_j)\ \frac{\Gamma \Rightarrow \Delta, \alpha \qquad \Gamma \Rightarrow \Delta, \beta}{\Gamma \Rightarrow \Delta, \alpha \wedge_j \beta}$$

$$(\vee_j \Rightarrow)\ \frac{\alpha, \Gamma \Rightarrow \Delta \qquad \beta, \Gamma \Rightarrow \Delta}{\alpha \vee_j \beta, \Gamma \Rightarrow \Delta} \qquad (\Rightarrow \vee_j)\ \frac{\Gamma \Rightarrow \Delta, \alpha, \beta}{\Gamma \Rightarrow \Delta, \alpha \vee_j \beta}$$

$$(\rightarrow_j \Rightarrow)\ \frac{\Gamma \Rightarrow \Delta, \alpha \qquad \beta, \Theta \Rightarrow \Lambda}{\alpha \rightarrow_j \beta, \Gamma, \Theta \Rightarrow \Delta, \Lambda} \qquad (\Rightarrow \rightarrow_j)\ \frac{\alpha, \Gamma \Rightarrow \Delta, \beta}{\Gamma \Rightarrow \Delta, \alpha \rightarrow_j \beta}$$

$$(\leftarrow_j \Rightarrow)\ \frac{\alpha, \Gamma \Rightarrow \Delta, \beta}{\alpha \leftarrow_j \beta, \Gamma \Rightarrow \Delta} \qquad (\Rightarrow \leftarrow_j)\ \frac{\Gamma \Rightarrow \Delta, \alpha \qquad \beta, \Theta \Rightarrow \Lambda}{\Gamma, \Theta \Rightarrow \Delta, \Lambda, \alpha \leftarrow_j \beta}$$

The jj-negated logical rules are as follows:

$$(\neg_j\wedge_j \Rightarrow)\ \frac{\neg_j\alpha, \Gamma \Rightarrow \Delta \qquad \neg_j\beta, \Gamma \Rightarrow \Delta}{\neg_j(\alpha \wedge_j \beta), \Gamma \Rightarrow \Delta} \quad (\Rightarrow \neg_j\wedge_j)\ \frac{\Gamma \Rightarrow \Delta, \neg_j\alpha, \neg_j\beta}{\Gamma \Rightarrow \Delta, \neg_j(\alpha \wedge_j \beta)}$$

$$(\neg_j\vee_j \Rightarrow)\ \frac{\neg_j\alpha, \neg_j\beta, \Gamma \Rightarrow \Delta}{\neg_j(\alpha \vee_j \beta), \Gamma \Rightarrow \Delta} \quad (\Rightarrow \neg_j\vee_j)\ \frac{\Gamma \Rightarrow \Delta, \neg_j\alpha \qquad \Gamma \Rightarrow \Delta, \neg_j\beta}{\Gamma \Rightarrow \Delta, \neg_j(\alpha \vee_j \beta)}$$

$$(\neg_j\rightarrow_j \Rightarrow)\ \frac{\neg_j\beta, \Gamma \Rightarrow \Delta, \neg_j\alpha}{\neg_j(\alpha \rightarrow_j \beta), \Gamma \Rightarrow \Delta} \quad (\Rightarrow \neg_j\rightarrow_j)\ \frac{\Gamma \Rightarrow \Delta, \neg_j\beta \qquad \neg_j\alpha, \Theta \Rightarrow \Lambda}{\Gamma, \Theta \Rightarrow \Delta, \Lambda, \neg_j(\alpha \rightarrow_j \beta)}$$

$$(\neg_j\leftarrow_j \Rightarrow)\ \frac{\Gamma \Rightarrow \Delta, \neg_j\beta \qquad \neg_j\alpha, \Theta \Rightarrow \Lambda}{\neg_j(\alpha \leftarrow_j \beta), \Gamma, \Theta \Rightarrow \Delta, \Lambda} \quad (\Rightarrow \neg_j\leftarrow_j)\ \frac{\neg_j\beta, \Gamma \Rightarrow \Delta, \neg_j\alpha}{\Gamma \Rightarrow \Delta, \neg_j(\alpha \leftarrow_j \beta)}$$

$$(\neg_j\neg_j \Rightarrow)\ \frac{\alpha, \Gamma \Rightarrow \Delta}{\neg_j\neg_j\alpha, \Gamma \Rightarrow \Delta} \quad (\Rightarrow \neg_j\neg_j)\ \frac{\Gamma \Rightarrow \Delta, \alpha}{\Gamma \Rightarrow \Delta, \neg_j\neg_j\alpha}$$

