Структура многозначности аналитических функций и аппроксимации Эрмита-Паде

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Program of the talk

- Equilibrium problems for vector potentials with matrix of interaction
- Rational approximants of analytic functions with branch points. (Pade aproximants, Nuttull-Stahl result, saddle points of the energy functional)
- Asymptotics of Hermite-Pade approximants for a vector of multivalued functions

Scalar case

- Logarithmic potential and energy

$$V^{\mu}(z) = \int \log \frac{1}{|z-t|}\, d\mu(t), \quad I(\mu) = \int \log V^{\mu}(x)\, d\mu(x)$$

of a measure μ, $\quad S(\mu) \subset \Delta$, $\quad \Delta$ is a compact in $\mathbb{C}$.

- Extremal measure

$$\exists! \lambda \; : \quad I(\lambda) = \min_{S(\mu)\subset\Delta,\ |\mu|=1} I(\mu), \qquad |\mu| = \int d\mu.$$

- Equilibrium of the potential

$$V^{\lambda}(x) = const, \quad x \in \Delta.$$

Energy of vector measures with matrix of interaction

- $\{\Delta_1, \ldots, \Delta_p\}$ compacts in $\mathbb{C}$
- $D = (d_{i,j})_{i,j=1}^p$ real symmetric (nonnegative def.) matrix
- Vector of measures

$$\vec{\mu} = (\mu_1, \ldots, \mu_p), \quad S(\mu_j) \subset \Delta_j, \quad j = 1, \ldots, p,$$

- Energy functional

$$I(\vec{\mu}) := \sum_{i=1}^{p} \sum_{j=1}^{p} d_{i,j} I(\mu_i, \mu_j),$$

where

$$I(\mu_i, \mu_j) := \int_{\Delta_i} \int_{\Delta_j} \log \frac{1}{|x-t|} \, d\mu_i(x) \, d\mu_j(t).$$

- A. Gonchar and E.Rakhmanov (1982).

Equilibrium of Vector Potential

- Extremal vector measure $\vec{\lambda}$

$$\exists ! \vec{\lambda} : \quad I(\vec{\lambda}) = \min_{S(\mu_j) \subset \Delta_j,\ |\mu_j|=1, j=1,\ldots,p} I(\vec{\mu})$$

- Equilibrium

$$U_j^{\vec{\lambda}}(x) = \sum_{i=1}^{p} d_{ij} V^{\lambda_i}(x) \begin{cases} = \kappa_j, & x \in S(\lambda_j) = \Delta_j^*, \\ \geq \kappa_j, & x \in \Delta_j \setminus \Delta_j^*. \end{cases}$$

- $\vec{U}^{\vec{\lambda}} = (U_1^{\vec{\lambda}}, \ldots, U_p^{\vec{\lambda}})$ is called the vector potential of the vector valued measure $\vec{\lambda}$ with respect to the interaction matrix D

Examples

- Condenser : $(|\mu_1| = 1, |\mu_2| = 1)$, $D := \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$
- Angelesco system : $(|\mu_1| = 1, |\mu_2| = 1)$, $D := \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$
- Nikishin system : $(|\mu_1| = 2, |\mu_2| = 1)$, $D := \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$
- Two intervals scalar case :

$$(|\mu_1| + |\mu_2| = 1),\ D := \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix},\ (\kappa_1 = \kappa_2)$$

Definition of HP-approximants

- Given $\vec{f} = (f_1, \ldots, f_p)$ a vector of p.s.

$$f_j(z) = \sum_{k=0}^{\infty} \frac{f_{j,k}}{z^k}, \qquad j = 1, \ldots, p.$$

- The HP - rational approximants (of type II)

$$\pi_{\vec{n}} = \left(\frac{Q_{\vec{n}}^{(1)}}{P_{\vec{n}}}, \ldots, \frac{Q_{\vec{n}}^{(p)}}{P_{\vec{n}}} \right), \ \deg P_{\vec{n}} \leq |\vec{n}| = n_1 + \cdots + n_p,$$

for the vector $\vec{f}$ and multi-index $\vec{n} \in \mathbb{N}^p$ are defined by

$$f_j(z) P_{\vec{n}}(z) - Q_{\vec{n}}^{(j)}(z) =: R_{\vec{n}}^{(j)}(z) = \emptyset \left(\frac{1}{z^{n_j+1}} \right), \qquad z \to \infty,$$

- C. Hermite, Sur la fonction exponentielle, C.R. Acad. Sci. Paris 77 (1873), 18-24; 74-79; 226-233.

