PIECEWISE-SMOOTH DYNAMICAL SYSTEMS WITH DISCRETE-TIME

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Outline

Introduction

Many scientific and practical researches are reduced to the studies of discrete-time and continuous dynamical systems. Also, it was proved that the dynamics of the mathematical model does not have the same character for discrete and continuous cases (for example, logistic map). Because of these reasons, the discrete-time dynamical systems also should be examined in parallel with the continuous case. Discrete-time dynamical systems generated by piecewise-smooth functions play important role in helping to understand the behavior of many important physical phenomena such as electrical circuits that have switches, mechanical devices in which components impact with each other (such as gear assemblies) or have free play, problems with friction, sliding, or squealing, many control systems, and models in the social and financial sciences[1] . Hence, the study of the dynamics associated with piecewise-smooth functions is one of the most essential problems in the theory of dynamical systems..

[1] M. di Bernardo, C.J. Budd, A.R. Champneys, P. Kowalczyk, *Piecewise-smooth Dynamical Systems: Theory and Applications*. Applied Math. Sci., **163** (2008).

§ **1.1 Basic notions**

Let X and Y are non empty subsets of real numbers. If f is a map from X to Y we will express that as $f : X \to Y, x \to f(x)$. Sometimes we will also express the map as a difference equation $x_{n+1} = f(x_n)$. If the function f depends on a parameter λ we write $f_\lambda(x)$ and say that f is a one-parameter family of maps.

For arbitrary maps f and g, the composition of f and g will be denoted as $f \circ g(x) = f(g(x))$. The n-fold composition of with itself will be denoted as $f^n(x) = \underbrace{f \circ \ldots \circ f}_{n-times}(x)$. For a given $x_0 (x_0 \in X)$, the sequence of points

$$x_1 = f(x_0), x_2 = f(x_1) = f^2(x_0), x_3 = f(x_2) = f(f^2(x_0)) = f^3(x_0), \ldots \tag{1}$$

are called the trajectory of the point or the discrete dynamical system generated by the function.

The main problem of the dynamical systems is to describe all limit points of the sequence $f^n(x)_{n\in\mathbb{N}}$ for any initial points x.

The point x is a fixed point for f if $f(x) = x$. The point x is a periodic point of period n if $f^n(x) = x$. The least positive n for which $f^n(x) = x$ is called the prime period of x. We denote the set of period points of period n by $\mathrm{Per}_n(f)$, and the set of fixed points by $\mathrm{Fix}(f)$. The set of all iterates of a periodic point form a periodic orbit. For a given function $f : \mathbb{R} \to \mathbb{R}$ the ω-limit set of $x \in \mathbb{R}$, denoted by $\omega(x, f)$ or $\omega(x)$, is the set of cluster points of the forward orbit $f^n(x)_{n\in\mathbb{N}}$ of the iterated function f.

Definition 1.1.4

Let p be a periodic point of prime period n.

- If $|(f^n)'(p)| < 1$, then p is attracting;
- If $|(f^n)'(p)| > 1$, then p is repelling;
- If $|(f^n)'(p)| = 1$, then p is indifferent.

Definition 1.1.5

Let $f : A \to B$ and $g : B \to B$ be two maps. f and g are said to be topologically conjugate if there exists a homeomorphism h such that, $h \circ f = g \circ h$. The homeomorphism h is called a topologically conjugacy.

The floor function of $x \in \mathbb{R}$ is defined by

$$\lfloor x \rfloor = \max m \in \mathbb{Z} : m \leq x.$$

We consider the dynamical system associated with the function $f : \mathbb{R} \Rightarrow \mathbb{R}$ defined by

$$f(x) \equiv f_\lambda(x) = \lfloor \lambda x \rfloor, \tag{2}$$

where $\lambda \in \mathbb{R}$ is a parameter.
For possible cases the parameter values: 1) $\lambda \leq 0$; 2) $0 < \lambda < 1$ 3) $\lambda \geq 1$ we have studied the dynamical systems generated by the floor function (2) and the limit points of the trajectiories is given for any initial point.

Let $0 < \lambda < 1$. Note that for each $\lambda \in (0, 1)$ there exists $m \in \mathbb{N}$ such that $\frac{m-1}{m} < \lambda \leq \frac{m}{m+1}$.