The kj-negated logical rules as follows:

$$(\neg_k\wedge_j \Rightarrow)\ \frac{\neg_k\alpha, \neg_k\beta, \Gamma \Rightarrow \Delta}{\neg_k(\alpha \wedge_j \beta), \Gamma \Rightarrow \Delta} \qquad (\Rightarrow \neg_k\wedge_j)\ \frac{\Gamma \Rightarrow \Delta, \neg_k\alpha \qquad \Gamma \Rightarrow \Delta, \neg_k\beta}{\Gamma \Rightarrow \Delta, \neg_k(\alpha \wedge_j \beta)}$$

$$(\neg_k\vee_j \Rightarrow)\ \frac{\neg_k\alpha, \Gamma \Rightarrow \Delta \qquad \neg_k\beta, \Gamma \Rightarrow \Delta}{\neg_k(\alpha \vee_j \beta), \Gamma \Rightarrow \Delta} \qquad (\Rightarrow \neg_k\vee_j)\ \frac{\Gamma \Rightarrow \Delta, \neg_k\alpha, \neg_k\beta}{\Gamma \Rightarrow \Delta, \neg_k(\alpha \vee_j \beta)}$$

$$(\neg_k\rightarrow_j \Rightarrow)\ \frac{\Gamma \Rightarrow \Delta, \neg_k\alpha \qquad \neg_k\beta, \Theta \Rightarrow \Lambda}{\neg_k(\alpha \rightarrow_j \beta), \Gamma, \Theta \Rightarrow \Delta, \Lambda} \qquad (\Rightarrow \neg_k\rightarrow_j)\ \frac{\neg_k\alpha, \Gamma \Rightarrow \Delta, \neg_k\beta}{\Gamma \Rightarrow \Delta, \neg_k(\alpha \rightarrow_j \beta)}$$

$$(\neg_k\leftarrow_j \Rightarrow)\ \frac{\neg_k\alpha, \Gamma \Rightarrow \Delta, \neg_k\beta}{\neg_k(\alpha \leftarrow_j \beta), \Gamma \Rightarrow \Delta} \qquad (\neg_k\leftarrow_j \Rightarrow)\ \frac{\Gamma \Rightarrow \Delta, \neg_k\alpha \qquad \neg_k\beta, \Theta \Rightarrow \Lambda}{\Gamma, \Theta \Rightarrow \Delta, \Lambda, \neg_k(\alpha \leftarrow_j \beta)}$$

$$(\neg_k\neg_j \Rightarrow)\ \frac{\Gamma \Rightarrow \Delta, \alpha}{\neg_k\neg_j\alpha, \Gamma \Rightarrow \Delta} \qquad (\Rightarrow \neg_k\neg_j)\ \frac{\alpha, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg_k\neg_j\alpha}$$

For the modal logical rules below we adopt the following convention:

- the letter π denotes a set which is empty or consists of exactly one formula from the list of formulas $\Box_j\beta, \neg_j\Diamond_j\beta, \neg_k\Box_j\beta$, where $k \neq j$;
- the letter δ denotes a set which is empty or consists of exactly one formula from the list $\Diamond_j\beta, \neg_j\Box_j\beta, \neg_k\Diamond_j\beta$, where $k \neq j$;
- the letter $\Lambda^\sharp$ stands for the set (possibly, empty) $\{\Box_j\Lambda_1, \neg_j\Diamond_j\Lambda_2, \neg_k\Box_j\Lambda_3\}$, where $k \neq j$;
- the letter $\Lambda^\flat$ stands for the set (possibly, empty) $\{\Diamond_j\Lambda_1, \neg_j\Box_j\Lambda_2, \neg_k\Diamond_j\Lambda_3\}$, where again $k \neq j$.

