A Labelled Sequent Calculus for HYPE

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HYPE is a system of non classical logic developed in [Lei18]:

"The system should be of interest

1. *as a logical and semantical system in its own right, with attractive logical and mathematical properties and with applications such as to the semantic paradoxes,*
2. *as a general logical framework in which different systems of logic can be studied simultaneously, and*
3. *as a logical background system for (certain) hyperintensional operators."*

The analysis brought on here is proof-theoretical, and will involve item 1 (without applications) and 2.

Why go proof-theoretical?

- HYPE's ability of capturing different logics is interesting for PT;
- It is not provided with a "nice" proof system;
- Challenging to find a way of representing such rich semantics;
- Formalizing reasoning in HYPE's model theory?

Language $\mathcal{L}$: $A ::= p \mid \top \mid \neg A \mid A \wedge A \mid A \vee A \mid A \rightarrow A$

Literals $\mathcal{L}_L$: $p_1, \overline{p_1}, p_2, \overline{p_2}, p_3, \overline{p_3}, \ldots$ (metavariable: v)

Abbreviations: $\overline{p} := \neg p$, $\overline{\overline{p}} := p$, $A \leftrightarrow B := (A \rightarrow B) \wedge (B \rightarrow A)$, $\bot := \neg\top$

Intuitively: Formulas are evaluated at possibly over- or under-determined states that can be fused together or be incompatible.

Models: $\mathcal{M} : \langle S, V, \circ, \perp \rangle$

- $S \neq \emptyset$ [non-empty set of states]
- $V : S \rightarrow \wp(\mathcal{L}_L)$ [valuation function]
- $\circ : S \times S \nrightarrow S$ [fusion (partial) function]
- $\perp \subseteq S \times S$ [incompatibility relation]

For every $s \in S$ there is a unique s^* [star image of s]

Relation of definedness of fused states: $D(s \circ s') := s \circ s' \in S$.

Partial order relation is definable: $s \leq s' := s \circ s' = s' \wedge D(s \circ s')$

Characteristics of D:

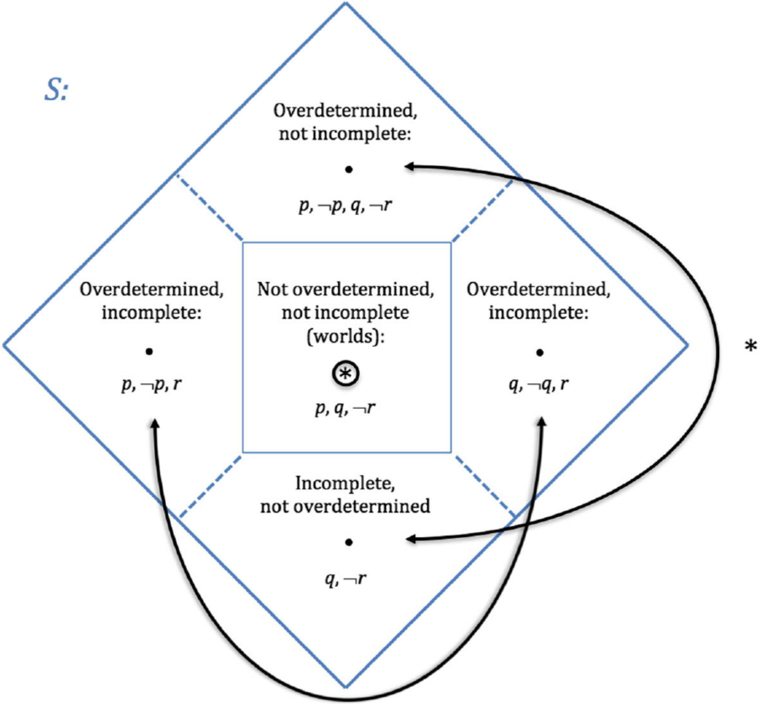
- D is reflexive and $D(s \circ s)$ implies $s \circ s = s$;
- If $D(s \circ s')$ then $D(s' \circ s)$ and $s \circ s' = s' \circ s$;
- If $D(s \circ s')$, then $V(s \circ s') \supseteq V(s) \cup V(s')$;
- If $D((s \circ s') \circ s'')$ then $D(s \circ ((s \circ s') \circ s''))$ and $s \circ ((s \circ s') \circ s'') = (s \circ s') \circ s''$.

