Rodrigues' descendants of a polynomial and Boutroux curves

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Rodriques formula

Around 1816 (Benjamin) Olinde Rodrigues discovered his famous formula

$$P_n(z) = \frac{1}{2^n n!} \frac{\mathrm{d}^n}{\mathrm{d}z^n} \left((z^2 - 1)^n\right) \tag{1}$$

for the Legendre polynomials which undoubtedly became a standard tool in the toolbox of classical orthogonal polynomials and special functions. (Later this formula was also rediscovered by Sir J. Ivory and C. G. Jacobi.)

Among other properties, the n-th Legendre polynomial $P_n(z)$ satisfies the linear ordinary differential equation

$$(1 - z^2)y'' - 2zy' + n(n+1)y = 0, \tag{2}$$

and the asymptotic of the zeros as $n \to \infty$ is described by classical results.

Main Problem

Imitating Rodrigues' approach, given a polynomial P of degree $d \geq 1$, let us consider a double-indexed family of polynomials determined by the Rodrigues-like expression

$$\mathbb{R}_{m,n,P}(z) := \frac{\mathrm{d}^m}{\mathrm{d}z^m}\left(P^n(z)\right), \; n = 0, 1, \dots \text{ and } m = 0, 1, \dots, nd.$$

These polynomials which we below call *Rodrigues' descendants* of P were apparently for the first time considered by N. Ciorânescu in 1933 where he, in particular, derived linear differential equations satisfied by them. In 1965, and, to the best of our knowledge, independently of N. Ciorânescu's work a linear differential equation satisfied by $\mathbb{R}_{n,n,P}(z)$ has been (re)discovered by J. M. Horner.

If $P = z^2 - 1$ and $m = n$, we get the above classical case of the Legendre polynomials up to a scalar factor.

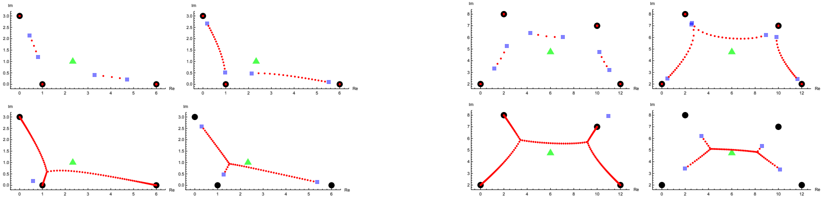

Figure: The zeros of $\mathbb{R}_{m,60,P}(z)$ shown by the small red dots. (The larger dots are the zeros of P, the triangle is the center of mass of the zero locus of P, and the squares are branch points of (3) and (8) in the z-plane when $\alpha = m/60$.) Both in the left and in the right subfigures, $m = 3$ (bottom left), $m = 18$ (bottom right), $m = 60$ (top right), and $m = 60(\deg P - 1)$ (top left).