Limiting distribution of the poles of R.A.

- multi-index $\vec{n}$ is normal $\Rightarrow$ $P_{\vec{n}}(z) = \prod_{k=1}^{|\vec{n}|}(z - z_{k,\vec{n}})$
- diagonal case $\vec{n} = (n, n, \ldots, n)$

- Poles of Rational Approximants $\rightarrow$ Singularities of the function
- Goal is to determine the limiting distribution of the zeros of the common denominator $P_{\vec{n}}$:

$$\nu_{P_{\vec{n}}} = \frac{1}{n}\sum_{k=1}^{pn}\delta(z - z_{k,\vec{n}}) \xrightarrow{*} ? \qquad n \to \infty.$$

Class of functions

- Functions with finite number of branch points

$$f \in \mathcal{A}(\overline{\mathbb{C}} \setminus A), \quad \sharp A < \infty.$$

- Domain of convergence of R.A. $\leftrightarrow$ Domain of holomorphicity.

  Limiting positions of the poles of R.A. $\leftrightarrow$ Cuts making functions singlevalued.

Pade approximants for functions with branch points

- $(p = 1)$ Nuttall conjecture $(f \in \mathcal{A}(\overline{\mathbb{C}} \setminus A), \quad \sharp A < \infty)$;
- Stahl theorem $(\mathrm{cap}(A) = 0)$
  - $\exists$ extremal domain $\Omega^* : f \in H(\Omega^*)$
  - $$\nu_{P_n} \xrightarrow{*} \lambda, \qquad S(\lambda) = \Delta := \partial\Omega^*,$$
    where λ is the extremal measure of Δ.
  - $$\lim_{n\to\infty} \pi_n(z) = f(z), \qquad z \in \Omega^* \text{ in capacity},$$

Characterization of the extremal domain of holomorph.

- Extremal compact $\Delta \in \mathcal{F}_f := \{\partial\Omega : f \in H(\Omega)\}$:
- Compact of minimal capacity (Nuttall):

$$Cap\Delta = \min_{\mathcal{F}_f} Cap\partial\Omega.$$

- Compact with S - (symmetry) property (Stahl):

$$\begin{cases} V^{\lambda} = \text{const.} \\ \dfrac{\partial V^{\lambda}}{\partial n_+} = \dfrac{\partial V^{\lambda}}{\partial n_-} \end{cases} \quad \text{a.e. on } \Delta,$$

- Support of saddle point of energy functional (Rakhmanov):

$$I(\lambda) \;=\; \max_{\mathcal{F}_f} \min_{S(\sigma)\subset\partial\Omega\in\mathcal{F}_f} I(\sigma)\,.$$

Statement of problem

- H-P approximants

$$f_j \in \mathcal{A}(\overline{\mathbb{C}} \setminus A_j), \quad \sharp A_j < \infty, \quad j = 1, \dots, p.$$

- $p := 2, \quad \vec{n} := (n, n), \quad \deg P_n \leq |\vec{n}| = 2n$:

$$f_j(z)P_n(z) - Q_n^{(j)}(z) =: R_n^{(j)}(z) = \emptyset\left(\frac{1}{z^{n+1}}\right), \quad z \to \infty, \quad j = 1, 2.$$

- The limiting distribution of the zeros of the common denominator $P_{\vec{n}}$:

$$\nu_{P_n} = \frac{1}{n}\sum_{k=1}^{2n} \delta(z - z_{k,\vec{n}}) \xrightarrow{*} \lambda \ ? \qquad n \to \infty.$$

