WDVV equations and invariant bi-Hamiltonian formalism

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First-order homogeneous operators

First-order homogeneous operators were introduced in 1983 by Dubrovin and Novikov for the Hamiltonian formalism of *quasilinear first-order PDEs* (or hydrodynamic PDEs)

$$u^i_t = v^i_j(\mathbf{u})u^j_x = P_1^{ij}\frac{\delta\mathcal{H}_1}{\delta u^j} \quad \mathcal{H}_1 = \int h(\mathbf{u})dx$$

$\mathbf{u} = (u^i(t,x))$, $i, j = 1,\ldots,n$ (n-components). The operators are of the form

$$\boxed{P_1^{ij} = g^{ij}(\mathbf{u})\partial_x + b^{ij}_k(\mathbf{u})u^k_x}$$

Homogeneity: $\deg\partial_x = 1$.

The interest for first-order homogeneous operators

- First examples for versions of the Euler equation of fluid dynamics.
- Dubrovin (1992): a solution F of the Witten–Dijkgraaf–Verlinde–Verlinde equation yields:
  - a bi-Hamiltonian pair of first-order homogeneous H.O. A_1, A_2 and, consequently, a quasilinear first-order system of PDEs
    $$u^i_t = V^i_j(\mathbf{u})u^j_x;$$
  - a Frobenius manifold;
- Invariants of bi-Hamiltonian pairs with respect to the Miura group have been introduced and studied (Dubrovin and Zhang, 2001).

Geometry of homogeneous operators

Any change of coordinates $\bar{u}^i = \bar{u}^i(u^j)$ will leave the 'form' of the system $\bar{u}^i_t = v^i_j(\bar{\mathbf{u}})\bar{u}^j_x$ and of the operator A_1 invariant. g^{ij} transforms as a *contravariant 2-tensor*; usually it is required that $\det(g^{ij}) \neq 0$; $\Gamma^j_{ik} = -g_{is}b^{sj}_k$ transforms as a *linear connection*.

Conditions on A_1 to be Hamiltonian:

- Skew-symmetry of $\{,\}_{A_1}$ is equivalent to: symmetry of g^{ij}, $\nabla[\Gamma]g = 0$;
- Jacobi identity of $\{,\}_{A_1}$ is equivalent to: g_{ij} flat pseudo-Riemannian metric and $\Gamma^j_{ik} = \Gamma^j_{ki}$, or Γ is the Levi-Civita connection of g.

It follows that the canonical form for such operators is $A_1 = \eta^{ij}\partial_x$, where η^{ij} are constants.

Witten–Dijkgraaf–Verlinde–Verlinde equations

The problem: in $\mathbb{R}^N$ find a function $F = F(t^1, \ldots, t^N)$ such that

1. $\dfrac{\partial^3 F}{\partial t^1 \partial t^\alpha \partial t^\beta} = \eta_{\alpha\beta}$ constant symmetric nondegenerate matrix
2. $c^\gamma_{\alpha\beta} = \eta^{\gamma\epsilon} \dfrac{\partial^3 F}{\partial t^\epsilon \partial t^\alpha \partial t^\beta}$ structure constants of an associative algebra
3. $F(c^{d_1} t^1, \ldots, c^{d_N} t^N) = c^{d_F} F(t^1, \ldots, t^N)$ quasihomogeneity $(d_1 = 1)$

If $e_1, \ldots, e_N$ is the basis of $\mathbb{R}^N$ then the algebra operation is

$$e_\alpha \cdot e_\beta = c^\gamma_{\alpha\beta}(\mathbf{t}) e_\gamma \quad \text{with unity} \quad e_1$$

WDVV equations of associativity

$$\eta^{\mu\lambda} \frac{\partial^3 F}{\partial t^\lambda \partial t^\alpha \partial t^\beta} \frac{\partial^3 F}{\partial t^\nu \partial t^\mu \partial t^\gamma} = \eta^{\mu\lambda} \frac{\partial^3 F}{\partial t^\nu \partial t^\alpha \partial t^\mu} \frac{\partial^3 F}{\partial t^\lambda \partial t^\beta \partial t^\gamma} \quad \text{(WDVV)}$$

Why study WDVV?