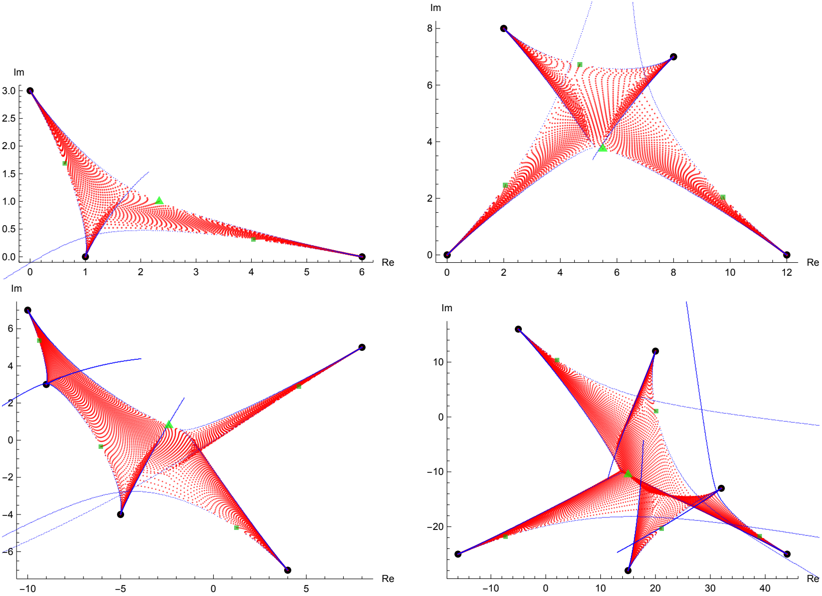

Figure: The union of all zeros of $\mathbb{R}_{m,30,P}(z)$ for $m = 0, 1, \ldots, 30 \deg P - 1$ shown by small red dots.

Main results

For any polynomial P and its Rodrigues' descendant $\mathbb{R}_{m,n,P}(z)$, denote by $\mu_{m,n,P}$ the *root-counting measure* of $\mathbb{R}_{m,n,P}(z)$ and by

$$\mathcal{C}_{m,n,P}(z) := \frac{\mathbb{R}'_{m,n,P}(z)}{(dn-m)\cdot \mathbb{R}_{m,n,P}(z)}$$

its *Cauchy transform*. Notice that $dn - m = \deg \mathbb{R}_{m,n,P}$. We say that a polynomial P is *strongly generic* if both P and P' have simple roots.

Theorem

For any strongly generic polynomial P and a given positive number $\alpha < \deg P$, there exists a weak limit

$$\mu_{\alpha,P} := \lim_{n\to\infty} \mu_{[\alpha n],n,P}.$$

Moreover, its Cauchy transform $\mathcal{C}_{\alpha,P}$ defined as the pointwise limit

$$\mathcal{C} := \mathcal{C}_{\alpha,P}(z) := \lim_{n\to\infty} \mathcal{C}_{[\alpha n],n,P}(z)$$

exists almost everywhere (a.e.) in $\mathbb{C}$ and satisfies the algebraic equation

$$\sum_{k=0}^{d} \frac{\alpha^{k-1}(\alpha-k)(d-\alpha)^{d-k}}{k!} P^{(k)}\mathcal{C}^{d-k} = 0. \tag{3}$$

Example

(i) For $P = z^2 + az + b$, equation (3) reduces to

$$(2-\alpha)(z^2+az+b)\mathcal{C}^2 + (\alpha-1)(2z+a)\mathcal{C} - \alpha = 0. \qquad (4)$$

(ii) For $P = z^3 + az^2 + bz + c$, it reduces to

$$(3-\alpha)^2(z^3+az^2+bz+c)\mathcal{C}^3+(\alpha-1)(3-\alpha)(3z^2+2az+b)\mathcal{C}^2+\alpha(\alpha-2)(3z+ \qquad (5)$$

Reinterpretation of formula (3) in Theorem 1 implies the following result.

Corollary

The Cauchy transform $\mathcal{C} := \mathcal{C}_{\alpha,P}(z)$ of the limiting measure $\mu_{\alpha,P}$ satisfies the equation

$$(d-\alpha)\mathcal{C} = \frac{\mathrm{d}}{\mathrm{d}z}\log P(z+\frac{\alpha}{d-\alpha}\mathcal{C}^{-1}). \qquad (6)$$

Proposition

For the asymptotic root-counting measure $\mu_{\alpha,P}$ as in Theorem 1, its scaled Cauchy transform $\mathcal{W}$ defined by

$$\mathcal{W} := \mathcal{W}_{\alpha,P} := \frac{d-\alpha}{\alpha}\mathcal{C}_{\alpha,P} \tag{7}$$

satisfies a.e. in $\mathbb{C}$ the algebraic equation

$$\sum_{k=0}^{d} \frac{\alpha-k}{k!} P^{(k)}\mathcal{W}^{d-k} = 0. \tag{8}$$

General sketch

Set

$$H(z,u) := \frac{1}{d-\alpha}(\log|P(u)| - \alpha \log|u-z|). \tag{9}$$

Let $\pi : \mathbb{C}_z \times \mathbb{C}_u \to \mathbb{C}_z$ be the standard projection, and $\mathcal{D} \subset \mathbb{C}_z \times \mathbb{C}_u$ be the *saddle point curve* of H. ($\mathcal{D}$ is a rational curve defined by an explicit algebraic equation (12) whose coefficients depend on P and α.) Further, let $U_{rel} \subset \mathcal{D}$ be the open set of relevant saddle points.