Example 1

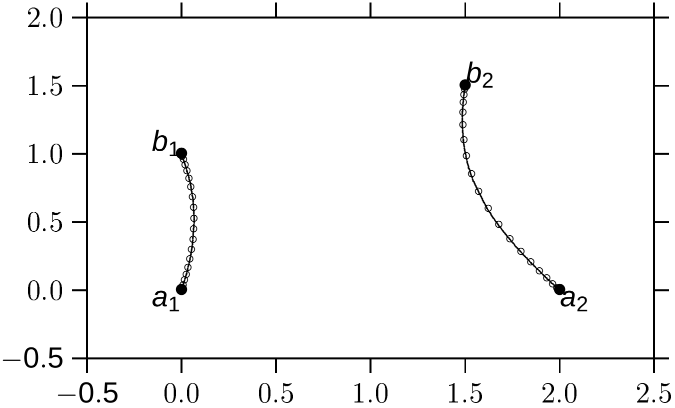


Рис.: $a_1 = 0$, $b_1 = i$, $a_2 = 2$, $b_2 = (3 + 3i)/2$: zeros of $P_{20,20}$

Example 2 (new phenomena)

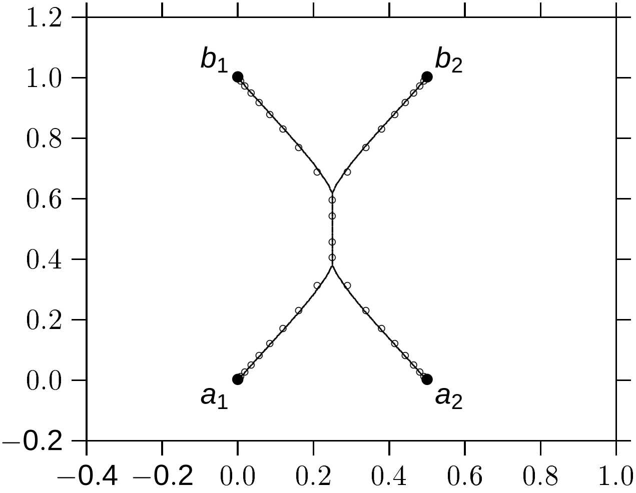


Рис.: $a_1 = i$, $b_1 = 0$, $a_2 = 1/2 + i$, $b_2 = 1/2$, zeros of $P_{20,20}$

Example 3 (new phenomena)

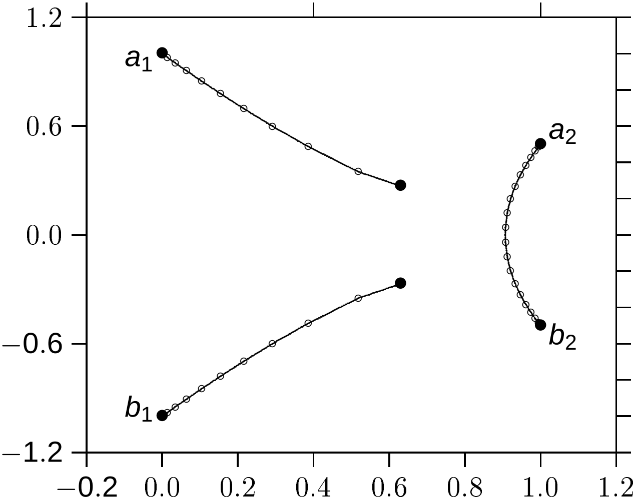


Рис.: $a_1 = i$, $b_1 = -1$, $a_2 = 1 + i/2$, $b_2 = 1 - i/2$.

New phenomena

- $S(\nu_{P_n}) \not\leftrightarrow$ cut joining branch points
- Extra interpolation points, i.e. finite zeros of

$$R_n^{(j)}(z) := f_j(z)P_n(z) - Q_n^{(j)}(z) = \emptyset\left(\frac{1}{z^{n+1}}\right), \quad z \to \infty, \quad j = 1, 2.$$

- We denote

$$\nu_{R_n^{(j)}} \xrightarrow{*} \mu_j\,, \quad j = 1, 2\,,$$
$$\nu_{P_n} \xrightarrow{*} \lambda\,.$$

- Goal (λ, μ_1, μ_2) !

Special classes

- Kalyagin (79 - Mat. Sb.), Nikishin (80 - Mat. Sb.)
- Gonchar and Rakhmanov (81 - Proc. Stekl.)
- Nuttall (81-84 - JAT)
- ....
- Apt., Kuijlaars and Van Assche (08 - IMRP)

General result $p = 2$

- Class of functions

$$\mathrm{f} := (f_1, f_2)\,, \quad f_j \in \mathcal{A}(\overline{\mathbb{C}} \backslash A_j)\,, \quad \sharp A_j < \infty, \quad j = 1, 2$$

A_j are br. points (+ .....)