Theorem 1.2.2

Let $\frac{m-1}{m} < \lambda \leq \frac{m}{m+1}$ for some $m \in \mathbb{N}$. Then

$$\lim_{n\to\infty} f^n(x) = \begin{cases} 0, & \textit{for all } x \in [0, +\infty), \\ k, & \textit{for all } x \in [\frac{k}{\lambda}, \frac{k+1}{\lambda}), \\ -m, & \textit{for all } x \in (-\infty, -\frac{m}{\lambda}), \end{cases}$$

where $k \in \{-1, -2, ..., -m\}$.

For the case $\lambda > 1$ we have the following theorem.

Theorem 1.2.3

Let $\frac{m+1}{m} \leq \lambda < \frac{m}{m-1}$ for some $m \in \mathbb{N}$. Then

$$\lim_{n\to\infty} f^n(x) = \begin{cases} k, & \textit{for all } x \in [\frac{k}{\lambda}, \frac{k+1}{\lambda}) \\ -\infty, & \textit{for all } x \in (-\infty, 0), \\ +\infty, & \textit{for all } x \in [-\frac{m}{\lambda}, +\infty), \end{cases}$$

where $k \in \{1, 2, ..., m-1\}$.

In this section, we have investigated two-dimensional dynamical system generated by (2). The operator is defined by:

$$A(z) = \begin{cases} x' = \lfloor \lambda y \rfloor \\ y' = \lfloor \lambda x \rfloor, \end{cases} \qquad z = (x, y) \in \mathbb{R}^2,$$

where $\lambda \in \mathbb{R}$ is a parameter.
The dynamical system of this operator is also studied for all parameter values. For $\lambda > 1$ the following theorem holds.

Theorem 1.3.3

Let $\frac{m+1}{m} \leq \lambda < \frac{m}{m-1}$ for some $m \in \mathbb{N}$. Then the following hold:

- If $z \in \{(x_0, y_0) | \frac{k}{\lambda} \leq x_0 < \frac{k+1}{\lambda}, y_0 < 0\} \bigcup \{(x_0, y_0) | \frac{k}{\lambda} \leq y_0 < \frac{k+1}{\lambda}, x_0 < 0\}$ then
$$\omega(z) = \{(k, -\infty), (-\infty, k)\};$$
where $k \in \{0, 1, 2, ..., m-1\}$.
- If $z \in \{(x_0, y_0) | x_0 < 0, y_0 \geq \frac{m}{\lambda}\} \bigcup \{(x_0, y_0) | y_0 < 0, x_0 \geq \frac{m}{\lambda}\}$ then
$$\omega(z) = \{(\infty, -\infty), (-\infty, \infty)\}.$$
- If $z \in \{(x_0, y_0) | \frac{k}{\lambda} \leq x_0 < \frac{k+1}{\lambda}, \frac{p}{\lambda} \leq y_0 < \frac{p+1}{\lambda}\}$ then
$$\omega(z) = \{(k, p), (p, k)\};$$
where $k \in \{0, 1, 2, ..., m-1\}$, $p \in \{0, 1, 2, ..., m-1\}$.
- If $z \in \{(x_0, y_0) | x_0 < 0, y_0 < 0\}$, then $\lim\limits_{n \to \infty} A^n(z) = (-\infty, -\infty)$.
- If $z \in \{(x_0, y_0) | x_0 \geq \frac{m}{\lambda}, y_0 \geq \frac{m}{\lambda}\}$, then $\lim\limits_{n \to \infty} A^n(z) = (\infty, \infty)$.

§ 2.1. The evolution operator with two equal dominated species

In this section, we give the mathematical model of population which consists of two species. We proved that both species will not extinct in the future. The operator is defined by two-dimensional quadratic stochastic operator (QSO).
Let S^{m-1} be the simplex:

$$S^{m-1} = \{x = (x_1, \dots, x_m) \in \mathbb{R}^m : x_i \geq 0,\ \sum_{i=1}^{m} x_i = 1\}.$$

Vertexes of the simplex are $x_1 = (1, 0, ..., 0)$, $x_2 = (0, 1, 0, ..., 0)$, ..., $x_m = (0, ..., 0, 1)$.
We denote $\mathrm{int} S^{m-1} = \{\mathbf{x} = (x_1, x_2, ..., x_m) \in S^{m-1} : x_i > 0 \ \ for \ \ all \ \ i \in 1, 2, ..., m\}$ and $\partial S^{m-1} = S^{m-1} \setminus \mathrm{int} S^{m-1}$.