1. Solutions are related with Gromov–Witten invariants
2. Solutions correspond to integrable hierarchies (B. Dubrovin)
3. Applications to Quantum Field Theory (?)

Solutions of WDVV and bi-Hamiltonian pairs

(B. Dubrovin, '90) Let F be a solution of WDVV equations with homogeneity degrees $d_1, \dots d_N$. Let us set

$$c^{\delta\gamma}_{\beta} = \eta^{\delta\alpha}\eta^{\gamma\epsilon}\frac{\partial^3 F}{\partial t^\epsilon \partial t^\alpha \partial t^\beta}.$$

Then, the two operators

$$P_1 = \eta^{ij}\partial_x, \qquad P_2 = g^{ij}\partial_x + \Gamma^{ij}_k u^k_x$$

where, after replacing $t^k \to u^k$:

$$g^{ij} = c^{ij}_k d_k u^k$$

are Hamiltonian and compatible $[P_1, P_2] = 0$, hence they define an integrable system of PDEs of the form $u^i_t = V^i_j u^j_x$.

WDVV equations in detail

Two canonical forms by linear transformations of $(t^2, \dots, t^N)$, if the weights d_i are distinct (Dubrovin, LNM 1996):

$d_F \neq 3$: By linear transformations preserving e_1:

$$\eta^{(1)}_{\alpha\beta} = \delta_{\alpha+\beta,N+1} = \begin{pmatrix} 0 & & 1 \\ & \iddots & \\ 1 & & 0 \end{pmatrix}$$

$$F = \tfrac{1}{2}(t^1)^2 t^N + \tfrac{1}{2} t^1 \sum_{\alpha=2}^{N-1} t^\alpha t^{N-\alpha+1} + f(t^2, \dots, t^N);$$

$d_F = 3$: By linear transformations preserving e_1:

$$\eta^{(2)}_{\alpha\beta} = \begin{pmatrix} c & & 1 \\ & \iddots & \\ 1 & & 0 \end{pmatrix} \tag{1}$$

$$c \neq 0,\ F = \tfrac{c}{6}(t^1)^3 + \tfrac{1}{2} t^1 \sum_{\alpha=2}^{N} (t^\alpha)^2 + f(t^2, \dots, t^N).$$

Main example: WDVV in the case $N = 3$

If $N = 3$ we have a single equation on $f = f(t^2, t^3) = f(x, t)$.
Two cases:

- $\eta^{(1)}_{\alpha\beta}$:

$$f_{ttt} = f_{xxt}^2 - f_{xxx} f_{xtt}$$

- $\eta^{(2)}_{\alpha\beta}$:

$$f_{ttt} = \frac{-f_{xxt}^2 + f_{xxx} f_{xtt} + \mu f_{xtt}^2}{\mu f_{xxt} - 1}$$

WDVV equations as hydrodynamic systems

Construction by O. Mokhov (1995). Let us introduce coordinates

$$a = f_{xxx}, \quad b = f_{xxt}, \quad c = f_{xtt}.$$

Then the compatibility conditions in the two cases are

$$\begin{cases} a_t = b_x, \\ b_t = c_x, \\ c_t = (b^2 - ac)_x \end{cases} \quad \text{and} \quad \begin{cases} a_t = b_x, \\ b_t = c_x, \\ c_t = \left(\frac{ac - b^2 + \mu c^2}{\mu b - 1}\right)_x \end{cases}$$

The system on the left is bi-Hamiltonian (Ferapontov, Galvao, Mokhov, Nutku CMP'98) by a third-order and a first-order Hamiltonian operator of Dubrovin–Novikov type. **What about the system on the right?**