The non-negated modal rules:

$$(\Box_j \Rightarrow)\ \frac{\alpha, \Gamma \Rightarrow \Delta}{\Box_j\alpha, \Gamma \Rightarrow \Delta} \qquad (\Rightarrow \Box_j)\ \frac{\pi \Rightarrow \Lambda^\flat, \alpha}{\pi \Rightarrow \Lambda^\flat, \Box_j\alpha}$$

$$(\Diamond_j \Rightarrow)\ \frac{\alpha, \Lambda^\sharp \Rightarrow \delta}{\Diamond_j\alpha, \Lambda^\sharp \Rightarrow \delta} \qquad (\Rightarrow \Diamond_j)\ \frac{\Gamma \Rightarrow \Delta, \alpha}{\Gamma \Rightarrow \Delta, \Diamond_j\alpha}$$

The jj-negated modal logical rules:

$$(\neg_j \Box_j \Rightarrow)\ \frac{\neg_j \alpha, \Lambda^\sharp \Rightarrow \delta}{\neg_j \Box_j \alpha, \Lambda^\sharp \Rightarrow \delta} \qquad (\Rightarrow \neg_j \Box_j)\ \frac{\Gamma \Rightarrow \Delta, \neg_j \alpha}{\Gamma \Rightarrow \Delta, \neg_j \Box_j \alpha}$$

$$(\neg_j \Diamond_j \Rightarrow)\ \frac{\neg_j \alpha, \Gamma \Rightarrow \Delta}{\neg_j \Diamond_j \alpha, \Gamma \Rightarrow \Delta} \qquad (\Rightarrow \neg_j \Diamond_j)\ \frac{\pi \Rightarrow \Lambda^\flat, \neg_j \alpha}{\pi \Rightarrow \Lambda^\flat, \neg_j \Diamond_j \alpha}$$

The kj-negated modal logical rules:

$$(\neg_k \Box_j \Rightarrow)\ \frac{\neg_k \alpha, \Gamma \Rightarrow \Delta}{\neg_k \Box_j \alpha, \Gamma \Rightarrow \Delta} \qquad (\Rightarrow \neg_k \Box_j)\ \frac{\pi \Rightarrow \Lambda^\flat, \neg_k \alpha}{\pi \Rightarrow \Lambda^\flat, \neg_k \Box_j \alpha}$$

$$(\neg_k \Diamond_j \Rightarrow)\ \frac{\neg_k \alpha, \Lambda^\sharp \Rightarrow \delta}{\neg_k \Diamond_j \alpha, \Lambda^\sharp \Rightarrow \delta} \qquad (\Rightarrow \neg_k \Diamond_j)\ \frac{\Gamma \Rightarrow \Delta, \neg_k \alpha}{\Gamma \Rightarrow \Delta, \neg_k \Diamond_j \alpha}$$

One can obtain sequent calculus for $\mathbf{MML}_n^{\mathbf{S4}}$ [1] from the set of above rules replacing in each of modal rule the letters π and δ, respectively, with the sets $\{\Box_j\Gamma_1, \neg_j\Diamond_j\Gamma_2, \neg_k\Box_j\Gamma_3\}$ and $\{\Diamond_j\Delta_1, \neg_j\Box_j\Delta_2, \neg_k\Diamond_j\Delta_3\}$ (where $k \neq j$).

One can obtain a hypersequent calculus for $\mathbf{MML}_n^{\mathbf{S5}}$ [1] from the sequent calculus for $\mathbf{MML}_n^{\mathbf{MNT4}}$ or $\mathbf{MML}_n^{\mathbf{S4}}$ as follows:

1. all the (internal) structural and logical non-modal rules should be presented into the hypersequent form;
2. one should add the rules (EW⇒), (⇒EW), and (Merge);
3. one should replace all the modal rules with the following ones.