QSO is a mapping $V : S^{m-1} \to S^{m-1}$ defined by

$$V(x_k) = \sum_{i,j=1}^{m} P_{ij,k} x_i x_j, \; k \in \{1, 2, ..., m\}.$$

Here, $P_{ij,k}$ are hereditary coefficients and satisfy

$$P_{ij,k} \geq 0, \quad P_{ij,k} = P_{ji,k}, \quad \sum_{k=1}^{m} P_{ij,k} = 1, \quad i,j \in \{1, 2, ..., m\} \tag{3}$$

Recall that a QSO $V : S^{m-1} \to S^{m-1}$ is called Volterra if one has

$$P_{ij,k} = 0 \quad when \quad k \notin \{i, j\}. \tag{4}$$

For a parameter $a \in [-1, 1]$ define the operator $V_a : S^1 \to S^1$ as

$$V_a : \begin{cases} x' = x(1 + ay) \\ y' = y(1 - ax) \end{cases}$$

For an initial point $z = (x, y) \in S^1$ we consider the trajectory $z^{(n)} = (x^{(n)}, y^{(n)}) = V^n(z)$ and

$$\lim_{n\to\infty} z^{(n)} = \begin{cases} (0, 1), & \text{if } \ a < 0 \\ (1, 0), & \text{if } \ a > 0 \end{cases} \tag{5}$$

Thus (5) means that if $a > 0$ (resp. $a < 0$) then the specie 2 (resp. 1) will extinct and the specie 1 (resp. 2) will dominate.

To ensure that both species will have equal domination we define an evolution operator, $V_{a,b} : z = (x, y) \in S^1 \to z' = (x', y') \in S^1$ by

$$V_{a,b}(z) = \begin{cases} V_a(z), & \text{if } \ x \le \frac{1}{2} \\ V_b(\overline{z}), & \text{if } \ x > \frac{1}{2} \end{cases} \tag{6}$$

where $\overline{z} = (y, x) \in S^1$.
We note that the probabilities $P_{ij,k}$ mentioned in (3) are independent on $x \in S^{m-1}$, but for operator (6) we have

$$P_{11,1} = 1 - P_{11,2} = P_{22,2} = 1 - P_{22,1} = 1$$

and the remaining coefficients depend on the points $z = (x, y)$ of the simplex S^1:

$$P_{12,1}(z) = 1 - P_{12,2}(z) = \begin{cases} \frac{1+a}{2}, & \text{if } \ x \le \frac{1}{2} \\ \frac{1-b}{2}, & \text{if } \ x > \frac{1}{2}. \end{cases} \tag{7}$$

In this section, we have studied the dynamical system of the operator (6) by reducing to one-dimensional function $f_{a,b} : [0,1] \to [0,1]$ with one discontinuity point:

$$f_{a,b}(x) = \begin{cases} x(1+a-ax), & 0 \le x \le \frac{1}{2} \\ x(1-b+bx), & \frac{1}{2} < x \le 1, \end{cases} \tag{8}$$

where by the symmetry of parameters we can assume that $a, b \in [0,1]$.
By using the notion topological conjugacy (Def. 1.1.5) we proved that it is sufficient to study the dynamical system generated by (8) for the parameters $a, b \in [0,1]$ with the condition $a \in [0,1]$, $a \le b$.
For $a \le b$ we consider the following possible cases: 1) $a = b = 0$; 2) $a = 0,\ b \ne 0$; 3) $a \ne 0,\ b \ne 0$.

Remark 2.2.1

If $a = b = 0$ then the function has the form $f_{0,0}(x) = x,\ 0 \le x \le 1$. This case is not interesting.

In the case $a = 0$, $b \neq 0$, we have completely investigated the dynamical systems. For the case $a \neq 0$, $b \neq 0$, we found an invariant set $A = (\frac{1}{2} - \frac{b}{4}, \frac{1}{2} + \frac{a}{4}]$ which $f(A) = A$ holds. Moreover, we have given the result that is sufficient for study dynamical systems in an invariant set. Because if we take an initial point outer of the invariant set, after several steps all points in the trajectory lies on the A.