Characteristics of $\perp$:

- $\perp$ is symmetric;
- If $v \in V(s)$ and $\overline{v} \in V(s')$, then $s \perp s'$;
- If $s \perp s'$, $D(s \circ s'')$ and $D(s' \circ s''')$, then $s \circ s'' \perp s' \circ s'''$.

Characteristics of $*$:

- $V(s^*) = \{\overline{v} \mid v \notin V(s)\}$;
- $s^{**} = s$;
- $s \not\perp s^*$;
- If $s \not\perp s'$, then $s' \leq s^*$.


S:
Overdetermined, not incomplete:
p, ¬p, q, ¬r
Overdetermined, incomplete:
p, ¬p, r
Not overdetermined, not incomplete (worlds):
p, q, ¬r
Overdetermined, incomplete:
q, ¬q, r
Incomplete, not overdetermined
q, ¬r
*

Satisfaction clauses:

- $s \vDash v$ iff $v \in V(s)$
- $s \vDash \top$
- $s \vDash \neg A$ iff for all s': if $s \not\perp s'$ then $s' \nvDash A$
  ("$s \vDash \neg A$ iff s is incompatible with every way of A being true")
- $s \vDash A \wedge B$ iff $s \vDash A$ and $s \vDash B$
- $s \vDash A \vee B$ iff $s \vDash A$ or $s \vDash B$
- $s \vDash A \rightarrow B$ iff for all s': if $s' \vDash A$ and $D(s \circ s')$, then $s \circ s' \vDash B$
  ("$s \vDash A \rightarrow B$ iff adding an A-state to s yields a B-state")

Logical consequence:
$\Gamma \vDash B$ iff for all HYPE models $\mathcal{M}$, for all $s \in S$, for all $A \in \Gamma$, it holds that if $s \vDash A$ then $s \vDash B$.

Logical Truth:
$\vDash B$ iff for all HYPE models $\mathcal{M}$, for all $s \in S$, $s \vDash B$.

$\vdash \top$

$\vdash A \to (B \to A)$

$\vdash (A \to (B \to C)) \to ((A \to B) \to (A \to C))$

$\vdash (A \wedge B) \to A \qquad \vdash (A \wedge B) \to B$

$\vdash A \to (A \vee B) \qquad \vdash B \to (A \vee B)$

$\vdash A \to (B \to (A \wedge B))$

$\vdash (A \to C) \to ((B \to C) \to ((A \vee B) \to C))$

$\vdash (A \wedge (B \vee C)) \leftrightarrow ((A \wedge B) \vee (A \wedge C))$

$\vdash (A \vee (B \wedge C)) \leftrightarrow ((A \vee B) \wedge (A \vee C))$

$\vdash \neg(A \wedge B) \leftrightarrow (\neg A \vee \neg B) \qquad \vdash \neg(A \vee B) \leftrightarrow (\neg A \wedge \neg B)$

$\vdash \neg\neg A \leftrightarrow A$

$$\frac{\vdash A \to B \quad \vdash A}{\vdash B} \text{ (Modus Ponens)}$$

$$\frac{\vdash A \to B}{\vdash \neg B \to \neg A} \text{ (Admissible Contraposition)}$$

Labelled calculi

A label x is an extralinguistic syntactic object bound to formulas of the logic's language. Labelled formulas $x\colon A, x\colon B, y\colon C, z\colon A, \dots$ and possible relational atoms $R^1_1 x$, $xR^2_1 y$, $R^3_1 xyz \dots$ enrich the language of the derivation.

Semantically, such structures have an interpretation more or less transparent with respect to the model's components. An useful interpretation for non classical logics of $x\colon A$ is that of world, or more in general, state variable x forcing A.

This, plus a good base calculus permits us to straightforwardly internalising the satisfaction clauses of connectives into rules. In sequent calculus this approach successfully preserves most of the properties we desire, is easily extendible, and is applied for many modal and intermediate logics, as in [NVP11].