The non-negated modal rules:

$$(\Box_j \Rightarrow)\ \frac{\varphi, \Gamma \Rightarrow \Delta \mid H}{\Box_j \varphi \Rightarrow \mid \Gamma \Rightarrow \Delta \mid H} \qquad (\Rightarrow \Box_j)\ \frac{\Rightarrow \varphi \mid H}{\Rightarrow \Box_j \varphi \mid H}$$

$$(\Diamond_j \Rightarrow)\ \frac{\varphi \Rightarrow \mid H}{\Diamond_j \varphi \Rightarrow \mid H} \qquad (\Rightarrow \Diamond_j)\ \frac{\Gamma \Rightarrow \Delta, \varphi \mid H}{\Gamma \Rightarrow \Delta \mid \Rightarrow \Diamond_j \varphi \mid H}$$

The jj-negated modal rules:

$$(\neg_j\Box_j \Rightarrow)\ \frac{\neg_j\varphi \Rightarrow \mid H}{\neg_j\Box_j\varphi \Rightarrow \mid H} \qquad (\Rightarrow \neg_j\Box_j)\ \frac{\Gamma \Rightarrow \Delta, \neg_j\varphi \mid H}{\Gamma \Rightarrow \Delta \mid \Rightarrow \neg_j\Box_j\varphi \mid H}$$

$$(\neg_j\Diamond_j \Rightarrow)\ \frac{\neg_j\varphi, \Gamma \Rightarrow \Delta \mid H}{\neg_j\Diamond_j\varphi \Rightarrow \mid \Gamma \Rightarrow \Delta \mid H} \qquad (\Rightarrow \neg_j\Diamond_j)\ \frac{\Rightarrow \neg_j\varphi \mid H}{\Rightarrow \neg_j\Diamond_j\varphi \mid H}$$

The kj-negated modal rules:

$$(\neg_k\Box_j \Rightarrow)\ \frac{\neg_k\varphi, \Gamma \Rightarrow \Delta \mid H}{\neg_k\Box_j\varphi \Rightarrow \mid \Gamma \Rightarrow \Delta \mid H} \qquad (\Rightarrow \neg_k\Box_j)\ \frac{\Rightarrow \neg_k\varphi \mid H}{\Rightarrow \neg_k\Box_j\varphi \mid H}$$

$$(\neg_k\Diamond_j \Rightarrow)\ \frac{\neg_k\varphi \Rightarrow \mid H}{\neg_k\Diamond_j\varphi \Rightarrow \mid H} \qquad (\Rightarrow \neg_k\Diamond_j)\ \frac{\Gamma \Rightarrow \Delta, \neg_k\varphi \mid H}{\Gamma \Rightarrow \Delta \mid \Rightarrow \neg_k\Diamond_j\varphi \mid H}$$

Soundness

All the rules of the sequent calculus $\mathbf{MML}_n^{\mathbf{MNT4}}$ are sound with respect to modal multilattices with Tarskian operators.

All the rules of the sequent calculus $\mathbf{MML}_n^{\mathbf{S4}}$ are sound with respect to modal multilattices with Kuratowski operators.

All the rules of the sequent calculus $\mathbf{MML}_n^{\mathbf{S5}}$ are sound with respect to modal multilattices with Halmos operators.

Let $\mathbf{L} \in \{\mathbf{MML}_n^{\mathbf{MNT4}}, \mathbf{MML}_n^{\mathbf{S4}}, \mathbf{MML}_n^{\mathbf{S5}}\}$. For each pair of finite sets Γ and Δ of $\mathscr{L}_{\mathrm{M}}$-formulas, it holds that if $\mathbf{L} \vdash \Gamma \Rightarrow \Delta$, then $\mathbf{L} \models \Gamma \Rightarrow \Delta$.

Algebraic completeness

Let $\mathbf{L} \in \{\mathbf{MML}_n^{\mathbf{MNT4}}, \mathbf{MML}_n^{\mathbf{S4}}, \mathbf{MML}_n^{\mathbf{S5}}\}$. For every pair of finite sets of $\mathscr{L}_{\mathrm{M}}$-formulas Γ and Δ, it holds that $\mathbf{L} \models \Gamma \Rightarrow \Delta$ iff $\mathbf{L} \vdash \Gamma \Rightarrow \Delta$.

Kripke semantics for $\mathbf{MML}_n^{S5}$

A Kripke frame $\mathcal{F}$ is a structure $(M, R_1, \ldots, R_n)$ where M is a non-empty set, each R_i is a binary relation on M, $1 \leqslant i \leqslant n$.

An **S5**-Kripke frame is a Kripke frame where each R_j, $1 \leqslant j \leqslant n$, is an equivalence relation.