Remark 2.2.3

Obviously, the set of fixed points is $\mathrm{Fix}(f) = \{0, 1\}$ for $ab \neq 0$. Besides we have $|f'(0)| = 1 + a > 1$, $|f'(1)| = 1 + b > 1$. Thus both fixed points are repeller. Therefore both species will always survive.

For the type of periodic points we have the theorem.

Theorem 2.2.2

If f has a periodic point then the point is a repelling.

The following example shows that Sharkovskii's theorem does not hold for discontinuous systems.

Example 2.2.1.

We have 3-period points when the function is

$$f_{\frac{1}{2},1}(x) = \begin{cases} \frac{1}{2}x(3-x), & 0 \leq x \leq \frac{1}{2}; \\ x^2, & \frac{1}{2} < x \leq 1. \end{cases} \tag{9}$$

3-periodic points of (9) are $\{p, f(p), f^2(p)\}$ where p is the root of the following equation in $[0, 1]$

$$x^6 - 11x^5 + 43x^4 - 65x^3 + 4x^2 + 76x - 32 = 0.$$

For discrete time dynamical system $x_{n+1} = f(x_n)$, for an orbit starting with x_0 the **Lyapunov exponent** can be defined as follows:

$$\lambda(x_0) = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \ln|f'(x_i)|. \tag{10}$$

The Lyapunov exponent "λ", is useful for distinguishing among the various types of orbits. It works for discrete as well as continuous systems. Let $a \neq 0$, $b \neq 0$.

Proposition 2.3.1

Let $x_0 \in [0,1] \setminus \{\frac{1}{2}\}$. Then the Lyapunov exponent $\lambda(x_0)$ is non negative for (8).

By Proposition 2.3.1 we have $\lambda \geq 0$, in the case

- $\lambda = 0$ it indicates that the trajectory converges to an indifferent fixed point.
- $\lambda > 0$ it follows that the dynamical system is chaotic, i.e., nearby points, no matter how close, will diverge to any arbitrary separation. All neighborhoods in the phase space will eventually be visited.

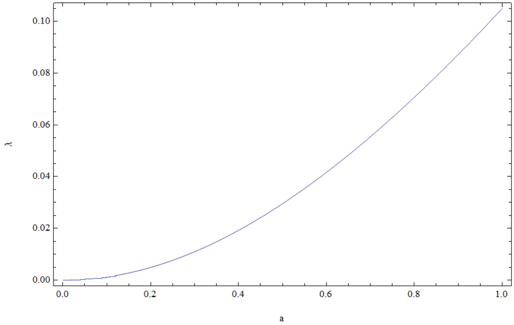


Figure 1 : Typical graphics of the Lyapunov exponent for the system function $f_{a,b}(x)$ for different values of the parameter b related to a: $b = a$.

Moreover, we have sketched bifurcation diagrams of the function (8) for some parameter values. A **bifurcation diagram**[2] shows the values visited or approached asymptotically (fixed points, periodic orbits, or chaotic attractors) of a system as a function of a bifurcation parameter in the system. It is usual to represent stable values with a solid line and unstable values with a dotted line, although often the unstable points are omitted.

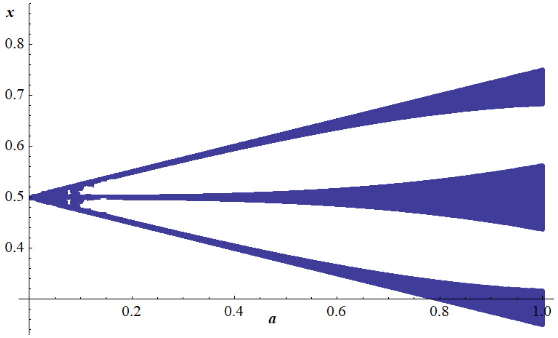


Figure 2 : Bifurcation diagram for the system function $f_{a,b}(x)$ for the case $b = a$.