Higher-order homogeneous operators

Higher order homogeneous operators were introduced in 1984 by Dubrovin and Novikov. We can consider the second-order and third-order homogeneous operators:

$$\begin{aligned} R_2^{ij} =& g_2^{ij}(\mathbf{u})\partial_x^2 + b_{2\,k}^{ij}(\mathbf{u})u_x^k\partial_x \\ &+ c_{2\,k}^{ij}(\mathbf{u})u_{xx}^k + c_{2\,km}^{ij}(\mathbf{u})u_x^k u_x^m, \end{aligned}$$

$$\begin{aligned} R_3^{ij} =& g_3^{ij}(\mathbf{u})\partial_x^3 + b_{3\,k}^{ij}(\mathbf{u})u_x^k\partial_x^2 \\ &+ [c_{3\,k}^{ij}(\mathbf{u})u_{xx}^k + c_{3\,km}^{ij}(\mathbf{u})u_x^k u_x^m]\partial_x \\ &+ d_{3\,k}^{ij}(\mathbf{u})u_{xxx}^k + d_{3\,km}^{ij}(\mathbf{u})u_x^k u_{xx}^m + d_{3\,kmn}^{ij}(\mathbf{u})u_x^k u_x^m u_x^n. \end{aligned}$$

Example: 3-component WDVV equation

The $\eta^{(1)}$-associativity (WDVV) equation:

$$f_{ttt} = f_{xxt}^2 - f_{xxx} f_{xtt}$$

can be presented by $a = f_{xxx}$, $b = f_{xxt}$, $c = f_{xtt}$ as

$$a_t = b_x, \quad b_t = c_x, \quad c_t = (b^2 - ac)_x.$$

From Ferapontov, Galvao, Mokhov, Nutku, CMP (1997), there are two local homogeneous Hamiltonian operators, first-order P_1 and third-order P_3:

bi-Hamiltonian structure of WDVV equations

$$P_1 = \begin{pmatrix} -\frac{3}{2}\partial_x & \frac{1}{2}\partial_x a & \partial_x b \\ \frac{1}{2}a\partial_x & \frac{1}{2}(\partial_x b + b\partial_x) & \frac{3}{2}c\partial_x + c_x \\ b\partial_x & \frac{3}{2}\partial_x c - c_x & (b^2 - ac)\partial_x + \partial_x(b^2 - ac) \end{pmatrix},$$

$$P_3 = \begin{pmatrix} 0 & 0 & \partial_x^3 \\ 0 & \partial_x^3 & -\partial_x^2 a\partial_x \\ \partial_x^3 & -\partial_x a\partial_x^2 & (\partial_x^2 b\partial_x + \partial_x b\partial_x^2 + \partial_x a\partial_x a\partial_x) \end{pmatrix}$$

P_1 and P_3 are completely determined by their leading coefficients:

$$g_1^{ij} = \begin{pmatrix} -3/2 & 1/2\,a & b \\ 1/2\,a & b & 3/2\,c \\ b & 3/2\,c & 2(b^2 - ac) \end{pmatrix}, \quad g_3^{ij} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & -a \\ 1 & -a & 2b + a^2 \end{pmatrix}$$

Canonical form

It can be proved that third-order homogeneous Hamiltonian operators have the canonical forms (Potemin '86, '97; Potemin–Balandin, '01; Doyle '95)

$$P_3^{ij} = \partial_x \circ (g_3^{ij} \partial_x + c_{3\,k}^{ij} u_x^k) \circ \partial_x,$$

The skew-symmetry and Jacobi property of the Poisson brackets defined by the Hamiltonian operators are equivalent to:

$$c_{3\,nkm} = \frac{1}{3}(g_{3\,nm,k} - g_{3\,nk,m}), \quad \text{c determined by g}$$

$$g_{3\,mk,n} + g_{3\,kn,m} + g_{3\,mn,k} = 0, \quad \text{g Monge metric}$$

$$c_{3\,mnk,l} = -g_3^{pq} c_{3\,pml} c_{3\,qnk}.$$

where $c_{3\,ijk} = g_{3\,iq} g_{3\,jp} c_{3\,k}^{pq}$.