Initial sequent

$$D(s \circ s'),\ s : P,\ \Gamma \Rightarrow \Delta,\ s \circ s' : P$$

Relation atoms rules

$$\frac{D(s \circ s),\ s \circ s = s,\ \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ \mathit{Refl}\ D \qquad \frac{s \not\perp s',\ s' \not\perp s,\ \Gamma \Rightarrow \Delta}{s \not\perp s',\ \Gamma \Rightarrow \Delta}\ \mathit{Sym} \not\perp$$

$$\frac{s = s,\ \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ \mathit{Refl} = \qquad \frac{D(s' \circ s),\ D(s \circ s'),\ s \circ s' = s' \circ s,\ \Gamma \Rightarrow \Delta}{D(s \circ s'),\ \Gamma \Rightarrow \Delta}\ \mathit{Sym}\ D$$

$$\frac{D(s \circ ((s \circ s') \circ s'')),\ D((s \circ s') \circ s''),\ s \circ ((s \circ s') \circ s'') = (s \circ s') \circ s'',\ \Gamma \Rightarrow \Delta}{D((s \circ s') \circ s''),\ \Gamma \Rightarrow \Delta}\ \mathit{Cond}\ D$$

$$\frac{s \not\perp s',\ D(s \circ s''),\ D(s' \circ s'''),\ s \circ s'' \not\perp s' \circ s''',\ \Gamma \Rightarrow \Delta}{D(s \circ s''),\ D(s' \circ s'''),\ s \circ s'' \not\perp s' \circ s''',\ \Gamma \Rightarrow \Delta}\ \mathit{Distr} \not\perp$$

$$\frac{\mathsf{s} \not\perp \mathsf{s}^*,\ \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ \mathit{Comp}\ * \qquad \frac{\mathsf{s}' \leq \mathsf{s}^*,\ \mathsf{s} \not\perp \mathsf{s}',\ \Gamma \Rightarrow \Delta}{\mathsf{s} \not\perp \mathsf{s}',\ \Gamma \Rightarrow \Delta}\ \mathit{Comp} \leq$$

$$\frac{D(\mathsf{s} \circ \mathsf{s}'),\ \mathsf{s} \circ \mathsf{s}' = \mathsf{s}',\ \mathsf{s} \leq \mathsf{s}',\ \Gamma \Rightarrow \Delta}{\mathsf{s} \leq \mathsf{s}',\ \Gamma \Rightarrow \Delta}\ \mathit{Ord} \qquad \frac{\mathsf{s} = \mathsf{s}^{**},\ \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ \mathit{Doub}$$

Labelling rules

$$\frac{\mathsf{s} = \mathsf{s}',\ \mathcal{A}(\mathsf{s}'/\mathsf{s}),\ \mathcal{A}(\mathsf{s}),\ \Gamma \Rightarrow \Delta}{\mathsf{s} = \mathsf{s}',\ \mathcal{A}(\mathsf{s}),\ \Gamma \Rightarrow \Delta}\ \mathit{Repl}$$

Connectives rules

$$\frac{\mathsf{s} \not\perp \mathsf{s}',\ \mathsf{s} : \neg A,\ \Gamma \Rightarrow \Delta,\ \mathsf{s}' : A}{\mathsf{s} \not\perp \mathsf{s}',\ \mathsf{s} : \neg A,\ \Gamma \Rightarrow \Delta}\ \mathsf{L}\neg \qquad \frac{\mathsf{s} \not\perp \mathbf{s},\ \mathbf{s} : A,\ \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta,\ \mathsf{s} : \neg A}\ \mathsf{R}\neg$$

$$\frac{\mathsf{s} : A,\ \mathsf{s} : B,\ \Gamma \Rightarrow \Delta}{\mathsf{s} : A \wedge B,\ \Gamma \Rightarrow \Delta}\ \mathsf{L}\wedge \qquad \frac{\Gamma \Rightarrow \Delta,\ \mathsf{s} : A \qquad \Gamma \Rightarrow \Delta,\ \mathsf{s} : B}{\Gamma \Rightarrow \Delta,\ \mathsf{s} : A \wedge B}\ \mathsf{R}\wedge$$