[2] https://en.wikipedia.org/wiki/Bifurcation diagram

§ 3.1. Two dimensional discontinuous quadratic stochastic operator

Consider a population consisting of two species, i.e., $m = 2$. For a parameter $a \in [0, 1]$ define the operator $W_a : S^1 \to S^1$ as

$$W_a = \begin{cases} x' = ax^2 + 2xy + y^2 \\ y' = (1 - a)x^2. \end{cases}$$

For a parameter $b \in [-1, 1]$ define the operator $W_b : S^1 \to S^1$ as

$$W_a = \begin{cases} x' = x(1 + by) \\ y' = y(1 - bx). \end{cases}$$

Define an evolution operator, $W_{a,b} : z = (x, y) \in S^1 \to S^1$ by

$$W_{a,b}(z) = \begin{cases} W_a(z), & if \quad x \leq \frac{1}{2} \\ W_b(z), & if \quad x > \frac{1}{2}. \end{cases}$$

Using the equality $x + y = 1$, W_a, W_b and $W_{a,b}$ operators can be reduced respectively to the functions $g_a(x) : [0, 1] \to [0, 1]$, $g_b(x) : [0, 1] \to [0, 1]$ and $g_{a,b}(x) : [0, 1] \to [0, 1]$ defined by

$$g_a(x) = (a - 1)x^1 + 1,$$

$$g_b(x) = x(1 + b - bx),$$

$$g(x) \equiv g_{a,b}(x) = \begin{cases} (a-1)x^2 + 1, & x \le \frac{1}{2} \\ x(1 + b - bx), & x > \frac{1}{2} \end{cases} \tag{11}$$

where $a \in [0, 1]$, $b \in [-1, 1]$.

The following three cases are possible: 1) $0 \leq a \leq 1$, $0 < b \leq 1$; 2) $0 \leq a \leq 1$, $b = 0$; 3) $0 \leq a \leq 1$, $-1 \leq b < 0$. For the cases 1) and 2) we have completely studied the dynamical system generated by (11). In the case $0 \leq a \leq 1$ and $-1 \leq b < 0$, we have the following.

Theorem 3.1.3

For the dynamical system generated by function (11) the followings hold:

1) g has unique fixed point $x = 1$.
2) if $a = 1$, then for any initial point $x^{(0)}$ the trajectory $x^{(n)}$ has the following limit
$$\lim_{n\to\infty} x^{(n)} = 1.$$
3) if $0 \leq a < 1$, then g does not have a periodic point of period two.
4) if $0 \leq a \leq \frac{2-b+2\sqrt{b^2+2\sqrt{b^2+1}-1}}{b}$, then g has periodic points of period three.

In § 3.2, the dynamical systems of a discontinuous three-dimensional Volterra operator is investigated. Define a QSO as the following:

$$V(\mathbf{x}) \equiv V_{a,b,c}(\mathbf{x}) = \begin{cases} V_1(\mathbf{x}), & \min\{x,y,z\} = x \quad \mathit{and} \quad x \neq y \;\mathit{or}\; \mathbf{x} = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}) \\ V_2(\mathbf{x}), & \min\{x,y,z\} = y \quad \mathit{and} \quad y \neq z \\ V_3(\mathbf{x}), & \min\{x,y,z\} = z \quad \mathit{and} \quad x \neq z \end{cases} \tag{12}$$

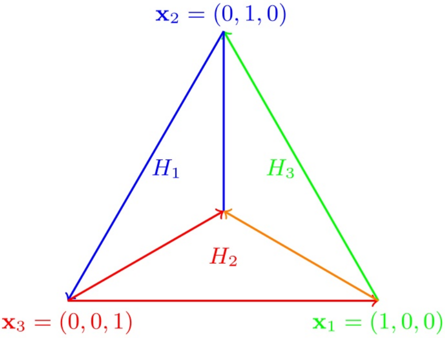
where $\mathbf{x} = (x,y,z) \in S^2$, $a,b,c \in [0,1]$ and

$$V_1 = \begin{cases} x' = x(1 + ay + bz) \\ y' = y(1 - ax + cz) \\ z' = z(1 - bx - cy) \end{cases} \qquad V_2 = \begin{cases} x' = x(1 - by - cz) \\ y' = y(1 + az + bx) \\ z' = z(1 - ay + cx) \end{cases}$$

$$V_3 = \begin{cases} x' = x(1 - az + cy) \\ y' = y(1 - bz - cx) \\ z' = z(1 + ax + by). \end{cases}$$