Denote by $\tilde{\pi} : U_{rel} \to O \subset \mathbb{C}_z$ the restriction of π to U_{rel} and define the *tropical trace* $\tilde{\pi}_* H(z) : O \to \mathbb{R} \cup \pm\infty$ as a piecewise-harmonic function obtained by taking the fiberwise maximum of $H(z,u)$.

Theorem

In the above notation, for any strongly generic polynomial P of degree $d \geq 2$, there exists a real number B (given by an explicit formula) such that

$$\lim_{n \to \infty} L_{\mu_{[\alpha n],n,P}}(z) = B + \tilde{\pi}_* H(z),$$

where the above relation is understood as the equality of L^1_{loc}-functions. Here L_μ stands for the logarithmic potential of a measure μ. Consequently,

$$\lim_{n \to \infty} \mathcal{C}_{\mu_{[\alpha n],n,P}}(z) = 2 \frac{\partial}{\partial z} \tilde{\pi}_* H(z),$$

and

$$\lim_{n \to \infty} \mu_{[\alpha n],n,P} = \mu_{\alpha,P} := \frac{2}{\pi} \frac{\partial^2}{\partial z \partial \bar{z}} \tilde{\pi}_* H(z),$$

where the latter two limits are understood in the sense of distributions and $\mu_{\alpha,P}$ is a positive measure.

Proposition

The Rodrigues' descendant $\mathbb{R}_{m,n,P/Q}(z)$ *of a rational function* $P(z)/Q(z)$ *satisfies the linear homogeneous differential equation*

$$\sum_{i=0}^{d} \sum_{j=0}^{i} \frac{m+d+(n-1)i-2nj}{(m+d-i)!\,(i-j)!\,j!} P^{(j)} Q^{(i-j)} y^{(d-i)} = 0 \qquad (10)$$

of order $d = \deg P + \deg Q$*.*

Proposition

The Rodrigues' descendant $\mathbb{R}_{m,n,P}(z)$ *of a polynomial* $P(z)$ *satisfies the linear homogeneous differential equation*

$$\sum_{i=0}^{d} \frac{(m-nd)-(i-d)(n+1)}{(d+m-i)!\,i!} P^{(i)} y^{(d-i)} = 0 \qquad (11)$$

of order $d = \deg P$*.*

Remark

Differential equations satisfied by $\frac{\mathrm{d}^m}{\mathrm{d}z^m}\left(Q_1^{N_1} Q_2^{N_2} \cdots Q_d^{N_d}\right)$, where $Q_1, \ldots, Q_d$ are polynomials in z and $N_1, \ldots, N_d$ are nonnegative integers, were previously derived by Ciorânescu. As Ciorânescu remarks, one of these differential equations looks strikingly similar to Pochhammer's generalized Gaussian differential equation. A special case was later rediscovered by J. M. Horner.

The original Rodrigues' formula inspires the following consequence of Proposition 7.

Defining affine Boutroux curves

Consider an irreducible affine algebraic curve $\Upsilon \subset \mathbb{C}_z \times \mathbb{C}_u$, where the product $\mathbb{C}_z \times \mathbb{C}_u$ is equipped with coordinates (z, u).

Denote by $\widehat{\Upsilon} \subset \mathbb{C}P^1_z \times \mathbb{C}P^1_u$ the closure of Υ. Let $\pi : \mathbb{C}_z \times \mathbb{C}_u \to \mathbb{C}_z$ (resp. $\pi : \mathbb{C}P^1_z \times \mathbb{C}P^1_u \to \mathbb{C}P^1_z$) be the standard projection onto the first coordinate. Additionally, denote by $\mathfrak{n} : \widetilde{\Upsilon} \to \widehat{\Upsilon}$ the normalisation map. (Recall that the smooth compact Riemann surface $\widetilde{\Upsilon}$ is birationally equivalent to Υ.)