- Class of compacts $\mathcal{F}_f := \{ F \,:\, f \in H(\Omega_F)\,,\ F := \partial\Omega_F \}$
- Class of vector compacts $\mathcal{F}^{(1)}_{\mathrm{f}} := \{ \mathrm{F} := (\mathrm{F}_1, \mathrm{F}_2, \mathrm{F}_3) \} :$

$$\begin{cases} \mathrm{F}_1 := F_1 \in \mathcal{F}_{f_1}\,, \quad \mathrm{F}_2 := \widetilde{F}_2 := F_2 \backslash F_1\,,\ F_2 \in \mathcal{F}_{f_2}\,, \\ \\ \mathrm{F}_3 := E \ :\ \left.\dfrac{f_{1+} - f_{1-}}{f_{2+} - f_{2-}}\right|_{F_1 \cap F_2} \in H(\Omega_E)\,,\ E = \partial\Omega_E\,. \end{cases}$$

- .....Dual class $\mathcal{F}^{(2)}_{\mathrm{f}}$

Example of $\mathcal{F}_{\mathrm{f}}^{(j)}$

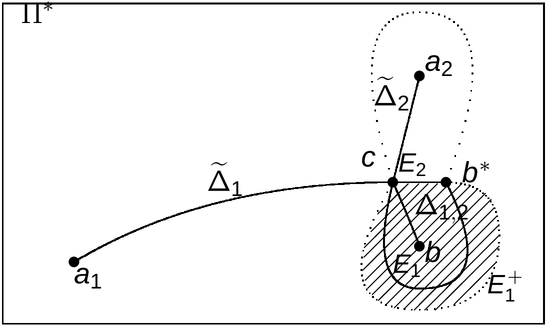
- $A_1 := \{a_1, b\} \quad A_2 := \{a_2, b\}$:
- 

Рис.: $F_1 := \Delta_{1,2} \bigcup \tilde{\Delta}_1,\ F_2 := \Delta_{1,2} \bigcup \tilde{\Delta}_2,\ E := E_1 \bigcup E_2$

Vector measures with matrix of interaction

- Class of vector measures on $\mathrm{F} \in \mathcal{F}_{\mathrm{f}}^{(j)}$ $j = 1, 2$:

$$\mathcal{L}_{\mathrm{F}} := \{(\sigma_1, \sigma_2, \sigma_3)\} \ : \ \begin{cases} S(\sigma_k) \subseteq \mathrm{F}_k \,, & k = 1, 2, 3 \\ \\ |\sigma_1| + |\sigma_2| = 2 \,, & |\sigma_1| - |\sigma_3| = 1 \end{cases} .$$

- Energy of vector measure with matrix of interaction D :

$$I_D(\vec{\sigma}) = \sum_i \sum_j d_{i,j} I(\sigma_i, \sigma_j).$$

- Matrix of interaction for $\mathcal{L}_{\mathrm{F}}$:

$$D := \begin{pmatrix} 2 & 1 & -1 \\ 1 & 2 & 1 \\ -1 & 1 & 2 \end{pmatrix} .$$

Statement of the result

- Theorem.
- 1) $\exists\,!\quad \vec{\lambda}^{(1)} := (\lambda_1, \tilde{\lambda}_2, \mu_1)$ and $\vec{\lambda}^{(2)} := (\lambda_2, \tilde{\lambda}_1, \mu_2)$:

$$I_D(\vec{\lambda}^{(j)}) \;=\; \max_{\mathrm{F}\in\mathcal{F}_{\mathrm{f}}^{(j)}} \min_{\vec{\sigma}\in\mathcal{L}_{\mathrm{F}}} I_D(\vec{\sigma})\,, \quad j = 1,2\,.$$

- 2) $n \to \infty$:

$$\nu_{P_n} \xrightarrow{*} \lambda := \lambda_1 + \tilde{\lambda}_2 = \lambda_2 + \tilde{\lambda}_1\,,$$
$$\nu_{R_n^{(j)}} \xrightarrow{*} \mu_j\,, \quad j = 1,2\,.$$