New results: projective invariance

Theorem Reciprocal transformations of projective type

$$d\tilde{x} = \Delta dx, \quad \tilde{u}^i = T^i(u^j) = (A^i_j u^j + A^i_0)/\Delta$$

with $\Delta = c_i u^i + c_0$ preserve the canonical form of third-order homogeneous operators (Ferapontov, Pavlov, V. JGP 2014). The leading terms are transformed as

$$g_{3\,ij} \to \frac{\tilde{g}_{3\,ij}}{\Delta^4}$$

where $\tilde{g}_{3\,ij}$ is of the same type as the initial metric; g_3 is identified with a quadratic line complex.

Classification

- $\mathbf{n = 1}$: trivial case, only ∂_x^3;
- $\mathbf{n = 2}$: affine classification: two nontrivial cases and one trivial case; projective classification: only

$$g_{3\,ij} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad A_3 = g_3^{ij}\partial_x^3$$

- $\mathbf{n = 3}$: C. Segre – Weiler classification of quadratic line complexes (Ferapontov, Pavlov, V. JGP 2014);
- $\mathbf{n = 4}$: classification of 4-dimensional subspaces of skew-symmetric 5×5 – matrices

$$\mathrm{span}(A^1, A^2, A^3, A^4) \subset \wedge^2 V, \tag{2}$$

under the action of $SL(5)$; from the projective classification of metabelian Lie algebrae (Galitski–Timashev 1999).
- $\mathbf{n \geqslant 5}$ **wild!**

Third-order operators and systems of conservation laws

Following a construction of Agafonov and Ferapontov (1996-2001) we associate to each system $u^i_t = (V^i)_{,j} u^j_x$ a congruence of lines in $\mathbb{P}^{n+1}$ with coordinates $[y^1, \ldots, y^{n+2}]$

$$y^i = u^i y^{n+1} + V^i y^{n+2}$$

A method introduced by Kersten, Krasil'shchik, Verbovetsky (JGP 2004) to characterize Hamiltonian operators yields the following compatibility conditions between the operator and quasilinear first-order systems:

$$g_{3\,im} V^m_j = g_{3\,jm} V^m_i, \tag{3}$$

$$c_{mkl} V^m_i + c_{mik} V^m_l + c_{mli} V^m_k = 0, \tag{4}$$

$$g_{3\,ks} V^k_{ij} = c_{smj} V^m_i + c_{smi} V^m_j. \tag{5}$$

WDVV: new results

When applied to the WDVV systems, the above equations allow to determine the third-order operators (Vašiček, V, Journal of High Energy Physics 2021):

- In the cases $N = 3$, $N = 4$, $N = 5$ both canonical forms of WDVV equations as quasilinear first-order systems of PDEs admit a third-order homogeneous Hamiltonian operator in canonical form.
- In the case $N = 3$ also the canonical form $\eta^{(2)}$ of WDVV equations as quasilinear first-order systems of PDEs admits a first-order homogeneous Hamiltonian operator. The operator is nonlocal of Ferapontov type.
- In the case $N = 3$ the bi-Hamiltonian pair is invariant with respect to $\partial/\partial t^1$-preserving affine coordinate changes in the WDVV space $(t^1, \ldots, t^N)$.

WDVV systems: new results

WDVV systems themselves turn out to have interesting projective goemetric properties:

Theorem. Every WDVV system (for $N = 3, 4, 5$), interpreted as a linear line congruence, has the following properties:

- The congruence is linear: there are n linear relations between u^i, V^i, $u^iV^j - u^jV^i$.
- The system is linearly degenerate, and non diagonalizable.
- The system admits non-local Hamiltonian, momentum and Casimirs.