Now consider the standard meromorphic 1-form

$$\Omega := u \, \mathrm{d}z$$

defined on $\mathbb{C}_z \times \mathbb{C}_u$ (resp. on $\mathbb{C}P^1_z \times \mathbb{C}P^1_u$).

Remark

One can easily show that the zero divisor of Ω on $\mathbb{C}P^1_z \times \mathbb{C}P^1_u$ is a copy of $\mathbb{C}P^1$ given by $u = 0$; (the closure of) its pole divisor is the union of two intersecting copies of $\mathbb{C}P^1$ given by $u = \infty$ and $z = \infty$.

Given a curve $\Upsilon \subset \mathbb{C}_z \times \mathbb{C}_u$ as above, consider the meromorphic 1-form

$$\Omega_\Upsilon := \Omega|_\Upsilon \quad (\text{resp.} \quad \Omega_{\widehat{\Upsilon}} := \Omega|_{\widehat{\Upsilon}})$$

obtained by the restriction of Ω to Υ (resp. to $\widehat{\Upsilon}$). Denote by $\widetilde{\Omega}$ the pullback of $\Omega_{\widehat{\Upsilon}}$ to $\widetilde{\Upsilon}$ under the normalisation map $\mathfrak{n} : \widetilde{\Upsilon} \to \widehat{\Upsilon}$. This form will be the key ingredient below.

Remark

The zero divisor of $\Omega_{\widehat{\Upsilon}}$ consists of the intersection points $\widehat{\Upsilon}$ with the line $u = 0$ and all the singularities of $\widehat{\Upsilon} \subset \mathbb{C}P^1_z \times \mathbb{C}P^1_u$. The pole divisor of $\Omega_{\widehat{\Upsilon}}$ consists of all non-singular points of the intersection of $\widehat{\Upsilon}$ with the union of the lines $z = \infty$ and $u = \infty$.

Further, given an irreducible affine curve $\Upsilon \subset \mathbb{C}_z \times \mathbb{C}_u$ as above and the corresponding meromorphic 1-form $\widetilde{\Omega}$ on $\widetilde{\Upsilon}$, consider the multi-valued primitive function

$$\Psi(p) = \int_{p_0}^{p} \widetilde{\Omega}.$$

$\Psi(p)$ is a well-defined uni-valued function on the universal covering of $\widetilde{\Upsilon} \setminus Pol$, where $Pol \subset \widetilde{\Upsilon}$ is the set of all poles of $\widetilde{\Omega}$ and $p_0 \in \widetilde{\Upsilon} \setminus Pol$ is some fixed base point. The next statement is trivial.

Lemma

In the above notation, $\widetilde{\Omega}$ has purely imaginary periods if and only if the multi-valued primitive function $\Psi(p)$ has a uni-valued real part $\mathrm{Re}\,\Psi(p)$. In other words, $\mathrm{Re}\,\Psi(p)$ is a well-defined uni-valued function on $\widetilde{\Upsilon} \setminus Pol$.

The following class of curves has been introduced by M.Bertola and extensively studied there in the context of hyperelliptic curves and orthogonal polynomials.

Definition

A plane affine irreducible curve $\Upsilon \subset \mathbb{C}_z \times \mathbb{C}_u$ is called an *affine Boutroux curve* (aBc, for short) if the meromorphic 1-form $\widetilde{\Omega}$ has purely imaginary periods on $\widetilde{\Upsilon}$.

Next we introduce yet another algebraic curve which will naturally reappear later in connection with the application of the saddle point method.

Definition

Given a polynomial P of degree d and $0 < \alpha < d$, we define its affine *saddle point curve* $\mathcal{D} := \mathcal{D}_{\alpha,P} \subset \mathbb{C}_z \times \mathbb{C}_u$ as the curve given by the equation:

$$\frac{P'(u)}{\alpha P(u)} - \frac{1}{u-z} = 0. \tag{12}$$

Following our notational conventions, we denote by $\widehat{\mathcal{D}}$ the closure of $\mathcal{D}$ in $\mathbb{C}P^1_z \times \mathbb{C}P^1_u$.