A valuation on a Kripke frame is a mapping $\models : \Pi \mapsto 2^M$ from the set of propositional variables to the power-set of M. A Kripre model is a pair $(\mathcal{F}, \models)$, where $\mathcal{F}$ is a Kripke frame, $\models$ is a valuation on it.

For a model $\mathscr{M}$ we say that it is *based* on the frame $\mathcal{F}$ if $\mathscr{M} = (\mathcal{F}, \models)$ for some valuation $\models$. We say that a model $\mathscr{M}$ is an **S5**-model if it is based on some **S5**-frame.

Kripke semantics for $\mathbf{MML_n^{S5}}$

$\Pi \cup \neg\Pi$ denotes the set of propositional variables joint with the set of negated propositional variables, that is for each $\pi \in \Pi$ and each j, $1 \leqslant j \leqslant n$, $\neg_j \pi \in \Pi$.

A paraconsistent valuation $\models^p$ on a Kripke frame is a mapping $\models^p : \Pi \cup \neg\Pi \mapsto 2^M$ from the set of propositional variables and negated propositional variable to the power-set of M. A paraconsistent Kripke model is a pair $(\mathcal{F}, \models^p)$, where $\mathcal{F}$ is a paraconsistent Kripke frame, $\models^p$ is a paraconsistent valuation. An **S5**-paraconsistent Kripke model is a paraconsistent Kripke model based on an **S5**-Kripke frame.

Kripke semantics for $\mathbf{MML_n^{S5}}$

An extension of a paraconsistent valuation on propositional formulas:

- $x \models^p \alpha \wedge_j \beta$ iff $x \models^p \alpha$ and $x \models^p \beta$,
- $x \models^p \alpha \vee_j \beta$ iff $x \models^p \alpha$ or $x \models^p \beta$,
- $x \models^p \alpha \rightarrow_j \beta$ iff $x \not\models^p \alpha$ or $x \models^p \beta$,
- $x \models^p \alpha \leftarrow_j \beta$ iff $x \models^p \alpha$ or $x \not\models^p \beta$,
- $x \models^p \neg_j(\alpha \wedge_j \beta)$ iff $x \models^p \neg_j\alpha$ or $x \models^p \neg_j\beta$,
- $x \models^p \neg_j(\alpha \vee_j \beta)$ iff $x \models^p \neg_j\alpha$ and $x \models^p \neg_j\beta$,
- $x \models^p \neg_j(\alpha \rightarrow_j \beta)$ iff $x \models^p \neg_j\beta$ and $x \not\models^p \neg_j\alpha$,
- $x \models^p \neg_j(\alpha \leftarrow_j \beta)$ iff $x \models^p \neg_j\alpha$ or $x \not\models^p \neg_j\beta$,
- $x \models^p \neg_j\neg_j\alpha$ iff $x \models^p \alpha$,
- $x \models^p \neg_k(\alpha \wedge_j \beta)$ iff $x \models^p \neg_k\alpha$ and $x \models^p \neg_k\beta$,
- $x \models^p \neg_k(\alpha \vee_j \beta)$ iff $x \models^p \neg_k\alpha$ or $x \models^p \neg_k\beta$,
- $x \models^p \neg_k(\alpha \rightarrow_j \beta)$ iff $x \models^p \neg_k\beta$ or $x \not\models^p \neg_k\alpha$,
- $x \models^p \neg_k(\alpha \leftarrow_j \beta)$ iff $x \models^p \neg_k\alpha$ and $x \not\models^p \neg_k\beta$,
- $x \models^p \neg_k\neg_j\alpha$ iff $x \not\models^p \alpha$.