According to the definition of the operator V (12), the domain consists of three non-intersecting subsets of S^2. They are:

$$H_1 = \{\mathbf{x} = (x, y, z) \in S^2 \,:\, \min\{x, y, z\} = x \quad and \quad x \neq y \quad or \quad \mathbf{x} = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})\};$$
$$H_2 = \{\mathbf{x} = (x, y, z) \in S^2 \,:\, \min\{x, y, z\} = y \quad and \quad y \neq z\};$$
$$H_3 = \{\mathbf{x} = (x, y, z) \in S^2 \,:\, \min\{x, y, z\} = z \quad and \quad x \neq z\}.$$



If $a = b = 0$ and $c \neq 0$ then the following two lemmas give information about $\omega(x^{(0)})$ of the trajectory (under the operator (12)).

Lemma 3.2.4

Let be QSO given by (12). Then for any it holds .

Lemma 3.2.5

The operator (12) does not have periodic cycle.

We can conclude the following theorem from Lemma 1 and Lemma 2.

Theorem 3.2.1

If and then the set either consists a single point, or is an infinite set.

If $b = c = 0$, $a \neq 0$, then we have the following proposition.

Theorem 3.2.2

Let V be QSO given by (12) (where $a \in (0, 1]$ is a fixed parameter). Then

$$\mathrm{Per}_2(V) = \{\emptyset\}$$

and

$$\mathrm{Per}_3(V) = \{\mathbf{x}_1^*, \mathbf{x}_2^*, \mathbf{x}_3^*\}$$

where

$$\mathbf{x}_1^* = \left(\frac{2}{-a+3+\sqrt{a^2-2a+9}}, \frac{2}{a+3+\sqrt{a^2-2a+9}}, \frac{2}{a+3+\sqrt{a^2-2a+9}}\right)$$
$$\mathbf{x}_2^* = \left(\frac{2}{a+3+\sqrt{a^2-2a+9}}, \frac{2}{a+3+\sqrt{a^2-2a+9}}, \frac{2}{-a+3+\sqrt{a^2-2a+9}}\right)$$
$$\mathbf{x}_3^* = \left(\frac{2}{a+3+\sqrt{a^2-2a+9}}, \frac{2}{-a+3+\sqrt{a^2-2a+9}}, \frac{2}{a+3+\sqrt{a^2-2a+9}}\right).$$

Part I. Articles

1) Rozikov U.A., Sattarov I.A., Usmonov J.B. **Journal of Applied Nonlinear Dynamics**, V.5, Issue 2, 2016, p.185-191. **(Scopus IF: 0.487)**
2) Usmonov J.B. **Uzb. Math. Jour.** 2019. No 2. p. 127-134.
3) Rozikov U.A., Usmonov J.B. **Qualitative Theory of Dynamical Systems**, V.19, Issue 2, 2020, 19 p. **(Web of Sciences IF: 1.400)**
4) Usmonov J.B., Kodirova M.A. **Bull. Ins. Math.**, 2020. No 3. p.98-107.
5) Usmonov J.B. **Uzb. Math. Jour.** 2021. No 2.

Part II. Abstracts in conference proceedings

1) Usmonov J.B. The dynamical system generated by the bounded floor function [f(x)]. Conference "Actual problems of differential equations and their applications", 15-17 December. 2017, Tashkent, Uzbekistan. p.121-122.
2) Usmonov J.B. On a two dimensional dynamical system generated by the floor function. Conference "New results of mathematics and their applications", 14-15 May, 2018. Samarkand, Uzbekistan. p.14-15.
3) Rozikov U.A., Usmonov J.B. Dynamical system of a piecewise-continuous function. Conference "Actual problems and implementation of the analysis", 4-5 October, 2019. Karshi, Uzbekistan. p.72-76.
4) Rozikov U.A., Usmonov J.B. On a piecewise-continuous dynamical system. **International** conference "Modern problems of geometry and topology and its applications", 21-23 November, 2019, Tashkent, Uzbekistan. p.75-76.
5) Usmonov J.B. The limit set of trajectories for a discontinuous Volterra operator. **International** conference "Modern problems of differential equations and related branches of mathematics", 12-13 March, 2020, Fergana, Uzbekistan. p.395-396.
6) Usmonov J.B. On the dynamics of discontinuous QSO Volterra. "41th **International** Conference on Quantum Probability and Related Topics", 28 March 1 April, 2021, UAE.

Thank you for your attention!