It turns out that $\mathcal{D} := \mathcal{D}_{\alpha,P}$ is closely related to the symbol curve $\Gamma := \Gamma_{\alpha,P}$. Namely, consider the birational transformation $\chi : \mathbb{C}P^1_z \times \mathbb{C}P^1_u \to \mathbb{C}P^1_z \times \mathbb{C}P^1_{\mathcal{C}}$ sending (z,u) to $(z,\mathcal{C})$ where $z \mapsto z$ and

$$\mathcal{C} = \frac{\alpha}{d-\alpha} \cdot \frac{1}{u-z} \iff u = z + \frac{\alpha}{d-\alpha} \cdot \mathcal{C}^{-1}. \qquad (13)$$

Under this change of variables equation (12) transforms into equation (6) which is equivalent to (3). Thus the restriction $\chi : \widehat{\mathcal{D}} \to \widehat{\Gamma}$ provides a birational isomorphism. (Observe that χ sends the complement of the line $z = u$ isomorphically to the complement of the line $z = \infty$.)

Given a branched covering $\nu : Y \to Y'$ of Riemann surfaces, and a function $f : Y \to \mathbb{R}$, we will define the induced function on Y' by taking the maximum of the values of f over each fiber. Notice that in the case of the usual trace one uses the summation/integration over the fiber. The basic idea of tropical geometry is to substitute the operation of summation/integration by the operation of taking the maximum, which provides a motivation for our terminology. It seems that this construction which regularly occurs in the study of the root asymptotic for polynomial sequences has not been given any special name yet.

Definition

Given a branched covering $\nu : Y \to Y'$ and a real-valued function $f : Y \to \mathbb{R}$, we define the *tropical trace* $\nu_* f : Y' \to \mathbb{R}$ of this pair as

$$\nu_* f(z) = \max_{y_i \in \nu^{-1}(z)} f(y_i).$$

The same definition extends to real-valued functions f defined on $Y \setminus S$, where S is a discrete set such that for any $s \in S$, $\lim_{z \to s} f(z)$ exists either as a real number or $\pm\infty$. (In other words, we allow f to attain values $\pm\infty$.)

About the proof

Our main tool will be the classical saddle point method. Let as above P be a monic polynomial of degree $d \geq 2$ and $\alpha \in (0, d)$. Slightly abusing our previous notation, let $\mu_n := \mu_{[\alpha n]-1,n,P}$ be the root-counting measure of the Rodrigues' descendant

$$\mathfrak{q}_n(z) := \mathbb{R}_{[\alpha n]-1,n,P}(z) = (P^n)^{([\alpha n]-1)}(z).$$

For any $z \in \mathbb{C}$, Cauchy's formula for higher order derivatives gives

$$\mathfrak{q}_n(z) = \frac{([\alpha n]-1)!}{2\pi i} \int_c \frac{P^n(u)\,\mathrm{d}u}{(u-z)^{[\alpha n]}}, \tag{14}$$

where c is any simple closed curve in $\mathbb{C}$ encircling z once in the counterclockwise direction. (Here we use the fact that P has no poles.)

The saddle point method allows us to analyze the asymptotic of (14) when $n \to \infty$. The degree of the polynomial $\mathfrak{q}_n(z)$ equals $d_n := dn - [\alpha n] + 1$. Below we will calculate the limit of the sequence $\{L_{\mu_n}(z)\}$ of logarithmic potentials of μ_n, where $L_{\mu_n}(z) := \frac{1}{d_n} \log |\mathfrak{q}_n(z)/a_n|$ and a_n is the leading coefficient of $\mathfrak{q}_n(z)$.

We show that the critical points of the integrand in (14) belong to the above saddle point curve $\mathcal{D} := \mathcal{D}_{\alpha,P}$ given by (12), which is birationally equivalent to the symbol curve $\Gamma := \Gamma_{\alpha,P}$ given by (3). Furthermore we will see that the critical points which will play an important role in our asymptotic calculation form an open subset $U \subset \mathcal{D}$.