Kripke semantics for $\mathbf{MML^{S5}_n}$

An extension of a paraconsistent valuation on modal formulas:

- $x \models^p \Box_j \alpha$ iff $\forall y(R_j(x,y) \Rightarrow y \models^p \alpha)$,
- $x \models^p \Diamond_j \alpha$ iff $\exists y(R_j(x,y)$ and $y \models^p \alpha)$,
- $x \models^p \neg_j \Box_j \alpha$ iff $\exists y(R_j(x,y)$ and $y \models^p \neg_j \alpha)$,
- $x \models^p \neg_j \Diamond_j \alpha$ iff $\forall y(R_j(x,y) \Rightarrow y \models^p \neg_j \alpha)$,
- $x \models^p \neg_k \Box_j \alpha$ iff $\forall y(R_j(x,y) \Rightarrow y \models^p \neg_k \alpha)$,
- $x \models^p \neg_k \Diamond_j \alpha$ iff $\exists y(R_j(x,y)$ and $y \models^p \neg_k \alpha)$,

A formula α is true in a paraconsistent model $\mathscr{M}$ iff for each $x \in \mathscr{M}$, $x \models^p \alpha$. A formula α is $\mathbf{MML^{S5}_n}$-valid in an **S5**-frame $\mathcal{F}$ iff it is true in every paraconsistent model $\mathscr{M}$ based on $\mathcal{F}$. A formula α is $\mathbf{MML^{S5}_n}$-valid iff it is $\mathbf{MML^{S5}_n}$-valid in every **S5**-frame.

Towards embeddings theorems

Let $n > 1$, $j, k \leqslant n$, and $j \neq k$. Then a mapping f from $\mathscr{L}_M$ to $\mathscr{L}$ is defined inductively as follows (propositional case):

- $f(\pi) := \pi$ and $f(\neg_j \pi) := \pi^j$ (where $\pi^j \in \Pi^j$), for each $\pi \in \Pi$,
- $f(\alpha \wedge_j \beta) := f(\alpha) \wedge f(\beta)$,
- $f(\alpha \vee_j \beta) := f(\alpha) \vee f(\beta)$,
- $f(\alpha \rightarrow_j \beta) := f(\alpha) \rightarrow f(\beta)$,
- $f(\alpha \leftarrow_j \beta) := f(\alpha) \leftarrow f(\beta)$,
- $f(\neg_j(\alpha \wedge_j \beta)) := f(\neg_j \alpha) \vee f(\neg_j \beta)$,
- $f(\neg_j(\alpha \vee_j \beta)) := f(\neg_j \alpha) \wedge f(\neg_j \beta)$,
- $f(\neg_j(\alpha \rightarrow_j \beta)) := f(\neg_j \beta) \leftarrow f(\neg_j \alpha)$,
- $f(\neg_j(\alpha \leftarrow_j \beta)) := f(\neg_j \beta) \rightarrow f(\neg_j \alpha)$,
- $f(\neg_j \neg_j \alpha) := f(\alpha)$,
- $f(\neg_k(\alpha \wedge_j \beta)) := f(\neg_k \alpha) \wedge f(\neg_k \beta)$,
- $f(\neg_k(\alpha \vee_j \beta)) := f(\neg_k \alpha) \vee f(\neg_k \beta)$,
- $f(\neg_k(\alpha \rightarrow_j \beta)) := f(\neg_k \alpha) \rightarrow f(\neg_k \beta)$,
- $f(\neg_k(\alpha \leftarrow_j \beta)) := f(\neg_k \alpha) \leftarrow f(\neg_k \beta)$,
- $f(\neg_k \neg_j \alpha) := \neg f(\alpha)$.

Towards embeddings theorems

Let $n > 1$, $j, k \leqslant n$, and $j \neq k$. Then a mapping f from $\mathscr{L}_M$ to $\mathscr{L}$ is defined inductively as follows (modal case):

- $f(\Box_j \alpha) := \Box f(\alpha)$,
- $f(\Diamond_j \alpha) := \Diamond f(\alpha)$,
- $f(\neg_j \Box_j \alpha) := \Diamond f(\neg_j \alpha)$,
- $f(\neg_j \Diamond_j \alpha) := \Box f(\neg_j \alpha)$,
- $f(\neg_k \Box_j \alpha) := \Box f(\neg_k \alpha)$,
- $f(\neg_k \Diamond_j \alpha) := \Diamond f(\neg_k \alpha)$.