$$\frac{\mathsf{s} : A,\ \Gamma \Rightarrow \Delta \qquad \mathsf{s} : B,\ \Gamma \Rightarrow \Delta}{\mathsf{s} : A \vee B,\ \Gamma \Rightarrow \Delta}\ \mathsf{L}\vee \qquad \frac{\Gamma \Rightarrow \Delta,\ \mathsf{s} : A,\ \mathsf{s} : B}{\Gamma \Rightarrow \Delta,\ \mathsf{s} : A \vee B}\ \mathsf{R}\vee$$

$$\frac{D(\mathsf{s}\circ\mathsf{s}'),\ \mathsf{s}\colon A\to B,\ \Gamma\Rightarrow\Delta,\ \mathsf{s}'\colon A, \qquad D(\mathsf{s}\circ\mathsf{s}'),\ \mathsf{s}\colon A\to B,\ \mathsf{s}\circ\mathsf{s}'\colon B,\ \Gamma\Rightarrow\Delta}{D(\mathsf{s}\circ\mathsf{s}'),\ \mathsf{s}\colon A\to B,\ \Gamma\Rightarrow\Delta}\ \mathsf{L}{\to}$$

$$\frac{D(\mathsf{s}\circ\mathbf{s}),\ \mathbf{s}\colon A,\ \Gamma\Rightarrow\Delta,\ \mathsf{s}\circ\mathbf{s}\colon B}{\Gamma\Rightarrow\Delta,\ \mathsf{s}\colon A\to B}\ \mathsf{R}{\to} \qquad \frac{}{\Gamma\Rightarrow\Delta,\mathsf{s}\colon\top}\ \mathsf{R}\top$$

Side conditions for rules R→, R¬ *are that* **s** *does not appear below the inference line.*

Structural rules, including Cut, admissible. Soundness straightforward, Completeness by automated construction of countermodels from failed proof search.

G3HYPE extensions

$$\frac{s_l \not\leq s_l, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ \textit{LeastComp}$$

$$\frac{s \leq s_g, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ \textit{Great} \leq \qquad \frac{s_g = s_l^*, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ \textit{Great} *$$

(One way of) Recapturing of LP and SK (G3HYPE + the rules above) by expressing $A \rightarrow B$ as $\neg A \vee B$:

$\Gamma \vDash_{\mathrm{LP}} \Delta$ iff $s_g : \Gamma \Rightarrow s_g : \Delta$
$\Gamma \vDash_{\mathrm{SK}} \Delta$ iff $s_l : \Gamma \Rightarrow s_l : \Delta$

Therefore, strict-tolerant logic consequence is definable:

$\Gamma \vDash_{\mathrm{ST}} \Delta$ iff $s_l : \Gamma \Rightarrow s_g : \Delta$

$$\frac{s \not\perp s, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta} \; Refl \not\perp$$

$$\frac{s = s',\ D(s \circ s'),\ \Gamma \Rightarrow \Delta}{D(s \circ s'),\ \Gamma \Rightarrow \Delta} \; Eq\ D \qquad\qquad \frac{s = s',\ s \not\perp s',\ \Gamma \Rightarrow \Delta}{s \not\perp s',\ \Gamma \Rightarrow \Delta} \; Eq \not\perp$$

(One way of) Recapturing of Classical logic CL (G3HYPE + the rules above):

$\Gamma \vDash_{CL} \Delta$ iff $s : \Gamma \Rightarrow s : \Delta$

(One way of) Recapturing of Intuitionistic logic J can be achieved by expressing $\neg A$ as $A \to \neg\top$.

First-order HYPE: QHYPE

From a semantical point of view, straightforward extension (constant domain) of HYPE. Dissecting it via labelled calculi [Lyo21] more interesting. In particular, we discover that the following axiomatization, without imposing Exchangeability of Equivalents, cannot prove all the instance of the constant domain axiom $\forall x(A \vee B) \rightarrow (\forall x A \vee B)$ for x not occurring free in B:

$\vdash \forall x(A \rightarrow B) \rightarrow (\forall x A \rightarrow \forall x B)$

$\vdash \forall x A \rightarrow A(t/x)$

$\vdash A(t/x) \rightarrow \exists x A$

$\vdash \forall x(A \rightarrow B) \rightarrow (A \rightarrow \forall y B(x/y))$ (x not free in A, and $y = x$ or y not free in B)

$\vdash \forall x(A \rightarrow B) \rightarrow (\exists y A(y/x) \rightarrow B)$ (x not free in B, and $y = x$ or y not free in A)