WDVV, $N = 3$, $\eta = \eta^{(2)}$, third-order P_3:

The system of PDEs has a third-order homogeneous Hamiltonian operator defined by the Monge metric

$$g_{3\,ij} = \begin{pmatrix} b(\mu b - 2) & (a+\mu c)(1-\mu b) & (\mu b - 1)^2 \\ (a+\mu c)(1-\mu b) & \mu(a+\mu c)^2 + 1 & \mu(a+\mu c)(1-\mu b) \\ (\mu b - 1)^2 & \mu(a+\mu c)(1-\mu b) & \mu(\mu b - 1)^2 \end{pmatrix}, \tag{6}$$

and has the following form:

$$P_3 = \begin{pmatrix} -\mu\partial_x^3 & 0 & \partial_x^3 \\ 0 & \partial_x^3 & \partial_x^2 \frac{a+\mu c}{\mu b-1}\partial_x \\ \partial_x^3 & \partial_x \frac{a+\mu c}{\mu b-1}\partial_x^2 & \frac{1}{2}(\partial_x^2 K \partial_x + \partial_x K \partial_x^2) \end{pmatrix}, \tag{7}$$

where $K = \frac{(a+\mu c)^2 + b(2-\mu b)}{(\mu b-1)^2}$.

WDVV, $N = 3$, $\eta = \eta^{(2)}$, first-order P_1:

The system of PDEs has a non-local first-order homogeneous Hamiltonian operator of Ferapontov type

$$P_1^{ij} = g_1^{ij}\partial_x + \Gamma^{ij}_k u^k_x + \alpha V^i_q u^q_x \partial_x^{-1} V^j_p u^p_x + \beta\left(V^i_q u^q_x \partial_x^{-1} u^j_x + u^i_x \partial_x^{-1} V^j_q u^q_x\right) + \gamma u^i_x \partial_x^{-1} u^j_x, \quad (8)$$

defined by the metric (in upper indices)

$$g^{ij} = \begin{pmatrix} b^2\mu^2 - a^2\mu - 2b\mu - 3 & a - ab\mu + bc\mu^2 - c\mu & 2b - b^2\mu + c^2\mu^2 \\ a - ab\mu + bc\mu^2 - c\mu & 2b - b^2\mu + c^2\mu^2 & \frac{c(ac\mu^2 - 2b^2\mu^2 + 4b\mu + c^2\mu^3 - 3)}{b\mu - 1} \\ 2b - b^2\mu + c^2\mu^2 & \frac{c(ac\mu^2 - 2b^2\mu^2 + 4b\mu + c^2\mu^3 - 3)}{b\mu - 1} & \frac{\delta}{(b\mu - 1)^2} \end{pmatrix}, \quad (9)$$

where

$$\delta = a^2c^2\mu^2 - 2ab^2c\mu^2 + 4abc\mu + 2ac^3\mu^3 - 4ac + b^4\mu^2 - 4b^3\mu - 3b^2c^2\mu^3 + 4b^2 + 6bc^2\mu^2 + c^4\mu^4 - 5c^2\mu$$

and $\alpha = -\mu^2, \beta = 0, \gamma = \mu$.

Invariance of the bi-Hamiltonian formalism

The invariance group of the WDVV equations with the quasihomogeneity constraint is the group of linear transformations that preserve the direction of $\partial/\partial t^1$:

$$\tilde{t}^\alpha = P^\alpha_\beta t^\beta + Q^\alpha, \qquad \det(P^\alpha_\beta) \neq 0, \quad P^\alpha_1 = \delta^\alpha_1 \tag{10}$$

Theorem

Let $N = 3$, and suppose that a WDVV system in first-order form $u^i_t = (V^i(\mathbf{u}))_x$ is bi-Hamiltonian with respect to a pair of compatible Hamiltonian operators A_1, A_2, where A_1 is a nonlocal first-order HHO and A_2 is a local third-order HHO. Then, the invariance transformation does not change the form of the bi-Hamiltonian pair A_1, A_2.