These facts enable us to identify the limit $L_\mu(z) := \lim_{n\to\infty} L_{\mu_n}(z)$ with the tropical trace of a natural harmonic function defined on U. Finally, applying the Laplace operator to L_μ, one obtains as an immediate consequence that the limiting asymptotic measure $\mu := \lim_{n\to\infty} \mu_n$ exists and that its Cauchy transform satisfies the algebraic equation (14).

Given $\alpha > 0$, define

$$s_n := n - \frac{[\alpha n]}{\alpha}, \tag{15}$$

where $0 \le s_n < 1/\alpha$ and set $m := [\alpha n]$. Consider the integral

$$I(z) = \int_\gamma \frac{P^n(u)\,\mathrm{d}u}{(u-z)^{[\alpha n]}}$$

over a curve segment γ that neither contains z nor the zeros of P.

Then

$$I_{P,m,s_n,\gamma}(z) := \int_\gamma \left(\frac{P^{1/\alpha}(u)}{u-z}\right)^m P^{s_n}(u)\,\mathrm{d}u = \int_\gamma e^{k(z,u)m}(P(u))^{s_n}\,\mathrm{d}u, \tag{16}$$

where

$$k(z,u) := \frac{1}{\alpha}\log P(u) - \log(u-z). \tag{17}$$

Clearly, for fixed z, $k(z,u)$ is holomorphic w.r.t the second variable $u \in \mathcal{O}$ if u avoids both z and the zeros of P.

Assume that

(i) $h(u)$ is any function holomorphic in a neighbourhood $\mathcal{O}$ of a simple curve γ;
(ii) $u^* \in \gamma$ is a saddle point of $h(u)$ and it is an inner point of γ;
(iii) $\forall u \in \gamma$ such that $u \neq u^*$, Re $h(u) <$ Re $h(u^*)$.

Finally, let $\ell \geq 2$ be the order of the saddle point u^*, i.e.,

$$h(u) = h(u^*) - h_0(u - u^*)^\ell (1 - \phi(u)), \tag{18}$$

where $h_0 \neq 0$ and $\phi(u)$ is a function which vanishes at u^* and is holomorphic in a small neighborhood of u^*.

Lemma

Using the above notation, for $m \in \mathbb{N}$ and $0 \le s \le A < \infty$, consider

$$I_{m,s,\gamma} := \int_\gamma e^{h(u)m} P^s(u) \, \mathrm{d}u.$$

Then, under the above assumptions (i)–(iii),

$$I_{m,s,\gamma} = e^{h(u^*)m} \left(\Gamma(\ell^{-1}) \frac{\beta_0(\epsilon_1 - \epsilon_2)}{m^{\frac{1}{\ell}}} + O\left(\frac{K(P)}{m^{\frac{2}{\ell}}} \right) \right), \tag{19}$$

where ϵ_1 and ϵ_2 are two distinct ℓ-th roots of unity depending only on γ and $K(P)$ is an upper bound of $|P^s(u)|$ in $\mathcal{O}$. Here Γ stands for the gamma function. The constant β_0 is given by

$$\beta_0 = \frac{1}{\ell} \cdot h_0^{-1/\ell}(P^s(u^*))$$

and the implicit constant in the remainder term $O(\dots)$ of (19) is independent of P, s, and m.

We are going to apply Lemma 12 to the integral (16). Note that

$$\text{Re } k(z,u) = \frac{1}{\alpha} \log|P(u)| - \log|(u-z)|$$

is defined independently of the above choice of a branch of the logarithm.

Corollary

For fixed z, assume that u^* and the path γ satisfy the assumptions (ii)–(iii) *of Lemma 12. Then, in the notation of (16), for any sequence $0 \le s_n < 1/\alpha$, one has*

$$\lim_{n\to\infty} |I_{P,m,s_n,\gamma}(z)|^{1/m} = e^{Re\ k(z,u^*)}. \tag{20}$$

This convergence is uniform in s.

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