$\vdash \neg\forall x A \leftrightarrow \exists x \neg A$

G3QHYPE: G3HYPE +

Initial sequent

$$D(\mathsf{s} \circ \mathsf{s}'),\ \vec{t} \in \mathcal{D}_{\mathsf{s}},\ \mathsf{s}\colon P(\vec{t}),\ \Gamma \Rightarrow \Delta,\ \mathsf{s} \circ \mathsf{s}' : P(\vec{t})$$

Relation atoms rules

$$\frac{y \in \mathcal{D}_{\mathsf{s}},\ \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ \mathit{NonEmp}\ \mathcal{D}$$

$$\frac{t \in \mathcal{D}_{\mathsf{s} \circ \mathsf{s}'},\ D(\mathsf{s} \circ \mathsf{s}'),\ t \in \mathcal{D}_{\mathsf{s}},\ \Gamma \Rightarrow \Delta}{D(\mathsf{s} \circ \mathsf{s}'),\ t \in \mathcal{D}_{\mathsf{s}},\ \Gamma \Rightarrow \Delta}\ \mathit{NonDec}\ \mathcal{D}$$

$$\frac{t \in \mathcal{D}_{\mathsf{s}'},\ \mathsf{s} \not\perp \mathsf{s}',\ t \in \mathcal{D}_{\mathsf{s}},\ \Gamma \Rightarrow \Delta}{\mathsf{s} \not\perp \mathsf{s}',\ t \in \mathcal{D}_{\mathsf{s}},\ \Gamma \Rightarrow \Delta}\ \mathit{Comp}\ \mathcal{D}$$

$$\frac{t \in \mathcal{D}_{\mathsf{s}},\ D(\mathsf{s} \circ \mathsf{s}'),\ t \in \mathcal{D}_{\mathsf{s}},\ \Gamma \Rightarrow \Delta}{D(\mathsf{s} \circ \mathsf{s}'),\ t \in \mathcal{D}_{\mathsf{s} \circ \mathsf{s}'},\ \Gamma \Rightarrow \Delta}\ \mathit{Const}\ \mathcal{D}$$

Connectives rules

$$\frac{t \in \mathcal{D}_{\mathsf{s}},\ \mathsf{s}\colon A(t/x),\ D(\mathsf{s}\circ\mathsf{s}'),\ \mathsf{s}\circ\mathsf{s}'\colon \forall x A,\ \Gamma \Rightarrow \Delta}{t \in \mathcal{D}_{\mathsf{s}},\ D(\mathsf{s}\circ\mathsf{s}'),\ \mathsf{s}\colon \forall x : A,\ \Gamma \Rightarrow \Delta}\ \mathsf{L}\forall$$

$$\frac{y \in \mathcal{D}_{\mathsf{s}\circ\mathbf{s}},\ D(\mathsf{s}\circ\mathbf{s}),\ \Gamma \Rightarrow \Delta,\ \mathsf{s}\circ\mathbf{s}\colon A(y/x)}{\Gamma \Rightarrow \Delta,\ \mathsf{s}\colon \forall x A}\ \mathsf{R}\forall$$

$$\frac{y \in \mathcal{D}_{\mathsf{s}\circ\mathsf{s}'},\ \mathsf{s}\colon A(y/x),\ \Gamma \Rightarrow \Delta}{\mathsf{s}\colon \exists x A,\ \Gamma \Rightarrow \Delta}\ \mathsf{L}\exists \qquad \frac{t \in \mathcal{D}_{\mathsf{s}},\ \Gamma \Rightarrow \Delta,\ \mathsf{s}\colon \exists x A,\ \mathsf{s}\colon A(t/x)}{t \in \mathcal{D}_{\mathsf{s}},\ \Gamma \Rightarrow \Delta,\ \mathsf{s}\colon \exists x A}\ \mathsf{R}\exists$$

Side conditions for rules NonEmpt $\mathcal{D}$, L$\exists$ *and* R$\forall$ *are that y and* $\mathbf{s}$ *do not appear below the inference line.*

Structural rules, including Cut, admissible. In G3QHYPE is without *Const* $\mathcal{D}$, is possible to derive all the previous axioms without the constant domain one.

Thank you!

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