Invariance in detail

The matrix $P = (P^\alpha_\beta)$ of the change of coordinates can be factorized as $P = T_1 \cdot T_2$ where

$$P = \begin{pmatrix} 1 & P^1_2 & P^1_3 \\ 0 & P^2_2 & P^2_3 \\ 0 & P^3_2 & P^3_3 \end{pmatrix}, \quad T_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & P^2_2 & P^2_3 \\ 0 & P^3_4 & P^3_3 \end{pmatrix}, \quad T_2 = \begin{pmatrix} 1 & P^1_2 & P^1_3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The matrix T_1 can be further factorized as $T_1 = R_1 \cdot E \cdot R_2$:

$$R_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \alpha & \beta \\ 0 & 0 & 1 \end{pmatrix}, \quad E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad R_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \gamma & \delta \\ 0 & 0 & 1 \end{pmatrix}.$$

when $P^3_2 \neq 0$ (when it is zero no factorization is needed). The above transformations *do not change* the form of the bi-Hamiltonian pair.

More canonical forms of η

Using *only* transformations of the type T_1, Mokhov and Pavlenko (TMP 2018) find four canonical forms for η:

$$\eta^1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & \lambda & 0 \\ 1 & 0 & \mu \end{pmatrix},\ \lambda^2 = 1; \quad \eta^3 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix};$$

$$\eta^2 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & \lambda & 0 \\ 1 & 0 & \mu \end{pmatrix},\ \lambda^2 = 1; \quad \eta^4 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \mu \end{pmatrix},\ \lambda^2 = 1,\ \mu^2 = 1.$$

We proved that all four canonical forms have a bi-Hamiltonian pair by a third-order and a first-order HHO. In the case η^1 the latter is local (found by Mokhov and Pavlenko), while in the cases η^2, η^3, η^4 it is nonlocal of Ferapontov type.

A distinguished example

This example was explicitly written by Dubrovin; Ferapontov associated it with the centroaffine geometry (equation of flat centroaffine metrics for surfaces in $\mathbb{R}^3$).

$$\eta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad f_{xxx}f_{yyy} - f_{xxy}f_{xyy} = 1, \tag{11}$$

and the system in first-order form reads:

$$a_t = b_x, \quad b_t = c_x, \quad c_t = \left(\frac{bc+1}{a}\right)_x. \tag{12}$$

The above system is bi-Hamiltonian by a pair as above, this time the first-order operator is characterized by $\alpha = \gamma = 0$, $\beta = -1$, which means that the Ferapontov operator is localizable by a reciprocal transformation.

WDVV, $N = 4$, $\eta = \eta^{(1)}$, third-order P_3

$$\eta^{(1)} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

WDVV equations are an overdetermined nonlinear system:

$$\begin{aligned}
&-2f_{xyz} - f_{xyy}f_{xxy} + f_{yyy}f_{xxx} = 0, \\
&-f_{xzz} - f_{xyy}f_{xxz} + f_{yyz}f_{xxx} = 0, \\
&-2f_{xyz}f_{xxz} + f_{xzz}f_{xxy} + f_{yzz}f_{xxx} = 0, \\
&f_{zzz} - (f_{xyz})^2 + f_{xzz}f_{xyy} - f_{yyz}f_{xxz} + f_{yzz}f_{xxy} = 0, \\
&f_{yyy}f_{xzz} - 2f_{yyz}f_{xyz} + f_{yzz}f_{xyy} = 0.
\end{aligned}$$

6-components WDVV system

We introduce new field variables u^k:

$$u^1 = f_{xxx}, u^2 = f_{xxy}, u^3 = f_{xxz}, u^4 = f_{xyy}, u^5 = f_{xyz}, u^6 = f_{xzz}.$$