Syntactical embedding theorem and its consequences

Syntactical embedding from $\mathbf{MML_n^{S5}}$ into S5

Let f be the above introduced mapping. Then, for each finite sets Ξ and Σ of $\mathscr{L}_M$-formulas and each hypersequent I, it holds that:

(1) $\mathbf{MML_n^{S5}} \vdash \Xi \Rightarrow \Sigma \mid I$ iff $\mathbf{S5} \vdash f(\Xi) \Rightarrow f(\Sigma) \mid f(I)$;
(2) $\mathbf{MML_n^{S5}} \setminus (Cut) \vdash \Xi \Rightarrow \Sigma \mid I$ iff $\mathbf{S5} \setminus (Cut) \vdash f(\Xi) \Rightarrow f(\Sigma) \mid f(I)$.

Cut elimination for $\mathbf{MML_n^{S5}}$

The rule (Cut) is admissible in the cut-free hypersequent calculus for $\mathbf{MML_n^{S5}}$.

Decidability for $\mathbf{MML_n^{S5}}$

$\mathbf{MML_n^{S5}}$ is decidable.

A few more syntactical embeddings

Let $n > 1$. Then a mapping g from $\mathscr{L}$ to $\mathscr{L}_M$ is defined inductively as follows:

(1) $g(\pi) := \pi$ and $g(\pi^j) := \neg_j \pi$, for each $\pi \in \Pi$, $\pi^j \in \Pi^j$, and $j \leqslant n$,
(2) $g(\alpha \wedge \beta) := g(\alpha) \wedge_j g(\beta)$, where j is a fixed positive integer such that $j \leqslant n$;
(3) $g(\alpha \vee \beta) := g(\alpha) \vee_j g(\beta)$, where j is a fixed positive integer such that $j \leqslant n$;
(4) $g(\alpha \to \beta) := g(\alpha) \to_j g(\beta)$, where j is a fixed positive integer such that $j \leqslant n$;
(5) $g(\alpha \leftarrow \beta) := g(\alpha) \leftarrow_j g(\beta)$, where j is a fixed positive integer such that $j \leqslant n$;
(6) $g(\neg\alpha) := \neg_k \neg_j g(\alpha)$, where j and k are two fixed positive integers such that $j, k \leqslant n$ and $j \neq k$,
(7) $g(\Box\alpha) := \Box_j g(\alpha)$, where j is a fixed positive integer such that $j \leqslant n$;
(8) $g(\Diamond\alpha) := \Diamond_j g(\alpha)$, where j is a fixed positive integer such that $j \leqslant n$.

A few more syntactical embeddings. Interpolation

Syntactical embedding from S5 into $\mathbf{MML_n^{S5}}$

For each finite sets Ξ and Σ of $\mathscr{L}$-formulas and each hypersequent I, it holds that:

(1) $\mathbf{S5} \vdash \Xi \Rightarrow \Sigma \mid I$ iff $\mathbf{MML_n^{S5}} \vdash g(\Xi) \Rightarrow g(\Sigma) \mid g(I)$;
(2) $\mathbf{S5} \setminus (Cut) \vdash \Xi \Rightarrow \Sigma \mid I$ iff $\mathbf{MML_n^{S5}} \setminus (Cut) \vdash g(\Xi) \Rightarrow g(\Sigma) \mid g(I)$.

Craig Interpolation

Let $n > 1$, $j, n \leqslant n$ and $j \neq k$. For any formulas α and β, if $\mathbf{MML_n^{S5}} \vdash \alpha \Rightarrow \beta$ and $V(\alpha) \cap V(\beta) \neq \emptyset$, then there exists a formula γ such that

(1) $\mathbf{MML_n^{S5}} \vdash \alpha \Rightarrow \gamma$ and $\mathbf{MML_n^{S5}} \vdash \gamma \Rightarrow \beta$,
(2) $V(\gamma) \subseteq V(\alpha) \cap V(\beta)$.

Semantical embedding theorem and its consequences

Semantical embedding from $\mathbf{MML_n^{S5}}$ into $\mathbf{S5}$

For any $\alpha \in \mathscr{F}_M$, α is $\mathbf{MML_n^{S5}}$-valid iff $f(\alpha)$ is $\mathbf{S5}$-valid.

Kripke completeness of $\mathbf{MML_n^{S5}}$

For any $\alpha \in \mathscr{F}_M$, $\mathbf{MML_n^{S5}} \vdash \Rightarrow \alpha$ iff α is $\mathbf{MML_n^{S5}}$-valid.

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Thank you for attention!