The compatibility conditions for this system can be written as a pair of *commuting* hydrodynamic type systems in conservative form:

$$\begin{cases} u^1_y = u^2_x, \\ u^2_y = u^4_x, \\ u^3_y = u^5_x, \\ u^4_y = \left(\frac{2u^5 + u^2 u^4}{u^1}\right)_x \\ u^5_y = \left(\frac{u^3 u^4 + u^6}{u^1}\right)_x \\ u^6_y = \left(\frac{2u^3 u^5 - u^2 u^6}{u^1}\right)_x \end{cases} \qquad \begin{cases} u^1_z = u^3_x, \\ u^2_z = u^5_x, \\ u^3_z = u^6_x, \\ u^4_z = \left(\frac{u^3 u^4 + u^6}{u^1}\right)_x \\ u^5_z = \left(\frac{2u^3 u^5 - u^2 u^6}{u^1}\right)_x \\ u^6_z = \Big((u^5)^2 - u^4 u^6 + \\ \qquad \frac{(u^3)^2 u^4 + u^3 u^6 - 2u^2 u^3 u^5 + (u^2)^2 u^6}{u^1}\Big)_x \end{cases}$$

WDVV as a bi-Hamiltonian system

Theorem: (Pavlov, V. LMP 2015) The leading term of a third-order Hamiltonian operator for *both* the previous hydrodynamic-type systems:

$$g_{ik}(\mathbf{u}) = \begin{pmatrix} (u^4)^2 & -2u^5 & 2u^4 & -(u^1u^4+u^3) & u^2 & 1 \\ -2u^5 & -2u^3 & u^2 & 0 & u^1 & 0 \\ 2u^4 & u^2 & 2 & -u^1 & 0 & 0 \\ -(u^1u^4+u^3) & 0 & -u^1 & (u^1)^2 & 0 & 0 \\ u^2 & u^1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Both systems are bi-Hamiltonian with the above homogeneous third-order Hamiltonian operator and a (compatible) local first-order Hamiltonian operator.

Remarks

- In the case $N = 3$ the compatibility (*very hard* computation) was checked with the new algorithm and software packages by Casati, Lorenzoni, V, Studies in Appl. Math. 2020; Casati, Lorenzoni, Valeri, V, Comp. Phys. Comm. 2021 (to appear).
- In the cases $N = 4$, $N = 5$ and $\eta^{(1)}$, $\eta^{(2)}$ WDVV equations as quasilinear first-order systems of PDEs admit a third-order homogeneous Hamiltonian operator in canonical form.
- Compatible first order operators are not known in the cases: $N = 4$, $\eta^{(2)}$; $N = 5$, $\eta^{(1)}$; $N = 5$, $\eta^{(2)}$.

Perspectives

WDVV equations	**Projective Geometry**
Third-order Hamiltonian operator	Quadratic Line Complex
Quasilinear system of PDEs	Linear Line Congruence
Comp. first-order Hamiltonian operator	???

The projective-geometric invariance of the corresponding hierarchies has *implications that are yet to be understood.* Initial analysis in original coordinates of the equation $f_{ttt} = f_{xxt}^2 - f_{xxx}f_{xtt}$ (Kersten, Krasil'shchik, Verbovetsky, V. TMP 2010) suggests that the key to understanding is there.

Remark: The equations for F-manifolds in the simplest case are endowed with a non-local homogeneous third-order Hamiltonian operator (Pavlov, V. JPA 2019).

Symbolic computations

Within the REDUCE CAS (now free software) we use the packages CDIFF and CDE, freely available at `https://reduce-algebra.sourceforge.io/`.

CDE (by RV) can compute symmetries and conservation laws, local and nonlocal Hamiltonian operators, Schouten brackets of local multivectors, Fréchet derivatives (or linearization of a system of PDEs), formal adjoints, Lie derivatives of Hamiltonian operators, anticommuting variables and super-PDEs.

Cooperation with AC Norman (Trinity College, Cambridge) to improvements and documentation of REDUCE's kernel.

A book, in cooperation with JS Krasil'shchik and AM Verbovetsky: *The symbolic computation of integrability structures for partial differential equations*, is published in the series Texts and Monographs in Symbolic Computation, Springer, 2018.

Thank you!

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